Research Article

Some New Properties of Convergence Uncertain Sequence in Measure

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Abstract: Some a new properties of convergence uncertain sequence in measure are introduced , further more we have that properties of convergence uncertain sequence in distribution were satisfied by using the relation between convergence sequence in measure and in distribution Also, we verify Kolmogorov inequality and some theorem that related with it. Finely new relation between convergence in mean and convergence in distribution were investigated.

Keyword: Convergence; uncertain variable. Uncertain sequence, uncertain measure

1. Introduction

Liu [2]founded in 2007uncertainty theory and Liu [3] refined it in 2010, an uncertain measure is first idea of uncertainty theory defined as a set function $U:\mathcal{F} \longrightarrow \mathcal{R}$saisfies he following axioms:

(1) (Normality Axiom):$U(\Gamma)=1$. (1)

(2) (Monotonicity Axiom): If $A_1 \subset A_2$, then $U(A_1) \le U(A_2)$ (2)

(3) (Self-duality Axiom):$U(A_1)+U(A_1^c)=1$ for any $A_1 \in \mathcal{F}$

(4) (Countable Subadditivity Axiom):If$\{A_i\}$ is countable sequence of events, then.

$$U(\bigcup_{i=1}^{\infty} A_i) \le \sum_{i=1}^{\infty} U(A_i)$$

(5) (Product Measure Axiom):The product measure$U$ is uncertain measure over the product σ -field$\mathcal{F}_1 \times \mathcal{F}_2 \times ... \times \mathcal{F}_n$ satisfying $U(\prod_{i=1}^{n} A_i) = \underset{1\le i\le n}{Min} U(A_i)$ for all $A_i \in \Gamma_i$,and $i=1,2,3,...,n$.

Definition (1-2) [1]

The triple $(\Gamma, \mathcal{F}, U)$ beuncertainty space such that Γ , $\mathcal{F}$,and U be a nonempty set,σ - field and uncertain measure respectively.

Definition (1-3) [2]

We say that the measurable function ω from an uncertainty space to the set of real numbers $\Re$ is uncertain variable.

Definition (1-4) [3]

We say that the uncertain variables $\omega_1, \omega_2, ..., \omega_n$ be independent if $U(\bigcap_{i=1}^{n}(\omega_n \in \mathrm{B}_i)) = \min_{1\le i\le n} U_i(\omega_i \in \mathrm{B}_i)$, for any Borel $\mathrm{B}_1, \mathrm{B}_2, ..., \mathrm{B}_n$ of real numbers $\Re$.

Definition (1-5) [6]

The expected value $\Im$ of uncertain variable ω defined by $\Im(\omega) = \int_0^{+\infty} \Im(\omega \ge \alpha) d\alpha - \int_{-\infty}^{0} \Im(\omega \le \alpha) d\alpha$, provided that at least $\int_0^{+\infty} \Im(\omega \ge \alpha) d\alpha < \infty$ or $\int_{-\infty}^{0} \Im(\omega \le \alpha) d\alpha < \infty$ .The variance of ω is defined by $\Im(\omega) = \Im(\omega - \Im(\omega))^2$.

Theorem (1-6) [5]

Suppose that the uncertain variable X . Then for any given numbers $\varepsilon > 0$ and $r > 0$, we have $U(|\omega| \geq \varepsilon) \leq \dfrac{\Im(|\omega|^r)}{\varepsilon^r}$ (3).

Definition (1-7)[7]

The uncertain sequence $\{\omega_n\}$ be convergent in measure to uncertain variable ω if $\lim\limits_{n\to\infty} U\{y \in \Gamma \mid |\omega_n(y) - \omega(y)| \geq \varepsilon\} = 0$, for every $\varepsilon > 0$.

Definition (1-8) [7]

The uncertain sequence $\{\omega_n\}$ be convergent in mean to uncertain variable ω if $\lim\limits_{n\to\infty} \Im\{y \in \Gamma \mid |\omega_n(y) - \omega(y)|\} = 0$.

