Research Article

Characterisation Of Nuclear Through Summability

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Abstract: This paper makes an attempt to analyze nuclear locally convex space, co-nuclear spaces and summability with the aim to identify some new properties of nuclear locally convex spaces and co-nuclear spaces. In this contest, a new character of nuclear locally convex spaces was found. It is proved that a nuclear locally convex space E is co-nuclear if E is boundedly summable.

Keywords: boundedly summable; co-nuclear Space, locally convex spaces nuclear space, summable.

1. Introduction

The theory of nuclear spaces was developed by A. Grothendieck in 1955[1-4]. A nuclear space is a topological vector space with many of the good properties of finite dimensional vector space. The topology on them can be defined by a family of seminorms, whose unit balls decrease rapidly in size. In other words, a nuclear space is a locally convex topological vector space such that for any seminorm p the natural map V to V_p is nuclear [1].

A co-nuclear space is also a locally convex space whose strong topological dual is a nuclear space [7].

Grothendieck established already the following important internal criterion: a metrizable locally convex space is nuclear if and only if every summable sequence in it is also absolutely summable [9-10]. Pietsch characterized nuclear spaces are those locally convex spaces for which appropriately topologized spaces of summable and absolutely summable sequences are the same algebraically [2].

The definition of summable families of vectors was given by E.H. Moore who was able to show that an infinite series of real or complex numbers converges unconditionally if and only if it is summable. The great advantage of this definition is that it can also be applied to uncountable families. The study of families in normed spaces was undertaken by T.H. Hildebrandt [2].

For a perfect sequence space Λ and a locally convex space E, A. Pietsch introduced the space Λ[E] of all weakly Λ-summable sequences in E and the space Λ(E) of all Λ summable sequences in E. He characterized the nuclearity of E in terms of the summability of its sequences[2-9].

In the theory of nuclear locally convex spaces there is a famous unsolved problem. This problem can be stated as follows:- "Is every nuclear space co-nuclear?"

In the present paper, an analysis of nuclear and co-nuclear spaces has been made through summability and identified that a nuclear locally convex space E is co-nuclear if E is boundedly summable.

Let A= {α} be an arbitrary index set. The class {f} of all finite subsets f of A forms a directed set with respect to the relation ⊇ for sets.

Let E be a locally convex topological vector space and $x_\alpha \in$ E for each α ϵ A. For each finite set f, we form the sum $S_f = \sum_{\alpha \in f} x_\alpha = \sum_f x_\alpha$.

This association of S_f with f defines a net in E since it is a mapping of a directed set into E.

Define the class $\vec{x} = (x_\alpha)_{\alpha \in A} = (x_\alpha)$ in E.

If E is a locally convex space, we shall write

$\ell_I^1[E] = \ell_I^1\{E\}$,

or $\ell_I^1(E) = \ell_I^1\{E\}$

according to the linear space $\ell_I^1\{E\}$ coincides with $\ell_I^1[E]$ or $\ell_I^1(E)$ algebraically.

Besides if the ε-topology and the π-topology also coincide we shall write

$\ell_I^1[E] \cong \ell_I^1\{E\}$ or, $\ell_I^1(E) \cong \ell_I^1\{E\}$ accordingly.

2. Definitions

2.1. Definition: A topological vector space is said to be locally convex if each point has fundamental system of convex neighborhoods [4].

2.2. Definition: A nuclear space is a locally convex topological vector space such that for any seminorm p the natural map from V to V_P is nuclear [7].

2.3. Definition: A locally convex space is called a co-nuclear space if its strong topological dual is a nuclear space [2-7].

2.4. Definition A class $\vec{x}= (x_\alpha)_{\alpha \in A}$ in a locally convex space E is said to be summable if $\{s_f\}$is a Cauchy net in E i.e. if given any neighborhood of 0,U ϵ u(E), $\exists f_0$ such that $f_1, f_2 \supseteq f_0 \Rightarrow s_{f_1} - s_{f_2} \in$ U [2-3].

2.5. Definition: A locally convex space E is called boundedly summable if for every bounded subset M of $\ell^1\{E\}$ there exists a bounded subset B of E such that

$\sum_{n=1}^{\infty} P_B(x_n) \leq 1$ for all $\vec{x}= (x_n) \in M$ [2].

3. Some facts

3.1. $\ell_I^1(E)$ is a vector subspace of $\ell_I^1[E]$.

3.2. $\ell_I^1\{E\} \subseteq \ell_I^1(E)$.

3.3. The identity map of $\ell_I^1\{E\}$ into $\ell_I^1(E)$ is continuous. Hence, the π-topology is finer than the ε-topology [9].

4. Theorems/Prepositions

4.1. Proposition: If E is a locally convex nuclear space, then for every index set I the relation

$\ell_I^1[E] \cong \ell_I^1(E) \cong \ell_I^1\{E\}$ holds.

