Multiplicative Triple Fibonacci Sequence of Third Order

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Abstract:
The Coupled Fibonacci Sequence are Firstly established by K.T. Atanassov in 1985. The abstractions of Fibonacci Triple Sequence are considered in 1987. Fibonacci Sequence stands as a kind of super sequence with Fabulous properties. This is the explosive development in the region of Fibonacci Sequence. Fibonacci was advanced by Leonardo de Pisa (whose nickname was Fibonacci, which means son of Bonacci) in 1202 as a result of his inspection on the growth of a population of rabbits. The consecutive Fibonacci numbers are produced by adding together the two previous numbers in the sequence, after specifying suitable initial conditions. In the last years Triple Fibonacci Sequence are hype, but Multiplicative Triple Sequence of Recurrence Relations are less known. Much work has been done to study on Fibonacci Triple Sequence in Additive form. In 1995, Multiplicative Coupled Fibonacci Sequence are contemplated. Our purpose of this paper to present some results of Multiplicative Triple Fibonacci Sequence of third order under one specific scheme.
This paper expanded out of a curiosity in the Fibonacci sequence and a craving to spread the results of Multiplicative Coupled Fibonacci sequence. Ever since Fibonacci (Leonardo of Pisa) wrote his Liber Abbaci in 1202, his fascinating sequence has transfixed men through the centuries, not only for its inborn mathematical riches, but also for its applications in art and nature. Indeed, it is almost true to say that the research produced by its nearly amounts to the quantity of off- spring generated by the mythical pair of rabbits who started Fibonacci off on the problem.

Keywords: Fibonacci Sequence, Multiplicative Triple Fibonacci sequence.

1. Introduction: The Fibonacci Triple Sequence is a new direction in generalization of Coupled Fibonacci sequence. Fibonacci sequence and their generalization have many attracting applications and properties to every field of science. Koshy's book [9] is a good origin for these applications. The Coupled Fibonacci Sequence was first inaugurated by K. T. Atnassov [4] and also examined many curious properties and a new guideline of generalization of Fibonacci Sequence [2, 5, 6].
J. Z. Lee and J. S. Lee established Firstly Additive Triple Sequence [3]. K. T. Atnassov delineate new notion for Additive Triple Fibonacci Sequence [7, 8] and called 3-Fibonacci Sequence or 3-F Sequence.
Let $\{\alpha_i\}_{i=0}^{\infty}$ $\{\beta_i\}_{i=0}^{\infty}$ $and\, \{\gamma_i\}_{i=0}^{\infty}$ be three infinite sequences and called 3-F Sequence or Triple Fibonacci Sequence with initial value a, b, c, d, e and f.
If $\alpha_0 = a, \beta_0 = b, \gamma_0 = c, \alpha_1 = d, \beta_1 = e, \gamma_1 = f$, then nine different schemes of Multiplicative Triple Fibonacci Sequence are as follows:

First Scheme:

$$\alpha_{n+2} = \beta_{n+1}.\gamma_n$$
$$\beta_{n+2} = \gamma_{n+1}.\alpha_n$$
$$\gamma_{n+2} = \alpha_{n+1}.\beta_n$$

Second Scheme:

$$\alpha_{n+2} = \gamma_{n+1}.\beta_n$$
$$\beta_{n+2} = \alpha_{n+1}.\gamma_n$$
$$\gamma_{n+2} = \beta_{n+1}.\alpha_n$$

Third Scheme:

$$\alpha_{n+2} = \alpha_{n+1}.\beta_n$$
$$\beta_{n+2} = \beta_{n+1}.\gamma_n$$
$$\gamma_{n+2} = \gamma_{n+1}.\alpha_n$$

Fourth Scheme:

$$\alpha_{n+2} = \beta_{n+1}.\alpha_n$$
$$\beta_{n+2} = \gamma_{n+1}.\beta_n$$
$$\gamma_{n+2} = \alpha_{n+1}.\gamma_n$$

Fifth Scheme:

$$\alpha_{n+2} = \alpha_{n+1}.\gamma_n$$
$$\beta_{n+2} = \beta_{n+1}.\alpha_n$$
$$\gamma_{n+2} = \gamma_{n+1}.\beta_n$$