Definition (1-9) [7]

We say that the uncertain sequence $\{\omega_n\}$ is convergent to uncertain variable in distribution ω if $\lim\limits_{n\to\infty} \psi_n(\omega) = \psi(\omega)$ for all ω at which $\psi(\omega)$ is continuous, and ψ_n , $n = 1,2,\ldots,k$ be an uncertainty distribution of uncertain variables $\omega_n, n = 1,2,\ldots,k$.

2. Convergence of independent uncertain variables sequence sum

Lemma (2-1) [5]

Suppose that $X_i, i = 1,2,\ldots,n$ be uncertain variables and $p > 0$. Then $\Im\left[\left|\sum_{i=1}^{n} \omega_i\right|\right]^p \leq n^p \sum_{i=1}^{n} \Im|\omega_i|^p$ (4)

Theorem (2-2) [7]

Suppose that $\{\omega_n\}$, and ω be an uncertain sequence and uncertain variable respectively. if $\omega_n \to \omega$ (in measure)as $n \to \infty$ .Then $\omega_n \to \omega$ as $n \to \infty$ (in distribution).

Theorem (2-3)

Suppose that $\{\omega_n\}$, and ω be an uncertain sequence and uncertain variable respectively such that

$$\sum_{n=1}^{\infty} U\{|\omega_n - \omega| \geq \varepsilon\} < \infty \qquad (5)$$

Then $\omega_n \to \omega$ as $n \to \infty$ in measure.

Proof:

Since $\{y \in \Gamma \mid |\omega_m(y) - \omega(y)| \geq \varepsilon\} \subset \{\bigcup_{m=n}^{\infty} \{y \in \Gamma \mid |\omega_m(y) - \omega(y)| \geq \varepsilon\}$

From (1), and (2) we have

$$U\{y \in \Gamma \mid |\omega_m(y) - \omega(y)| \geq \varepsilon\} \leq U\{\bigcup_{m=n}^{\infty} \{y \in \Gamma \mid |\omega_m(y) - \omega(y)| \geq \varepsilon\}$$

$$\leq \sum_{m=n}^{\infty} U\{|\omega_m - \omega| \geq \varepsilon\}$$

Furthermore, $\lim\limits_{n\to\infty} U\{y \in \Gamma \mid |\omega_m(y) - \omega(y)| \geq \varepsilon\} \leq \lim\limits_{n\to\infty} \sum_{m=n}^{\infty} U\{|\omega_n - \omega| \geq \varepsilon\} = 0$

Corollary(2-4)

Suppose that $\{\omega_n\}$, and ω be an uncertain sequence, uncertain variables respectively such that

$$\sum_{n=1}^{\infty} U\{|\omega_n - \omega| \geq \varepsilon\} < \infty \qquad (6)$$

Then $\omega_n \to \omega$ as $n \to \infty$ in distribution.

Proof:-

From theorem (2-2) and from theorem (2-3) according to (5), we have $\omega_n \to \omega$ as $n \to \infty$ in distribution

Theorem (2-5)

Suppose that $\{\omega_n\}$, and ω be an uncertain sequence, uncertain variables respectively such that $\sum_{n=1}^{\infty} \Im\{|\omega_n - \omega|^2 \geq \varepsilon\}$ is a finite. Then $\omega_n \to \omega$ as $n \to \infty$ in measure.

