Some New Properties of Convergence Uncertain Sequence in Measure

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Abstract: Some new properties of convergence uncertain sequence in measure are introduced, further more we have that properties of convergence uncertain sequence in distribution were satisfied by using the relation between convergence sequence in measure and in distribution. Also, we verify Kolmogorov inequality and some theorem that related with it. Finely new relation between convergence in mean and convergence in distribution were investigated.

Keyword: Convergence, uncertain variable. Uncertain sequence, uncertain measure

1. Introduction

Liu [2] founded in 2007 uncertainty theory and Liu [3] defined it in 2010, an uncertain measure is first idea of uncertainty theory defined as a set function \( \mu: \mathcal{F} \rightarrow \mathcal{R} \) satisfying the following axioms:

1. (Normality Axiom): \( \mu(\Gamma) = 1 \).
2. (Monotonicity Axiom): If \( A_1 \subseteq A_2 \), then \( \mu(A_1) \leq \mu(A_2) \).
3. (Self-duality Axiom): \( \mu(A_i) + \mu(A_i^c) = 1 \) for any \( A_i \in \mathcal{F} \).
4. (Countable Subadditivity Axiom): If \( \{ A_i \} \) is a countable sequence of events, then:

\[
\mu\left( \bigcup_{i=1}^{\infty} A_i \right) \leq \sum_{i=1}^{\infty} \mu(A_i)
\]

5. (Product Measure Axiom): The product measure \( \mu \) is an uncertain measure over the product space \( \mathcal{F}_1 \times \mathcal{F}_2 \times \cdots \times \mathcal{F}_n \) satisfying:

\[
\mu(\bigcap_{i=1}^{n} A_i) = \min \{ \mu(A_i) \} \text{ for all } A_i \in \mathcal{F}_i \text{ and } i = 1, 2, 3, \ldots, n.
\]

Definition (1-2) [1]

The triple \( (\Gamma, \mathcal{F}, \mu) \) is an uncertainty space such that \( \Gamma, \mathcal{F} \), and \( \mu \) be a nonempty set, \( \sigma \)-field and uncertain measure respectively.

Definition (1-3) [2]

We say that the measurable function \( \omega \) from an uncertainty space to the set of real numbers \( \mathcal{R} \) is an uncertain variable.

Definition (1-4) [3]

We say that the uncertain variables \( \omega_1, \omega_2, \ldots, \omega_n \) be independent if

\[
\mu(\bigcap_{i=1}^{n} (\omega_i \in B_i)) = \min_{i=1}^{n} \mu_i(\omega_i \in B_i), \text{ for any Borel } B_1, B_2, \ldots, B_n \text{ of real numbers } \mathcal{R}.
\]

Definition (1-5) [6]

The expected value \( \mathcal{E} \) of uncertain variable \( \omega \) defined by

\[
\mathcal{E}(\omega) = \int_{-\infty}^{\infty} \mathcal{E}(\omega \geq \alpha) d\alpha - \int_{-\infty}^{0} \mathcal{E}(\omega \leq \alpha) d\alpha,
\]

provided that at least \( \int_{-\infty}^{\infty} \mathcal{E}(\omega \geq \alpha) d\alpha < \infty \) or \( \int_{-\infty}^{0} \mathcal{E}(\omega \leq \alpha) d\alpha < \infty \). The variance of \( \omega \) is defined by

\[
\mathcal{V}(\omega) = \mathcal{E}(\omega - \mathcal{E}(\omega))^2.
\]

Theorem (1-6) [5]
Suppose that the uncertain variable $X$. Then for any given numbers $\varepsilon > 0$ and $r > 0$, we have

$$U(\omega \geq \varepsilon) \leq \frac{\mathbb{E}(\omega^r)}{\varepsilon^r}$$  \hspace{1cm} (3).