Sixth Scheme:

$$\alpha_{n+2} = \gamma_{n+1}.\alpha_n$$
$$\beta_{n+2} = \alpha_{n+1}.\beta_n$$
$$\gamma_{n+2} = \beta_{n+1}.\gamma_n$$

Seventh Scheme:

$$\alpha_{n+2} = \alpha_{n+1}.\alpha_n$$
$$\beta_{n+2} = \beta_{n+1}.\beta_n$$
$$\gamma_{n+2} = \gamma_{n+1}.\gamma_n$$

Eighth Scheme:

$$\alpha_{n+2} = \beta_{n+1}.\beta_n$$
$$\beta_{n+2} = \gamma_{n+1}.\gamma_n$$
$$\gamma_{n+2} = \alpha_{n+1}.\alpha_n$$

Ninth Scheme:

$$\alpha_{n+2} = \gamma_{n+1}.\gamma_n$$
$$\beta_{n+2} = \alpha_{n+1}.\alpha_n$$
$$\gamma_{n+2} = \beta_{n+1}.\beta_n$$

O.P. Sikhwal, M. Singh, S. Bhatnagar [1] studied numerous results of second order.

In this paper, we put forward some results on Multiplicative Triple Fibonacci Sequence of third order under two specific schemes.

2. Multiplicative Triple Fibonacci Sequence of third order:

Let $\{\alpha_i\}_{i=0}^{\infty}$ $\{\beta_i\}_{i=0}^{\infty}$ and $\{\gamma_i\}_{i=0}^{\infty}$ be three infinite sequences and called 3-F Sequence or Triple Fibonacci Sequence with initial value a, b, c, d, e, f, g, h and i be given.

If $\alpha_0 = a, \beta_0 = b, \gamma_0 = c, \alpha_1 = d, \beta_1 = e, \gamma_1 = f, \alpha_2 = g, \beta_2 = h, \gamma_2 = i$ then twenty seven different schemes of Multiplicative Triple Fibonacci Sequence are as follows:

First Scheme:

$$\alpha_{n+3} = \beta_{n+2}.\gamma_{n+1}.\alpha_n$$
$$\beta_{n+3} = \gamma_{n+2}.\alpha_{n+1}.\beta_n$$
$$\gamma_{n+3} = \alpha_{n+2}.\beta_{n+1}.\gamma_n$$

Second Scheme:

$$\alpha_{n+3} = \alpha_{n+2}.\alpha_{n+1}.\alpha_n$$
$$\beta_{n+3} = \beta_{n+2}.\beta_{n+1}.\beta_n$$
$$\gamma_{n+3} = \gamma_{n+2}.\gamma_{n+1}.\gamma_n$$

Third Scheme:

$$\alpha_{n+3} = \alpha_{n+2}.\gamma_{n+1}.\beta_n$$
$$\beta_{n+3} = \beta_{n+2}.\alpha_{n+1}.\gamma_n$$
$$\gamma_{n+3} = \gamma_{n+2}.\beta_{n+1}.\alpha_n$$

Fourth Scheme:

$$\alpha_{n+3} = \gamma_{n+2}.\beta_{n+1}.\alpha_n$$
$$\beta_{n+3} = \alpha_{n+2}.\gamma_{n+1}.\beta_n$$
$$\gamma_{n+3} = \beta_{n+2}.\alpha_{n+1}.\gamma_n$$

Fifth Scheme:

$$\alpha_{n+3} = \alpha_{n+2}.\beta_{n+1}.\gamma_n$$
$$\beta_{n+3} = \beta_{n+2}.\gamma_{n+1}.\alpha_n$$
$$\gamma_{n+3} = \gamma_{n+2}.\alpha_{n+1}.\beta_n$$

Sixth Scheme:

$$\alpha_{n+3} = \alpha_{n+2}.\alpha_{n+1}.\beta_n$$
$$\beta_{n+3} = \beta_{n+2}.\beta_{n+1}.\gamma_n$$
$$\gamma_{n+3} = \gamma_{n+2}.\gamma_{n+1}.\alpha_n$$

Seventh Scheme:

$$\alpha_{n+3} = \alpha_{n+2}.\beta_{n+1}.\alpha_n$$
$$\beta_{n+3} = \beta_{n+2}.\gamma_{n+1}.\beta_n$$
$$\gamma_{n+3} = \gamma_{n+2}.\alpha_{n+1}.\gamma_n$$

