Existence Of Ψ - Bounded Solutions for Lyapunov Systems

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ABSTRACT: In this research paper, the researchers present an indispensable and adequate condition for the existence of Ψ-bounded solution for the linear non- homogeneous Lyapunov matrix differential system on R. Besides, it is given a result in connection with the asymptotic behaviour of the Ψ- bounded solutions of a linear non- homogeneous Lyapunov matrix differential equation.

1. INTRODUCTION

Differential equations provide a common description of experimental evalu- ation phenomena and in most of the cases, mathematical models are analyzed with regard to differential equations. In fact, the boundedness of solutions is strongly related to the examination of numerical discretization for the differ- ential equations. In this paper, we define Ψ - bounded solution for the matrix differential equation and establish a required indispensable and adequate con- dition for the existence of Ψ - bounded solutions of matrix differential systemfor the linear Lyapunov system on R of the form

$$Z^{J}(\tau) = A(\tau)Z(\tau) + Z(\tau)B(\tau) + R^{2}(\tau) + F(\tau) \tag{1.1}$$

This paper investigates the existence of at least one Ψ - bounded solution for the linear matrix differential equation on R of the form

$$Z^{J}(\tau) = A(\tau)Z(\tau) + R^{2}(\tau) + F(\tau)$$

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and then using vectorization operator and Kronecker product of matrices, we try to give the solution to the same problems for the linear Lyapunov matrix differential systems on R of the form (1.1)and has at least one Ψ - bounded solution on R for every continuous and Ψ - integrable matrix function F on R. where A,B are an ($n \times n$) matrices and Z is a column vectors of orders ($n \times 1$) respectively.

This paper is organized as follows: In section 2, we can provide some basic definitions, notations, hypothesis and results that are useful and we present the general solution of (1.1). Section 3 presents a criteria for the existence of atleast one Ψ - bounded solutions of a linear non-homogeneous Lyapunov matrix differential equation(1.1)

Kronecker product of linear systems and its applications in two-point bound- ary value problems were first introduced by Murty and Fausett [12] in 2002. Many results followed after this basic paper in control theory and in systems analysis in [11]. Recently, the indispensable of at least one Ψ-bounded solu- tion of equation (1.1) on R for distinct types of functions have been studied in [2],[3],[4],[5],[6],[7],[8][9]. In [7–9], Kasi Viswanadh V.Kanuri etl., present the novel concept of Ψ-boundedness of solutions, Ψ being a continuous matrix- valued function, allows a better identification of various types of asymptotic behavior of the solutions on R. Kasi Viswanadh V. Kanuri, R. Suryanarayana

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K. N. Murty [7] provide sufficient conditions for the existence and uniquenessof at least one Ψ - bounded solution for the linear differential systems on time scales. Recently Kasi Viswanadh V Kanuri, Y. Wu, K.N. Murty [8] present a crite- rion for the existence of $(\Phi \otimes \Psi)$ bounded solution of linear first order Kronecker product of system of differential equations.

Thus, the results can be attained, analyzed and extended the recent results concerning the boundedness of solutions of the equation (1.1). The methodused in our research paper is prominently based on the technique and process of Kronecker product of matrices (it has been effectively applied in similar prob- lems [4]-[8]) and on a decomposition of the underlying space at the initial moment [4]-[9] for finite- dimensional spaces and in general case of Banach spaces).

2. PRELIMINARIES

In this section, we present some basic definitions, notations, hypothesis and results which are useful.

Definition 2.1. *Any set of n-linearly independent solutions $\rho_1, \rho_2, ...\rho_n$ of*

$$\rho'(\tau) = A(\tau)\rho(\tau)$$

is called a fundamental set of solutions and the matrix with $\rho_1, \rho_2, ..., \rho_n$ as its columns is called a fundamental matrix for the equation (1.2) and is denoted by Φ . The fundamental matrix Φ is non-singular.