Definition (1-7) [7]
The uncertain sequence $\{\omega_n\}$ be convergent in measure to uncertain variable $\omega$ if

$$\lim_{n \to \infty} U\{y \in \Gamma \mid \|\omega_n(y) - \omega(y)\| \geq \varepsilon\} = 0, \text{ for every } \varepsilon > 0.$$

Definition (1-8) [7]
The uncertain sequence $\{\omega_n\}$ be convergent in mean to uncertain variable $\omega$ if

$$\lim_{n \to \infty} \mathbb{E}\{y \in \Gamma \mid \|\omega_n(y) - \omega(y)\|\} = 0.$$

Definition (1-9) [7]
We say that the uncertain sequence $\{\omega_n\}$ is convergent to uncertain variable in distribution $\omega$ if

$$\lim_{n \to \infty} \mathbb{P}\{y \in \Gamma \mid \|\omega_n(y) - \omega(y)\|\} = 0.$$

2. Convergence of independent uncertain variables sequence sum

Lemma (2-1) [5]
Suppose that $X_i, i = 1, 2, \ldots, n$ be uncertain variables and $p > 0$. Then

$$\mathbb{E}\left[\sum_{i=1}^{n} \omega_i\right]^p \leq n^p \sum_{i=1}^{n} \mathbb{E}[\omega_i]^p$$  \hspace{1cm} (4).

Theorem (2-2) [7]
Suppose that $\{\omega_n\}$, and $\omega$ be an uncertain sequence and uncertain variable respectively. If $\omega_n \rightarrow \omega$ (in measure) as $n \to \infty$. Then $\omega_n \rightarrow \omega$ as $n \to \infty$ (in distribution).

Theorem (2-3)
Suppose that $\{\omega_n\}$, and $\omega$ be an uncertain sequence and uncertain variable respectively such that

$$\sum_{n=1}^{\infty} U\{\|\omega_n - \omega\| \geq \varepsilon\} < \infty$$  \hspace{1cm} (5).

Then $\omega_n \to \omega$ as $n \to \infty$ in measure.

Proof:
Since $\{y \in \Gamma \mid \|\omega_n(y) - \omega(y)\| \geq \varepsilon\} \subset \bigcup_{m=n}^{\infty} \{y \in \Gamma \mid \|\omega_m(y) - \omega(y)\| \geq \varepsilon\}$

From (1), and (2) we have

$$U\{y \in \Gamma \mid \|\omega_m(y) - \omega(y)\| \geq \varepsilon\} \leq U\bigcup_{m=n}^{\infty} \{y \in \Gamma \mid \|\omega_m(y) - \omega(y)\| \geq \varepsilon\}$$

$$\leq \sum_{m=n}^{\infty} U\{\|\omega_m - \omega\| \geq \varepsilon\}$$

Furthermore, $\lim_{n \to \infty} U\{y \in \Gamma \mid \|\omega_n(y) - \omega(y)\| \geq \varepsilon\} \leq \lim_{n \to \infty} \sum_{m=n}^{\infty} U\{\|\omega_n - \omega\| \geq \varepsilon\} = 0$

Corollary (2-4)
Suppose that $\{\omega_n\}$, and $\omega$ be an uncertain sequence, uncertain variables respectively such that

$$\sum_{n=1}^{\infty} U\{\|\omega_n - \omega\| \geq \varepsilon\} < \infty$$  \hspace{1cm} (6).
Then $\omega_n \rightarrow \omega$ as $n \rightarrow \infty$ in distribution.

**Proof:**
From theorem (2-2) and from theorem (2-3) according to (5), we have $\omega_n \rightarrow \omega$ as $n \rightarrow \infty$ in distribution.