Eighth Scheme:

$$\alpha_{n+3} = \beta_{n+2}.\alpha_{n+1}.\alpha_n$$
$$\beta_{n+3} = \gamma_{n+2}.\beta_{n+1}.\beta_n$$
$$\gamma_{n+3} = \alpha_{n+2}.\gamma_{n+1}.\gamma_n$$

Ninth Scheme:

$$\alpha_{n+3} = \alpha_{n+2}.\alpha_{n+1}.\gamma_n$$
$$\beta_{n+3} = \beta_{n+2}.\beta_{n+1}.\alpha_n$$
$$\gamma_{n+3} = \gamma_{n+2}.\gamma_{n+1}.\beta_n$$

Tenth Scheme:

$$\alpha_{n+3} = \alpha_{n+2}.\gamma_{n+1}.\alpha_n$$
$$\beta_{n+3} = \beta_{n+2}.\alpha_{n+1}.\beta_n$$
$$\gamma_{n+3} = \gamma_{n+2}.\beta_{n+1}.\gamma_n$$

Eleventh Scheme:

$$\alpha_{n+3} = \gamma_{n+2}.\alpha_{n+1}.\alpha_n$$
$$\beta_{n+3} = \alpha_{n+2}.\beta_{n+1}.\beta_n$$
$$\gamma_{n+3} = \beta_{n+2}.\gamma_{n+1}.\gamma_n$$

Twelfth Scheme:

$$\alpha_{n+3} = \beta_{n+2}.\beta_{n+1}.\gamma_n$$
$$\beta_{n+3} = \gamma_{n+2}.\gamma_{n+1}.\alpha_n$$
$$\gamma_{n+3} = \alpha_{n+2}.\alpha_{n+1}.\beta_n$$

Thirteenth Scheme:

$$\alpha_{n+3} = \beta_{n+2}.\gamma_{n+1}.\beta_n$$
$$\beta_{n+3} = \gamma_{n+2}.\alpha_{n+1}.\gamma_n$$
$$\gamma_{n+3} = \alpha_{n+2}.\beta_{n+1}.\alpha_n$$

Fourteenth Scheme:

$$\alpha_{n+3} = \gamma_{n+2}.\beta_{n+1}.\beta_n$$
$$\beta_{n+3} = \alpha_{n+2}.\gamma_{n+1}.\gamma_n$$

$$\gamma_{n+3} = \beta_{n+2}.\alpha_{n+1}.\alpha_n$$

Fifteenth Scheme:

$$\alpha_{n+3} = \beta_{n+2}.\gamma_{n+1}.\gamma_n$$
$$\beta_{n+3} = \gamma_{n+2}.\alpha_{n+1}.\alpha_n$$
$$\gamma_{n+3} = \alpha_{n+2}.\beta_{n+1}.\beta_n$$

Sixteenth Scheme:

$$\alpha_{n+3} = \gamma_{n+2}.\beta_{n+1}.\gamma_n$$
$$\beta_{n+3} = \alpha_{n+2}.\gamma_{n+1}.\alpha_n$$
$$\gamma_{n+3} = \beta_{n+2}.\alpha_{n+1}.\beta_n$$

Seventeenth Scheme:

$$\alpha_{n+3} = \gamma_{n+2}.\gamma_{n+1}.\beta_n$$
$$\beta_{n+3} = \alpha_{n+2}.\alpha_{n+1}.\gamma_n$$
$$\gamma_{n+3} = \beta_{n+2}.\beta_{n+1}.\alpha_n$$

Eighteenth Scheme:

$$\alpha_{n+3} = \beta_{n+2}.\gamma_{n+1}.\alpha_n$$
$$\beta_{n+3} = \gamma_{n+2}.\alpha_{n+1}.\beta_n$$
$$\gamma_{n+3} = \alpha_{n+2}.\beta_{n+1}.\gamma_n$$

Nineteenth Scheme:

$$\alpha_{n+3} = \beta_{n+2}.\alpha_{n+1}.\beta_n$$
$$\beta_{n+3} = \gamma_{n+2}.\beta_{n+1}.\gamma_n$$
$$\gamma_{n+3} = \alpha_{n+2}.\gamma_{n+1}.\alpha_n$$