Let R^n be the Euclidean n- space. For $\rho = (\rho_1, \rho_2, \rho_3, \dots, \rho_n)^T \in \mathrm{R}^n$, let $\|\rho\| = \max\{|\rho_1|, |\rho_2|, |\rho_3|, \dots, |\rho_n|\}$ be the norm of ρ.

Let $K_{m\times n}$ be the linear space of all $m \times n$ matrices with real entries.

For a $n \times n$ real matrix $A = (a_{ij})$, we define the norm $|A| = \sup_{\|\rho\|\leq 1} \|A\rho\|$.

It is well-known that $|A| = \max_{1\leq i\leq n|} \{\sum_{j=1}^{n} |a_{ij}|\}$.

Let $\Psi_i : \mathrm{R} \to (-\infty, \infty)$, $i = 1, 2, \dots n$, be continuous functions and

$$\Psi = \mathrm{diag}[\Psi_1, \Psi_2, \dots \Psi_n].$$

Let the vector space R^n be represented as a direct sum of three sub spaces Ω_-, Ω_0, Ω_+ such that a solution $\eta(\tau)$ of (1.1) is Ψ-bounded on R if and only if $y(0) \in \eta_0$ and Ψ-bounded on R if and only if $\eta(0) \in \Omega_- \oplus \Omega_0$. Also, let ξ_-, ξ_0, ξ_+ denote the corresponding projection of R^n onto Ω_-, Ω_0, Ω_+ respectively.

Definition 2.2. *A function $f : R \to R^{n\times n}$ is said to be Ψ- bounded on R if $\Psi(\tau)f(\tau)$ is bounded on Ri.e.,*

$$sup_{\tau\in R} \| \Psi(\tau)f(t) \| < +\infty$$

Extend this definition for matrix functions.

Definition 2.3. *A matrix function $K : R \to K_{n\times n}$ is said to be Ψ- bounded on R ifthe matrix function ΨK is bounded on R*

$$i.e., sup_{\tau\geq 0} \| \Psi(\tau)K(\tau) \| < +\infty$$

Definition 2.4. *A matrix function $K : R \to K_{n\times n}$ is said to be Ψ- bounded on R ifthe matrix function $\Psi(\tau)K(\tau)$ is bounded on R,*

i.e., there exists
$m > 0$ such that $\| \Psi(\tau)K(\tau) \| < m$, for all $\tau \in R$

Definition 2.5. *A function $f : R \to R^{n\times n}$ is said to be Lebesgue Ψ integrable on R iff f is measurable and $\Psi(\tau)f(\tau)$ is Lebesgue integrable on R*
i.e.,

$$\int_0^\infty \| \Psi(\tau)f(\tau) \| d\tau < \infty$$

Extend this definition for matrix functions.

Definition 2.6. *A function $K : R \to R^{n\times n}$ is said to be Lebesgue Ψ integrable on R if K is measurable and $\Psi(\tau)K(\tau)$ is Lebesgue integrable on R*
i.e.,

$$\int_0^\infty \| \Psi(\tau)K(\tau) \| dt < \infty$$

Definition 2.7. *The vectorization operator $Vec : K_{m\times n} \to R^{mn}$, defined by*

$$VecA = (a_{11}, a_{21}, \ldots\ldots a_{m1}, a_{12}, a_{22}, \ldots\ldots a_{mn})^*$$

where $A = a_{ij} \in K_{m\times n}$, is called the vectorization operator.