**Theorem (2-5)**
Suppose that $\{\omega_n\}$, and $\omega$ be an uncertain sequence, uncertain variables respectively such that

$$\sum_{n=1}^{\infty} \mathbb{I} \left[ |\omega_n - \omega| \geq \varepsilon \right] \text{ is finite. Then } \omega_n \rightarrow \omega \text{ as } n \rightarrow \infty \text{ in measure.}$$

**Proof:**
Since $\{ x \in \Gamma \mid |\omega_n(x) - \omega(x)| \geq \varepsilon \} \subseteq \bigcup_{m=1}^{\infty} \{ x \in \Gamma \mid |\omega_m(x) - \omega(x)| \geq \varepsilon \}$

From (2) that

$$U \{ y \in \Gamma \mid |\omega_n(y) - \omega(y)| \geq \varepsilon \} \leq U \{ y \in \Gamma \mid |\omega_m(y) - \omega(y)| \geq \varepsilon \}$$

Then

$$U \{ y \in \Gamma \mid |\omega_n(y) - \omega(y)| \geq \varepsilon \} \leq \frac{\mathbb{I} \left[ \sum_{m=n}^{\infty} |\omega_m - \omega| \right]^2}{\varepsilon^2} \leq \frac{\mathbb{E} \left[ \omega_m - \omega \right]^2}{\varepsilon^2} \sum_{m=n}^{\infty} \mathbb{I} \left[ |\omega_m - \omega| \right]$$

Furthermore, $\lim_{n \rightarrow \infty} U \{ y \in \Gamma \mid |\omega_n(y) - \omega(y)| \geq \varepsilon \} \leq \lim_{n \rightarrow \infty} \frac{\mathbb{E} \left[ \omega_m - \omega \right]^2}{\varepsilon^2} \sum_{m=n}^{\infty} \mathbb{I} \left[ |\omega_m - \omega| \right] = 0$

**Corollary (2-6)**
Suppose that $\{\omega_n\}$, $\omega$ be an uncertain sequence, and uncertain variables respectively such that Then

$$\sum_{n=1}^{\infty} \mathbb{I} \left[ |\omega_n - \omega| \geq \varepsilon \right] < \infty \quad (7)$$

Then $\omega_n \rightarrow \omega$ as $n \rightarrow \infty$ in distribution.

**Proof:**
From theorem (2-2) and from theorem (2-4) according to (6), we have $\omega_n \rightarrow \omega$ as $n \rightarrow \infty$ in distribution.

**Theorem (2-7) (Kolmogrov inequality)**
Suppose that $\omega_i, i = 1, 2, \ldots, n$ be uncertain variables such that $W_n = \sum_{i=1}^{n} \omega_i$. If $\mathbb{I} [\omega_i^2] < \infty, i = 1, 2, \ldots, n$ then

$$U \{ \max_{1 \leq i \leq n} |W_i - \mathbb{I} (W_i)| \geq \varepsilon \} \leq \frac{n^2}{\varepsilon^2} \sum_{i=1}^{n} \text{Var} (\omega_i)$$

for any number $\varepsilon > 0$.

**Proof:**
From theorem (1-6) according to (3) and lemma (2-1) according to (4), we have

$$U \{ \max_{1 \leq i \leq n} |W_i - \mathbb{I} (W_i)| \geq \varepsilon \} \leq \frac{\mathbb{I} \left[ \max_{1 \leq i \leq n} |W_i - \mathbb{I} (W_i)| \right]^2}{\varepsilon^2} \leq \frac{1}{\varepsilon^2} \sum_{i=1}^{n} \mathbb{I} \left[ |\omega_i - \mathbb{I} (\omega_i)| \right]^2 \leq \frac{n^2}{\varepsilon^2} \sum_{i=1}^{n} \text{Var} (\omega_i)$$

**Theorem (2-8)**
Suppose that \{\omega_i\} be an uncertain sequence. If \(\sum_{i=1}^{n} \text{Var}(\omega_i)\) is a finite. Then \(\sum_{i=1}^{\infty} ((\omega_i) - \mathcal{I}(\omega_i))\) convergent in measure.