Twentieth Scheme:

$$\alpha_{n+3} = \alpha_{n+2}.\beta_{n+1}.\beta_n$$
$$\beta_{n+3} = \beta_{n+2}.\gamma_{n+1}.\gamma_n$$
$$\gamma_{n+3} = \gamma_{n+2}.\alpha_{n+1}.\alpha_n$$

Twenty First Scheme:

$$\alpha_{n+3} = \beta_{n+2}.\beta_{n+1}.\alpha_n$$
$$\beta_{n+3} = \gamma_{n+2}.\gamma_{n+1}.\beta_n$$
$$\gamma_{n+3} = \alpha_{n+2}.\alpha_{n+1}.\gamma_n$$

Twenty Second Scheme:

$$\alpha_{n+3} = \alpha_{n+2}.\gamma_{n+1}.\gamma_n$$
$$\beta_{n+3} = \beta_{n+2}.\alpha_{n+1}.\alpha_n$$
$$\gamma_{n+3} = \gamma_{n+2}.\beta_{n+1}.\beta_n$$

Twenty Third Scheme:

$$\alpha_{n+3} = \gamma_{n+2}.\alpha_{n+1}.\gamma_n$$
$$\beta_{n+3} = \alpha_{n+2}.\beta_{n+1}.\alpha_n$$
$$\gamma_{n+3} = \beta_{n+2}.\gamma_{n+1}.\beta_n$$

Twenty Fourth Scheme:

$$\alpha_{n+3} = \gamma_{n+2}.\gamma_{n+1}.\alpha_n$$
$$\beta_{n+3} = \alpha_{n+2}.\alpha_{n+1}.\beta_n$$
$$\gamma_{n+3} = \beta_{n+2}.\beta_{n+1}.\gamma_n$$

Twenty Fifth Scheme:

$$\alpha_{n+3} = \beta_{n+2}.\alpha_{n+1}.\gamma_n$$
$$\beta_{n+3} = \gamma_{n+2}.\beta_{n+1}.\alpha_n$$
$$\gamma_{n+3} = \alpha_{n+2}.\gamma_{n+1}.\beta_n$$

Twenty Sixth Scheme:

$$\alpha_{n+3} = \beta_{n+2}.\beta_{n+1}.\beta_n$$
$$\beta_{n+3} = \gamma_{n+2}.\gamma_{n+1}.\gamma_n$$
$$\gamma_{n+3} = \alpha_{n+2}.\alpha_{n+1}.\alpha_n$$

Twenty Seventh Scheme:

$$\alpha_{n+3} = \gamma_{n+2}.\gamma_{n+1}.\gamma_n$$
$$\beta_{n+3} = \alpha_{n+2}.\alpha_{n+1}.\alpha_n$$
$$\gamma_{n+3} = \beta_{n+2}.\beta_{n+1}.\beta_n$$

First few terms of first scheme (1.1) are as below:

N	α_n	β_n	γ_n
0	a	b	c
1	d	e	f
2	g	h	i
3	hfa	idb	gec
4	i^2d^2b	g^2e^2c	h^2f^2a
5	$g^4e^3c^2$	$h^4f^3a^2$	$i^4d^3b^2$

3. Main Result: We obtain some results of Multiplicative Triple Fibonacci Sequence of third order for first scheme:

Theorem: For every integer $n \geq 0$

(a) $\alpha_{n+9} = \alpha_{n+6}^4.\beta_{n+5}^3.\gamma_{n+4}^2$

(b) $\beta_{n+9} = \beta_{n+6}^4.\gamma_{n+5}^3.\alpha_{n+4}^2$

(c) $\gamma_{n+9} = \gamma_{n+6}^4.\alpha_{n+5}^3.\beta_{n+4}^2$

Proof: We prove the result by induction method

(a) If $n = 0$ then $\alpha_9 = \beta_8.\gamma_7.\alpha_6$ (By first scheme)

$= \gamma_7.\alpha_6.\beta_5.\gamma_7.\alpha_6$ (By first scheme)