Lemma 2.1. *The vectorization operator $Vec : K_{n\times n} \to R^{n^2}$ is a linear and one to one operator. In addition, Vec and Vec^{-1} are continuous operators.*

Proof. The fact that the vectorization operator is linear and one to one oper-ator. Now, for $A = (a_{ij}) \in K_{n\times n}$, we have $\| Vec(A) \| = max_{1\le i\le n}| a_{ij} | \le max_{1\le i\le n} \sum_{j=1}^{n} | a_{ij} | = | A |$. Thus, the vectorization operator is continuous and $\| Vec \| \le 1$. In addition, for $A = I_n$, we have $\| Vec(I_n) \| = | I_n |$ and then $\| Vec \| = 1$. We have $\| Vec^{-}(u) \| = max_{1\le i\le n} \sum_{j=0}^{n-1} | u_{n,j+i} | \le n.max_{1\le i\le n^2} | u_i | = n.u$. Thus, $\| Vec^{-1} \|$ is a continuous operator

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Theorem 2.1. *Let $A \in R$ be an $n \times n$ matrix-valued function on R and suppose that $f : R \longrightarrow R^n$ is continuous. Let $\tau_0 \in R$ and $\eta_0 \in R^n$. Then the initial value problem*

$$\eta'(\tau) = A(\tau)\eta(\tau) + f(\tau), \eta(\tau_0) = \eta_0$$

has a unique solution $\eta : R \longrightarrow R^n$. Moreover, this solution is given by

$$\eta(\tau) = \Phi_A(\tau, \tau_0)\eta_0 + \int_{\tau_0}^{\tau} \Phi_A(\tau, s)f(s)ds$$

where $\Phi_A(\tau, \tau_0)$ is a fundamental matrix.

Theorem 2.2. *Let $P(\tau)$ and $Q(\tau)$ be fundamental matrices for the dynamical sys-tems*

$$Z'(\tau) = A(\tau)Z(\tau) \tag{2.1}$$

$$Z'(\tau) = Z(\tau)B(\tau) \tag{2.2}$$

$\tau \in T^+$, respectively. Then the matrix $W(\tau) = (Q^*(\tau) \otimes P(\tau))$ is a fundamentalmatrix for the system

$$Z'(\tau) = (I_n \otimes A(\tau) + B^*(\tau) \otimes I_n)Z(\tau) \tag{2.3}$$

In addition ,$P(0) = I_n$ and $Q(0) = I_n$ then $W(0) = I_n2$.

Proof. Using the above properties of the Kronecker product

$$\begin{aligned}
W'(\tau) &= (Q^*(\tau) \otimes P(\tau))' \\
&= (Q^*)'(\tau) \otimes P(\tau) + Q^*(\tau) \otimes P'(t) \\
&= (Q')^*(\tau) \otimes P(\tau) + Q^*(\tau) \otimes P'(\tau)) \\
&= ((Q(\tau)B(\tau))^*(\tau) \otimes P(\tau) + Q^*(\tau) \otimes A(\tau)P(\tau)) \\
&= (B^*(\tau)Q^*(\tau) \otimes P(\tau) + Q^*(\tau) \otimes A(\tau)P(\tau)) \\
&= (B^*(\tau) \otimes I_n)(Q^*(\tau) \otimes P(\tau)) + (I_n \otimes A(\tau))(Q^*(\tau) \otimes P(\tau)) \\
&= (B^*(\tau) \otimes I_n) + (I_n \otimes A(\tau))(Q^*(\tau) \otimes P(\tau))
\end{aligned}$$

Therefore, $W'(\tau) = (B^*(\tau) \otimes I_n) + (I_n \otimes A(\tau))W(\tau)$,
for all $\tau \in R$.

On the other hand, the matrix $Z(\tau)$ is an invertible matrix for all $\tau \geq 0$, since $P(\tau)$ and $Q(\tau)$ are non singular matrices. Thus the matrix W is a fundumentalmatrix of R. Also $W(0) = P(0) \otimes Q(0) = I_n \otimes I_n = I_n2$

Then, the matrix $(P(\tau) \otimes Q(\tau))$ is an invertible matrix for all $\tau \in R$. Thus $(P(\tau) \otimes Q(\tau))$ is the fundamental matrix of (1.1). Also $W(0) = P(0) \otimes Q(0) = I_n \otimes I_n = I_n2$

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Theorem 2.3. *The matrix function $P(\tau)$ is a solution of (1.1) if and only if thevector valued function $\rho(\tau) = Vec(P(\tau))$ is a solution of the differential system*

$$\rho'(\tau) = (I_n \otimes A(\tau) + B^*(\tau) \otimes I_n)x(\tau) + R^2(\tau) + f(\tau). \tag{2.4}$$

where $f(\tau) = Vec(F(\tau))$. The above system (2.1) is the corresponding kroneckerproduct system associated with (1.1).