Proof:

Since $\{y \in \Gamma \mid |\omega_n(y) - \omega(y)| \geq \varepsilon\} \subset \{\bigcup_{m=n}^{\infty}\{y \in \Gamma \mid |\omega_m(y) - \omega(y)| \geq \varepsilon\}$

From (2) that

$$U\{y \in \Gamma \mid |\omega_n(y) - \omega(y)| \geq \varepsilon\} \leq U\{\bigcup_{m=n}^{\infty}\{y \in \Gamma \mid |\omega_m(y) - \omega(y)| \geq \varepsilon\}$$

From theorem (1-6) according to (3) and lemma (2-1) according to (4)

$$U\{y \in \Gamma \mid |\omega_n(y) - \omega(y)| \geq \varepsilon\} \leq U\{\bigcup_{m=n}^{\infty}\{y \in \Gamma \mid |\omega_m(y) - \omega(y)| \geq \varepsilon\} \leq \frac{\Im\{\left|\bigcup_{m=n}^{\infty}\{y \in \Gamma \mid |\omega_m - \omega|\right|^2\}}{\varepsilon^2}$$

$$= \frac{\Im\{\left|\sum_{m=n}^{\infty}|\omega_m - \omega|\right|^2\}}{\varepsilon^2} \leq \frac{n^2}{\varepsilon^2}\sum_{m=n}^{\infty}\Im|\omega_m - \omega|^2$$

Furthermore , $\lim_{n\to\infty} U\{y \in \Gamma \mid |\omega_n(y) - \omega(y)| \geq \varepsilon\} \leq \lim_{n\to\infty}\frac{n^2}{\varepsilon^2}\sum_{m=n}^{\infty}\Im|\omega_m - \omega|^2 = 0$

Corollary (2-6)

Suppose that $\{\omega_n\}$, ω be an uncertain sequence, and uncertain variables respectively such that Then

$$\sum_{n=1}^{\infty} \Im\{|\omega_n - \omega|^2 \geq \varepsilon\} < \infty \quad (7)$$

.Then $\omega_n \to \omega$ as $n \to \infty$ in distribution.

Proof:-

From theorem (2-2) and from theorem (2-4) according to (6), we have $\omega_n \to \omega$ as $n \to \infty$ in distribution.

Theorem (2-7) (Kolmogrov inequality)

Suppose that $\omega_i, i = 1,2,\ldots,n$ be uncertain variables such that $W_n = \sum_{i=1}^{n}\omega_i$. If $\Im[\omega_i^2] < \infty$, $i = 1,2,\ldots,$ then $U\{\underset{1\leq i\leq n}{Max}|W_i - \Im(W_i)| \geq \varepsilon\} \leq \frac{n^2}{\varepsilon^2}\sum_{i=1}^{n}Var(\omega_i)$ for any number $\varepsilon > 0$.

Proof:

From theorem (1-6) according to (3) and lemma (2-1) according to (4), we have

$$U\{\underset{1\leq i\leq n}{Max}|W_i - \Im(W_i)| \geq \varepsilon\} \leq \frac{\Im\left[\left|\underset{1\leq i\leq n}{Max}|W_i - \Im(W_i)|\right|^2\right]}{\varepsilon^2} \leq \frac{1}{\varepsilon^2}\Im\left|\sum_{i-1}^{n}|\omega_i - \Im(\omega_i)|\right|^2$$

$$\leq \frac{n^2}{\varepsilon^2}\sum_{i=1}^{n}\Im|\omega_i - E(\omega_i)|^2 = \frac{n^2}{\varepsilon^2}\sum_{i=1}^{n}\Im(\omega_i - \Im(\omega_i))^2 = \frac{n^2}{\varepsilon^2}\sum_{i=1}^{n}Var(\omega_i)$$

Theorem (2-8)

Suppose that $\{\omega_i\}$ be an uncertain sequence. If $\sum_{i=1}^{n} Var(\omega_i)$ is a finite. Then $\sum_{i=1}^{\infty}((\omega_i)-\Im(\omega_i))$ (8) convergent in measure.