$= \alpha_6^2.\beta_5.\gamma_7^2$

$= \alpha_6^2.\beta_5.\alpha_6.\beta_5.\gamma_4.\alpha_6.\beta_5.\gamma_4$ (By first scheme)

$= \alpha_6^4.\beta_5^3.\gamma_4^2$

Thus, the result is true for $n = 0$

Let us assume that the result is true for some integer $n \geq 0$. Then

$\alpha_{n+10} = \beta_{n+9}.\gamma_{n+8}.\alpha_{n+7}$ (By first scheme)

$= \gamma_{n+8}.\alpha_{n+7}.\beta_{n+6}.\gamma_{n+8}.\alpha_{n+7}$ (By first scheme)

$= \alpha_{n+7}^2.\beta_{n+6}.\gamma_{n+8}^2$

$= \alpha_{n+7}^2 . \beta_{n+6} . (\alpha_{n+7} . \beta_{n+6} . \gamma_{n+5})^2$ (By first scheme)
$= \alpha_{n+7}^4 . \beta_{n+6}^3 . \gamma_{n+5}^2$
Hence the result is true for all integers $n \geq 0$
Similarly, we can give the proof of part (b) and (c)
Theorem: For every integer $n \geq 0$

(a) $\prod_{k=0}^{n} \alpha_{2k+10} = \prod_{k=0}^{n} \beta_{2k+9} . \gamma_{2k+8} . \alpha_{2k+7}$
(b) $\prod_{k=0}^{n} \beta_{2k+10} = \prod_{k=0}^{n} \gamma_{2k+9} . \alpha_{2k+8} . \beta_{2k+7}$
(c) $\prod_{k=0}^{n} \gamma_{2k+10} = \prod_{k=0}^{n} \alpha_{2k+9} . \beta_{2k+8} . \gamma_{2k+7}$

Proof. We prove the above result by induction method:
(a) For $n = 0$ then $\alpha_{10} = \beta_9 . \gamma_8 . \alpha_7$ this is true by first scheme.

Let us assume the result is true for $n = l$
Hence $\prod_{k=0}^{l} \alpha_{2k+10} = \prod_{k=0}^{l} \beta_{2k+9} . \gamma_{2k+8} . \alpha_{2k+7}$
Now for $n = l + 1$
Then $\prod_{k=0}^{l+1} \beta_{2k+9} . \gamma_{2k+8} . \alpha_{2k+7} = \beta_{2(l+1)+9} . \gamma_{2(l+1)+8} . \alpha_{2(l+1)+7} . \prod_{k=0}^{l} \beta_{2k+9} . \gamma_{2k+8} . \alpha_{2k+7}$
$= \alpha_{2(l+1)+10} . \prod_{k=0}^{l} \alpha_{2k=10}$ (By induction hypothesis)
$= \prod_{k=0}^{l+1} \alpha_{2k=10}$
Thus, the result is true for $n = l + 1$. Hence by induction method the result is true for any positive integer n.
Similar proof can be given for remaining parts (b) and (c).

Theorem: For every integer $n \geq 0$

(a) $\prod_{k=0}^{n} \alpha_{3k+10} = \prod_{k=0}^{n} \beta_{3k+9} . \gamma_{3k+8} . \alpha_{3k+7}$
(b) $\prod_{k=0}^{n} \beta_{3k+10} = \prod_{k=0}^{n} \gamma_{3k+9} . \alpha_{3k+8} . \beta_{3k+7}$
(c) $\prod_{k=0}^{n} \gamma_{3k+10} = \prod_{k=0}^{n} \alpha_{3k+9} . \beta_{3k+8} . \gamma_{3k+7}$

This can also prove by induction.

Theorem: For every integer $n \geq 0$

(a) $\prod_{k=0}^{n} \alpha_{pk+10} = \prod_{k=0}^{n} \beta_{pk+9} . \gamma_{pk+8} . \alpha_{pk+7}$
(b) $\prod_{k=0}^{n} \beta_{pk+10} = \prod_{k=0}^{n} \gamma_{pk+9} . \alpha_{pk+8} . \beta_{pk+7}$
(c) $\prod_{k=0}^{n} \gamma_{pk+10} = \prod_{k=0}^{n} \alpha_{pk+9} . \beta_{pk+8} . \gamma_{pk+7}$

This can also prove by induction.