**Proof:**

Since \(\sum_{i=1}^{\infty} ((\omega_i) - \mathcal{I}(\omega_i))\) convergent in measure if and only if \(\lim_{n \to \infty} \sum_{i=1}^{n} ((\omega_i) - \mathcal{I}(\omega_i)) = 0\) in measure if and only if \(\lim_{n \to \infty} U\left\{\bigcup_{k=0}^{n} \left\{\sum_{i=n}^{n+k} (\omega_i - \mathcal{I}(\omega_i)) \right\} \geq \varepsilon\right\} = 0\)

For any \(\varepsilon > 0\),

\[
U\left\{\bigcup_{k=0}^{n} \left\{\sum_{i=n}^{n+k} (\omega_i - \mathcal{I}(\omega_i)) \right\} \geq \varepsilon\right\} = \lim_{m \to \infty} U\left\{\bigcup_{k=0}^{m} \left\{\sum_{i=n}^{n+m} (\omega_i - \mathcal{I}(\omega_i)) \right\} \geq \varepsilon\right\}
\]

\[
= \lim_{m \to \infty} U\left\{\max_{0 \leq k \leq m} \sum_{i=n}^{n+k} (\omega_i - \mathcal{I}(\omega_i)) \geq \varepsilon\right\} \leq \lim_{m \to \infty} \frac{n^2}{\varepsilon} \sum_{i=n}^{n+m} \mathcal{I}(\omega_i) - \mathcal{I}(\omega_i))
\]

\[
= \lim_{m \to \infty} \frac{n^2}{\varepsilon} \sum_{i=n}^{n+m} \text{Var}(\omega_i) = \lim_{n \to \infty} \frac{n^2}{\varepsilon} \sum_{i=n}^{\infty} \text{Var}(\omega_i) = 0, \text{from } \sum_{i=1}^{\infty} \text{Var}(X_i) \text{ is finite.}
\]

**Theorem (2-9)**

Suppose that \{\omega_i\} be an uncertain sequence. If \(\sum_{i=1}^{\infty} \text{Var}(\omega_i)\) is a finite. Then \(\sum_{i=1}^{\infty} ((\omega_i) - \mathcal{I}(\omega_i))\) converges in distribution.

**Proof:**

From theorem (2-2) and theorem (2-8) according to (8), we have \(\sum_{i=1}^{\infty} ((\omega_i) - \mathcal{I}(\omega_i))\) convergent to uncertain variable \(\omega\) in distribution.

**Theorem (2-10)**

Suppose that \{\omega_n\}, \omega be uncertain sequence, and uncertain variables respectively. If \(\omega_n \to \omega\) as \(n \to \infty\) in mean, then \(\omega_n \to \omega\) as \(n \to \infty\) in distribution.

**Proof:**

Since \(\omega_n \to \omega\) as \(n \to \infty\) in mean means \(\omega_n \to \omega\) as \(n \to \infty\) in measure, and from theorem (2-2), thus \{\omega_n\} is convergence uncertain sequence in distribution to uncertain variable \(\omega\).

**Example (2-11)[4]**

If \(\omega_n \to \omega\) as \(n \to \infty\) in distribution, then \(\omega_n \to \omega\) as \(n \to \infty\) in mean.

For example, suppose that the uncertainty space and \(\Gamma = \{y_1, y_2\}\) with \(U\{y_1\} = U\{y_2\} = 0.5\) and

\[
\omega_i(y_j) = \begin{cases} 
    b, & \text{if } y = y_1 \\
    -b, & \text{if } y = y_2
\end{cases}
\]

, and uncertainty distribution of \(\omega_i\) and \(\omega\) be \(\psi(\omega) = \begin{cases} 
    0, & \omega < -b \\
    0.5, & -b \leq \omega < b \\
    1, & \omega \geq b
\end{cases}\), that is \(\lim_{n \to \infty} \psi_n(\omega) = \psi(\omega)\) in distribution.

Putting \(\omega_n = -\omega\) we have \(|\omega_n - \omega| = 2b\) for all \(n = 1,2,3,\ldots, y = y_1, y_2\). Thus

\[
\mathbb{E}[|\omega_n - \omega|] = \int_{0}^{2b} 1 dx = 2b
\]

so that \(\lim_{n \to \infty} \mathbb{E}[|\omega_n - \omega|] = 2b\).

That is, \(\omega_n \to \omega\) as \(n \to \infty\) in mean.
3. Conclusion

The purpose of this paper is to obtain new properties of uncertain variable convergence in measure, also Kolmogorov inequality is verified.

References