Proof. similar

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Theorem 2.4. *The matrix function $Z(\tau)$ is a solution on R of (1.1) if and only ifthe vector valued function $z(\tau) = Vec(Z(\tau))$ is a solution of the differential system*

$$z'(\tau) = (I_n \otimes A(\tau) + B^*(\tau) \otimes I_n)z(\tau) + R^2(\tau) + f(\tau). \tag{2.5}$$

where $f(\tau) = Vec(F(\tau))$ and $R^2(\tau) = VecR^2(\tau)$, on the same interval R. Theabove system (2.2) is the corresponding kronecker product system associated with (1.1).

Proof. Using Kronecker product notation, the vectorization operator Vec and the above properties, we can rewrite the equality(1.1) in the equivalent form

$$VecZ'(\tau) = (I_n \otimes A(\tau) + B^*(\tau) \otimes I_n)VecZ(\tau) + VecR^2(\tau) + Vecf(\tau),$$

for all $\tau \geq 0$.

If we denote $VecZ(\tau) = z(\tau), VecF(\tau) = f(\tau)$ and $VecR^2(\tau) = R^2(\tau)$ andthen, the above equality becomes

$$z'(\tau) = (I_n \otimes A(\tau) + B^*(\tau) \otimes I_n)z(\tau) + R^2(\tau) + f(\tau),$$

for almost all $\tau \geq 0$.

The proof is now complete. Q

Theorem 2.5. *The matrix function $Z(\tau)$ is Ψ - bounded on R of (1.1) if and only if the vector function $Vec(Z(\tau))$ is $(I_n \otimes \Psi)$ - bounded on R.*

Proof. similar

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Theorem 2.6. *If A is a continuous $n \times n$ real matrix on R then, the system $\rho'(\tau) = A(\tau)\rho(\tau) + R^2(\tau) + f(\tau)$ has at least one Ψ bounded solution on R for every continuous and Ψ- bounded function f on R if and only if for the fundamental matrix $Q(\tau)$ of the system $P'(\tau) = A(\tau)P(\tau)$ there exists a positive constant σ such that, for $\tau \geq 0$,*

$$\int_{-\infty}^{\tau} |\Psi(\tau)Q(\tau)\xi_- Q^{-1}(s)\Psi^{-1}(s)(R^2(s) + f(s))|ds + \int_0^{\tau} |\Psi(\tau)Q(\tau)\xi_0 Q^{-1}(s)\Psi^{-1}(s)(R^2(s) + f(s))|ds + \int_{\tau}^{\infty} |\Psi(\tau)Q(\tau)\xi_1 Q^{-1}(s)\Psi^{-1}(s)(R^2(s) + f(s))|ds \leq \sigma\rho. \quad (2.6)$$

Here ξ_-, ξ_0 and ξ_1 are supplementary projections for the system $Z'(\tau) = A(\tau)Z(\tau)$.