Proof:

Since $\sum_{i=1}^{\infty}((\omega_i)-\Im(\omega_i))$ convergent in measure if and only if $\lim_{n\to\infty}\sum_{i=n}^{\infty}((\omega_i)-\Im(\omega_i))=0$ in measure if and only if $\lim_{n\to\infty} U\left\{\bigcup_{k=0}^{\infty}\left\{\sum_{i=n}^{n+k}(\omega_i-\Im(\omega_i)\right\}\geq\varepsilon\right\}=0$

For any $\varepsilon>0$,

$$U\left\{\bigcup_{k=0}^{\infty}\left\{\sum_{i=n}^{n+k}(\omega_i-\Im(\omega_i)\right\}\geq\varepsilon\right\}=\lim_{m\to\infty}U\left\{\bigcup_{k=0}^{m}\left\{\left|\sum_{i=n}^{n+k}(\omega_i-\Im(\omega_i)\right|\right\}\geq\varepsilon\right\}$$

$$=\lim_{m\to\infty}U\left\{\underset{0\leq k\leq m}{Max}\left|\sum_{i=n}^{n+k}(\omega_i-\Im(\omega_i)\right|\geq\varepsilon\right\}\leq\lim_{m\to\infty}\frac{n^2}{\varepsilon^2}\sum_{i=n}^{n+m}\Im(\omega_i-\Im(\omega_i))$$

$$=\lim_{m\to\infty}\frac{n^2}{\varepsilon^2}\sum_{i=n}^{n+m}Var(\omega_i)=\lim_{n\to\infty}\frac{n^2}{\varepsilon^2}\sum_{i=n}^{\infty}Var(\omega_i)=0$$,from $\sum_{i=n}^{\infty}Var(X_i)$ is finite.

Theorem (2-9)

Suppose that $\{\omega_i\}$ be an uncertain sequence. If $\sum_{i=1}^{n}Var(\omega_i)$ is a finite. Then $\sum_{i=1}^{\infty}((\omega_i)-\Im(\omega_i))$ converges in distribution.

Proof:-

From theorem (2-2) and theorem (2-8) according to (8), we have $\sum_{i=1}^{\infty}((\omega_i)-\Im(\omega_i))$ convergent to uncertain variable ω in distribution.

Theorem (2-10)

Suppose that $\{\omega_n\}$, ω bean uncertain sequence, and uncertain variables respectively. If $\omega_n\to\omega$ as $n\to\infty$ in mean, then $\omega_n\to\omega$ as $n\to\infty$ in distribution.

Proof:

Since $\omega_n\to\omega$ as $n\to\infty$ in mean means $\omega_n\to\omega$ as $n\to\infty$ in measure, and from theorem (2-2), thus $\{\omega_n\}$ is convergence uncertain sequence in distribution to uncertain variable ω.

Example (2-11)[4]

If $\omega_n\to\omega$ as $n\to\infty$ in distribution, then $\omega_n\nrightarrow\omega$ as $n\to\infty$ in mean

For example, suppose that the uncertainty space and $\Gamma=\{y_1,y_2\}$ with $U\{y_1\}=U\{y_2\}=0.5$ and

$$\omega_i(y_j)=\left\{\begin{matrix} b & ,ify=y_1 \\ -b & ,ify=y_2\end{matrix}\right\}$$

,and uncertainty distribution of ω_i and ω be $\psi(\omega)=\begin{cases}0 & ,\omega<-b\\ 0.5 & ,-b\leq\omega<b\\ 1 & ,\omega\geq b\end{cases}$, that is $\lim_{n\to\infty}\psi_n(\omega)=\psi(\omega)$ in distribution, put $\omega_n=-\omega$ we have $|\omega_n-\omega|=2b$ for all $n=1,2,3,\ldots,.\ y=y_1,y_2$. Thus $E[|\omega_n-\omega|]=\int_0^{2b}1dx=2b$ so that $\lim_{n\to\infty}\Im[|\omega_n-\omega|]=2b$.

That is, $\omega_n\nrightarrow\omega$ as $n\to\infty$ in mean.

3. Conclusion

The rustles of this paper is to obtain new properties of uncertain variable convergence in measure, also Kolmgorov inequality is verified.

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