Proof:

Let E be a locally convex nuclear space and I be any index set. By nuclearity we know that, given any neighborhood ∪ ϵ u(E), ∃V ϵu(E) with V <∪ and a sequence $\{\cup_n\} \subseteq E_{V^0}$ with

$\sum_{n=1}^{\infty} \| u_n \|_{V^0} \leq 1$ …………………………..……(1)

such that

$P_\cup(x) \leq \sum_{n=1}^{\infty} | < u_n, x > |$ for all x ϵ E ...……….(2)

Now let the family x = $(x_i)_{i\epsilon I}$be an arbitrary element of $\ell_I^1[E]$. Then,

$\sum_I | < x_i, \cup_n > | \leq \| u_n \|_{V^0} \, \varepsilon_V(x)$ ……………………..(3)

Hence, $\pi_\cup(x) = \sum_I P_\cup(x_i) \leq \sum_I \sum_{n=1}^{\infty} | < u_n, x_i > |$ by (2)

$= \sum_{n=1}^{\infty}(\sum_I | < u_n, x_i > |)$

$\leq \varepsilon_V(x).\sum_{n=1}^{\infty} \| u_n \|_{V^0}$ by (3)

$\leq \varepsilon_V(x)$ by (1)…………………….……....(4)

Then, x = $(x_i) \in \ell_I^1\{E\}$ and it follows that all weakly summable classes in E are absolutely summable. Hence, in view of facts 3.1 and 3.2, it gives the following equality.

$\ell_I^1[E] = \ell_I^1(E) = \ell_I^1\{E\}$.

It also follows from (4) that the ε-topology is finer than the π -topology. Hence, the two topologies coincide and prove the required relation.

4.2. Proposition: Let E be a locally convex space. If there exists an infinite index set I for which the relation

(i) $\ell_I^1[E] = \ell_I^1\{E\}$.

(ii) $\ell_I^1(E) = \ell_I^1\{E\}$

holds then E is nuclear[2].

Proof:

Let us first observe that for every locally convex space E the relation

$\ell_I^1\{E\} \subseteq \ell_I^1(E) \subseteq \ell_I^1[E]$ holds.

Hence (i) ⇒ (ii)

It is assumed that (ii) holds. Then on $\ell_I^1(E) = \ell_I^1\{E\}$ the ε-topology coincides with the π- topology and hence given any neighborhood U ϵu(E), ∃V ϵu(E), with V < U such that

$\pi_U(x) \leq \varepsilon_V(x)$ for all x = $(x_i) \in \ell_I^1(E)$………………………(1)

Consider an arbitrary finite class $[x_n(V),\eta]$ from E_V and let η = $\{n_1, n_2,…,n_k\}$. Let $\boldsymbol{m} = \{m_1, m_2, ..., m_k\}$ be a finite subset of I such that card.($\boldsymbol{m}$) = card.(η). We now construct a class $[z_i, I] \in \ell_I^1(E)$ by setting $z_i = 0$ for $I \notin \boldsymbol{m}$ and $z_{m_h} = x_{n_h}$ for h = 1, 2, …, k.

Then, $\sum_\eta P(x_n(U)) = \sum_\eta P_U(x_n)$

$= \sum_I P_U(z_i)$

$= \pi_U[z_i, I]$ ……………………………........ (2)

and $\sup\{\sum_\eta | < x_n(V), u > | : u \in V^0\} = \sup\{\sum_I | < z_i, u > | : u \in V^0\}$

$= \varepsilon_V[z_i, I]$ ………..... (3)

Hence, it follows from (1) that

$\sum_\eta P(x_n(U)) \leq \sup\{\sum_\eta | < x_n(V), u > | : u \in V^0\}$..… (4)

From (4) it follows that the canonical map of E_V onto E_U is absolutely summing and hence E is a nuclear space.

4.3. Theorem: A locally convex space E is nuclear iff for at least one (for every) infinite index set I the relation

$\ell_I^1[E] \cong \ell_I^1\{E\}$

or, $\ell_I^1(E) \cong \ell_I^1\{E\}$ holds.

4.4. Proposition: Every co-nuclear space is boundedly summable. If E is co-nuclear then $\ell_I^1[E] = \ell_I^1(E) = \ell_I^1\{E\} = \ell_I^1 < E$ holds for every index set I [2].

4.5. Proposition: E is co-nuclear if and only if E is boundedly summable and

$\ell_I^1[E] = \ell_I^1\{E\}$

or $\ell_I^1(E) = \ell_I^1\{E\}$ for all(one) infinite index set I [2].

4.6. Proposition: A nuclear space E is co-nuclear if it is boundedly summable.

Proof : It follows from proposition (4.5).

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