Theorem: For every integer $n \geq 0, q \geq 0$

(a) $\prod_{k=0}^{n} \alpha_{2k+q+3} = \prod_{k=0}^{n} \beta_{2k+q+2} . \gamma_{2k+q+1} . \alpha_{2k+q}$
(b) $\prod_{k=0}^{n} \beta_{2k+q+3} = \prod_{k=0}^{n} \gamma_{2k+q+2} . \alpha_{2k+q+1} . \beta_{2k+q}$
(c) $\prod_{k=0}^{n} \gamma_{2k+q+3} = \prod_{k=0}^{n} \alpha_{2k+q+2} . \beta_{2k+q+1} . \gamma_{2k+q}$

Proof. We prove the above result by induction method:
(a) For $n = 0$ then $\alpha_{q+3} = \beta_{q+2} . \gamma_{q+1} . \alpha_q$ which is true by first scheme.

Let us assume the result is true for $n = l$
Hence $\prod_{k=0}^{l} \alpha_{2k+10} = \prod_{k=0}^{l} \beta_{2k+9} . \gamma_{2k+8} . \alpha_{2k+7}$
Now for $n = l + 1$

Then $\prod_{k=0}^{l+1}\beta_{2k+q+2}\cdot\gamma_{2k+q+1}.\alpha_{2k+q} = \beta_{2(l+1)+q+2}.\gamma_{2(l+1)+q+1}.\alpha_{2(l+1)+q}.\prod_{k=0}^{l}\beta_{2k+q+2}\cdot\gamma_{2k+q+1}.\alpha_{2k+q}$
$= \alpha_{2(l+1)+q+3}.\prod_{k=0}^{l}\alpha_{2k+q+3}$ (By induction hypothesis)
$= \prod_{k=0}^{l+1}\alpha_{2k+q+3}$ = L.H.S.
Thus, the result is true for $n = l+1$. Hence by induction method the result is true for any positive integer n.
Similar proof can be given for remaining parts (b) and (c).

Theorem: For every integer $n \geq 0, p \geq 0, q \geq 0$

(a) $\prod_{k=0}^{n}\alpha_{pk+q+3} = \prod_{k=0}^{n}\beta_{pk+q+2}\cdot\gamma_{pk+q+1}.\alpha_{pk+q}$
(b) $\prod_{k=0}^{n}\beta_{pk+q+3} = \prod_{k=0}^{n}\gamma_{pk+q+2}\cdot\alpha_{pk+q+1}.\beta_{pk+q}$
(c) $\prod_{k=0}^{n}\gamma_{pk+q+3} = \prod_{k=0}^{n}\alpha_{pk+q+2}\cdot\beta_{pk+q+1}.\gamma_{pk+q}$

Proof. We prove the above result by induction method:
(a) For $n = 0$ then $\alpha_{q+3} = \beta_{q+2}.\gamma_{q+1}.\alpha_q$ this is true by first scheme.

Let us assume the result is true for $n = l$
Hence $\prod_{k=0}^{l}\alpha_{pk+10} = \prod_{k=0}^{l}\beta_{pk+9}\cdot\gamma_{pk+8}.\alpha_{pk+7}$
Now for $n = l+1$
Then $\prod_{k=0}^{l+1}\beta_{pk+q+2}\cdot\gamma_{pk+q+1}.\alpha_{pk+q} ==$
$\beta_{p(l+1)+q+2}.\gamma_{p(l+1)+q+1}.\alpha_{p(l+1)+q}.\prod_{k=0}^{l}\beta_{pk+q+2}\cdot\gamma_{pk+q+1}.\alpha_{pk+q}$
$= \alpha_{p(l+1)+q+3}.\prod_{k=0}^{l}\alpha_{pk+q+3}$ (By induction hypothesis)
$= \prod_{k=0}^{l+1}\alpha_{pk+q+3}$ = L.H.S.
Thus, the result is true for $n = l+1$. Hence by induction method the result is true for any positive integer n.
Similar proof can be given for remaining parts (b) and (c).