Proof. We prove this theorem by means of Banach fixed point theorem. Consider

$S_\Psi = \{Z : R \rightarrow K_{n\times n}$, Z is continuous and Ψ - bounded on R

S_Ψ is Banach space with respect to the norm $| Z | = sup_{\tau\in R} \| \Psi(\tau)Z(\tau) \|$.Let $S_\rho = \{Z \in S_\Psi \mid |Z|_\Psi \leq \rho\}$. For $Z \in S_\Psi$,

Now,

$$\int_{-\infty}^{0} |(P(\tau)\xi_- P^{-1}(s)(R^2(s) + f(s))|ds + \int_0^{\tau} |(P(\tau)\xi_0 P^{-1}(s)(R^2(s) + f(s))|ds + \int_{\tau}^{\infty} |(P(\tau)\xi_1 P^{-1}(s)(R^2(s) + f(s))|ds.$$

From hypotheses, T exists and is continuous differentiable on R. For $Z \in S_\rho$ and $\tau \in R$, we have

$$\int_{-\infty}^{0} |\Psi(\tau)(P(\tau)\xi - P_1^{-1}(s)\Psi^{-1}(s)\Psi(s)(C(s)+f(s))|ds+$$

$$\int_{0}^{\tau} |\Psi(\tau)(P(\tau)\xi_0 P_1^{-1}(s)\Psi^{-1}(s)\Psi(s)(C(s)+f(s))|ds+$$

$$\int_{\tau}^{\infty} |\Psi(\tau)(P(\tau)\xi_1^{\wedge} P^{-1}(s)\Psi^{-1}(s)\Psi(s)(C(s)+f(s))|ds.$$

implies

$$\int_{-\infty}^{0} |\Psi(\tau)(P(\tau)\xi - P_1^{-1}(s)\Psi^{-1}(s) \parallel \Psi(s)(C(s)+f(s))|ds+$$

$$\int_{0}^{\tau} |\Psi(\tau)(P(\tau)\xi_0 P_1^{-1}(s)\Psi^{-1}(s) \parallel \Psi(s)(C(s)+f(s))|ds+$$

$$\int_{\tau}^{\infty} |\Psi(\tau)(P(\tau)\xi^{\wedge} P^{-1}(s)\Psi^{-1}(s) \parallel \Psi(s)(C(s)+f(s))|ds.$$

$\leq \sigma\rho$

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Theorem 2.7. *Suppose that:*

1. *The fundamental matrix* $P(\tau)$ *of the system* $Z'(\tau) = A(\tau)Z(\tau)$ *satisfies the condition(2.6)for all* $t \geq 0$,
2. *The continuous and* Ψ *bounded function* $f : R \to R^n$ *is such that*

$$lim_{\tau \to \infty} \| \Psi(\tau)\rho(\tau) \| = 0.$$

Then , every Ψ *bounded solution* ρ *of the system* $\rho'(\tau) = A(\tau)\rho(\tau) + f(\tau)$ *is suchthat*

$$lim_{\tau \to \infty} \| \Psi(\tau)\rho(\tau) \| = 0.$$

3. EXISTENCE OF Ψ - BOUNDED SOLUTIONS FOR THE NON-HOMOGENEOUS LYAPUNOV SYSTEMS

In this section we present the existence of Ψ bounded solutions for the non-homogeneous Lyapunov matrix differential equation(1.1).

Theorem 3.1. *Let* $A(\tau)$ *and* $B(\tau)$ *be continuous* $n \times n$ *real matrix function on R and let P and Q be the fundamental matrices of the homogeneous linear equations (2.1) and (2.2) respectively for which P(0) = Q(0) =* I_n. *Then, the equation (1.1) has at least one* Ψ *bounded solution on R for every continuous and* Ψ *bounded matrix function* $F : R \to R_{n\times n}$ *if and only if there exists supplementary projections* $\xi_-, \xi_0, \xi_1 \in K_{n\times n}$ *and a positive constant* σ *such that, for all* $\tau \geq 0$,

$$\int_{-\infty}^{\tau} |(Q^*(\tau) \otimes (\Psi(\tau)P(\tau))\hat{\xi}_-((Q^*)^{-1}(s) \otimes (P^{-1}(s)\Psi^{-1}(s)))|ds + \int_{*}^{\tau} |(Q^*(\tau) \otimes \Psi(\tau)P(\tau))\hat{\xi}_0((Q^*)^{-1}(s) \otimes (P^{-1}(s)\Psi^{-1}(s)))|ds + \int_{\tau}^{\infty} |(Q^*(\tau) \otimes (\Psi(\tau)P(\tau))\hat{\xi}_1((Q^*)^{-1}(s) \otimes (P^{-1}(s)\Psi^{-1}(s)))|ds \leq \sigma. \tag{3.1}$$