Theorem: For every integer $n \geq 2$,
$$(\alpha_0\beta_0\gamma_0)^n(\alpha_1\beta_1\gamma_1)^{n+1}(\alpha_2\beta_2\gamma_2)^{n+2} = (\alpha_3\beta_3\gamma_3)^{n-2}(\alpha_5\beta_5\gamma_5)$$
Proof: We prove the above result by induction method:
For $n = 2$ then
$(\alpha_0\beta_0\gamma_0)^2(\alpha_1\beta_1\gamma_1)^3(\alpha_2\beta_2\gamma_2)^4 = (\alpha_1\beta_1\gamma_1)(\alpha_2\beta_2\gamma_2)^2(\alpha_3\beta_3\gamma_3)^2$ (By second scheme)
$= (\alpha_2\beta_2\gamma_2)(\alpha_3\beta_3\gamma_3)(\alpha_4\beta_4\gamma_4)$ (By second scheme)
$= (\alpha_5\beta_5\gamma_5)$
The result is true for $n = 2$
Let us assume the result is true for $n = l$
Hence
$$(\alpha_0\beta_0\gamma_0)^l(\alpha_1\beta_1\gamma_1)^{l+1}(\alpha_2\beta_2\gamma_2)^{l+2} = (\alpha_3\beta_3\gamma_3)^{l-2}(\alpha_5\beta_5\gamma_5)$$
Now for $n = l+1$
Then $(\alpha_0\beta_0\gamma_0)^{l+1}(\alpha_1\beta_1\gamma_1)^{l+2}(\alpha_2\beta_2\gamma_2)^{l+3}$
$=(\alpha_0\beta_0\gamma_0)^l(\alpha_0\beta_0\gamma_0)(\alpha_1\beta_1\gamma_1)^{l+1}(\alpha_1\beta_1\gamma_1)(\alpha_2\beta_2\gamma_2)^{l+2}(\alpha_2\beta_2\gamma_2)$
$= (\alpha_0\beta_0\gamma_0)^l(\alpha_1\beta_1\gamma_1)^{l+1}(\alpha_2\beta_2\gamma_2)^{l+2}(\alpha_0\beta_0\gamma_0)(\alpha_1\beta_1\gamma_1)(\alpha_2\beta_2\gamma_2)$
$= (\alpha_3\beta_3\gamma_3)^{l-2}(\alpha_5\beta_5\gamma_5)(\alpha_0\beta_0\gamma_0)(\alpha_1\beta_1\gamma_1)(\alpha_2\beta_2\gamma_2)$ (By hypothesis)
$= (\alpha_3\beta_3\gamma_3)^{l-2}(\alpha_5\beta_5\gamma_5)(\alpha_3\beta_3\gamma_3)$ (By second scheme)
$= (\alpha_3\beta_3\gamma_3)^{l-1}(\alpha_5\beta_5\gamma_5)$
Thus, the result is true for $n = l+1$. Hence by induction method the result is true for any positive integer $n \geq 2$.

Theorem: For every integer $n \geq 0$:

$$(a)\ \alpha_{6n+4} = \frac{\prod_{p=0}^{6n+3}\alpha_p}{\prod_{p=0}^{6n}\alpha_p}$$

$$(b)\ \beta_{6n+4} = \frac{\prod_{p=0}^{6n+3}\beta_p}{\prod_{p=0}^{6n}\beta_p}$$

$$(c)\ \gamma_{6n+4} = \frac{\prod_{p=0}^{6n+3}\gamma_p}{\prod_{p=0}^{6n}\gamma_p}$$

Theorem: For every integer $n \geq 0$:

$$(a)\ \alpha_{ln+m} = \frac{\prod_{p=0}^{6n+m-1}\alpha_p}{\prod_{p=0}^{6n+m-4}\alpha_p}$$

$$(b)\ \beta_{ln+m} = \frac{\prod_{p=0}^{6n+m-1}\beta_p}{\prod_{p=0}^{6n+m-4}\beta_p}$$

$$(c)\ \gamma_{ln+m} = \frac{\prod_{p=0}^{6n+m-1}\gamma_p}{\prod_{p=0}^{6n+m-4}\gamma_p}$$

4. Conclusion:

Much work has been achieved on Multiplicative Triple Fibonacci Sequence. In this paper, we have to characterize some results of Multiplicative Triple Fibonacci Sequence of third order under one specific scheme.

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