Proof. First, we prove the "only if" part. suppose that the system (1.1) hasat least one Ψ-bounded solution on R for every continuous Ψ bounded matrix function $F : R \to K_{n\times n}$. Let $f : R \to R^{n^2}$ be a continuous and $I_n \otimes \Psi$ -bounded function on R. From theorem (2.5),it follows that the matrix function

$F(\tau) = Vec^{-1}(f(\tau))$ is continuous and bounded on R. From the hypothesis, theequation

$$Z' = A(\tau)Z(\tau) + Z(\tau)B(\tau) + Vec^{-1}(f(\tau))$$

has at least one Ψ bounded solution Z(t) on R.

From theorem (2.4) and (2.5),it follows that the vector valued function $z(\tau) = Vec(z(\tau))$ is a $I_n \otimes \Psi$ - bounded solution on R of the differential system(2.5).Thus, this system has at least one $I_n \otimes \Psi$ bounded solution on R for every continuous and $I_n \otimes \Psi$ bounded function f on R. From the Theorem (2.6) , there is a positive constant K such that the fundamental matrix W(t) of the equation (2.6) satisfies the condition

$$\int_{-\infty}^{0} |(I_n \otimes \Psi(\tau))W(\tau)\xi_{-}W^{-1}(s)(I_n \otimes \Psi(s))^{-1} ds| + \int_{0}^{\tau} |(I_n \otimes \Psi(\tau))W(\tau)\xi_0 W^{-1}(s)(I_n \otimes \Psi(s))^{-1} ds| + \int_{\tau}^{\infty} |(I_n \otimes \Psi(\tau))W(\tau)\xi_{+}W^{-1}(s)(I_n \otimes \Psi(s))^{-1} ds| \leq k$$

for all $\tau \geq 0$.

By theorem (2.2), we have $W(\tau) = Q^*(\tau) \otimes P(\tau)$.

Now the above equation becomes

$$\int_{-\infty}^{0} |(I_n \otimes \Psi(\tau))(Q^*(\tau) \otimes P(\tau))\xi_{-}(Q^*(s) \otimes P(s))^{-1}(I_n \otimes \Psi(s))^{-1} ds| +$$

$$\int_{*}^{\tau} |(I_n \otimes \Psi(\tau))(Q^{\wedge *}(\tau) \otimes P(\tau))\xi_0(Q^*(s) \otimes P(s))^{-1}(I_n \otimes \Psi(s))^{-1} ds| +$$

$$\int_{\tau}^{\infty} |(I^n \otimes \Psi(\tau))(Q^*(\tau) \otimes P(\tau))\xi_{+}(Q^*(s) \otimes P(s))^{-1}(I_n \otimes \Psi(s))^{-1} ds| \leq \sigma$$

for all $\tau \geq 0$.

$$\int_{-\infty}^{0} |(Q^*(\tau) \otimes \Psi(t)P(\tau))\xi_{-}(Q^{\wedge *})^{-1}(s) \otimes P^{-1}(s)\Psi^{-1}(s) ds| +$$

$$\int_{*}^{\tau} |(Q^{\wedge *}(\tau) \otimes \Psi(\tau)P(\tau))\xi_0(Q^*)^{-1}(s) \otimes P^{-1}(s)\Psi^{-1}(s) ds +$$

0

$$\int_{\tau}^{\infty} |(Q^{*}(\tau) \otimes \Psi(\tau)P(\tau))P_1(Q^{*})^{-1}(s) \otimes P^{-1}(s)\Psi^{-1}(s)ds| \leq \sigma$$

The above equation can be written as

$$\int_{-\infty}^{0} |(Q^{*}(\tau)(Q^{*})^{-1}(s) \otimes (\Psi(\tau)P(\tau))\widehat{\xi}_{-}P^{-1}(s)\Psi^{-1}(s)ds| +$$

$$\int_{0}^{\tau} |(Q^{*}(\tau)(Q^{*})^{-1}(s) \otimes (\Psi(\tau)P(\tau))\widehat{\xi}_{0}P^{-1}(s)\Psi^{-1}(s)ds| +$$

$$\int_{\tau}^{0} |(Q^{*}(\tau)(Q^{*})^{-1}(s) \otimes (\Psi(\tau)P(\tau))\widehat{\xi}_{1}P^{-1}(s)\Psi^{-1}(s)ds| \leq \sigma$$

Now, we prove the "if" part. Suppose that equation(2.6) holds for some $\sigma > 0$ and for all $t \geq 0$.

Let $F : R \to K_{n\times n}$ is continuous and Ψ - bounded matrix function on R. From theorem (2.5), it follows that the vector valued function $f(\tau) = Vec(F(\tau))$ is continuous and $I_n \otimes \Psi$ bounded function on R. From this, equation(2.6), it follows that the differential system (3.1) has at least one $I_n \otimes \Psi$ bounded solution on R . Let $z(\tau)$ be the solution. From theorem(2.4) and theorem(2.5), it follows that the matrix function $Z(\tau) = Vec^{-1}(z(\tau))$ is a bounded solution on of the equation (1.1) (because $F(\tau) = Vec(f(\tau))$). Thus, the differential equation (1.1) has at least one bounded solution on for every continuous and bounded solution F on R . The proof is now complete.

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Theorem 3.2. *Suppose that:*

1) The fundamental matrices $P(\tau)$ and $Q(\tau)$ of (2.1) and (2.2) respectively ($P(0) = Q(0) = I_n$) satisfy the condition (3.1) for some $\sigma \geq 0$ and for all $\tau \geq 0$.

2) The continuous matrix function $F : R \to K_{n\times n}$ a continuous and Ψ- bounded matrix function on R satisfies the condition

$$\lim_{\tau\to\infty} |\Psi(\tau)F(\tau)| = 0.$$

Then, every Ψ- bounded solution $Z(\tau)$ of (1.1) satisfies the condition

$$\lim_{t\to\infty} |\Psi(t)Z(\tau)| = 0.$$

Proof. Let $Z(\tau)$ be a Ψ- bounded solution of (1.1). From theorem(2.4)and the- orem(2.5), it follows that the function $X(\tau) = Vec(x(\tau))$ is a $I_n \otimes \Psi$ - bounded solution on R of the differential system

$$z'(\tau) = (I_n \otimes A(\tau) + B^{*}(\tau) \otimes I_n)z(\tau) + f(\tau). \tag{3.2}$$

where $f(\tau) = Vec(F(\tau))$.

Also, from the proof of theorem(2.7), we have

(3.3) $$\| (I_n \otimes \Psi(\tau).f(\tau)) \|_{R^{n^2}} \leq |\Psi(\tau)F(\tau)|, \tau \geq 0.$$

then

$$\lim_{\tau \to \infty} \| (I_n \otimes \Psi(\tau).z(\tau)) \|_{R^{n^2}} = 0.$$

Now, from the proof of theorem(2.7) again, we have

(3.4) $$|\Psi(\tau)Z(\tau)| \leq n \| (I_n \otimes \Psi(\tau).z(\tau)) \|_{R^{n^2}}, \tau \geq 0$$

and then

$$\lim_{\tau \to \infty} |\Psi(\tau)Z(\tau)| = 0.$$

The proof is now complete. Q

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