Research Article

Exploring the causal influence of beliefs on mathematical competencies using an integrated approach of Interval-valued Intuitionistic Fuzzy Relational Map and TOPSIS

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Abstract
Achievement in mathematics is widely recognised as a sign of intelligence and cognitive development. Students' involvement in acquiring mathematical competencies is not just automatic, but needs conscious efforts from the part of students. The intrinsic factor that motivates the students to plunge into activities that develop mathematical competencies is self-belief or belief system. The relationship between personal beliefs and competencies acquired is so important that it attracts the interest of researchers to create an 'intellectual and competent' community. In this paper, the causal influence of mathematics related beliefs on mathematical competencies is studied using a hybrid fuzzy model called IVIFRM-TOPSIS. This expert based method is more comprehensive in its approach as the uncertain information is modelled with interval-valued intuitionistic fuzzy sets. This study exhibits the efficiency of the fuzzy hybrid model in analysing the causal relationship and ranking the order of influence.

Key words
Fuzzy relational map, intuitionistic fuzzy set, interval-valued, aggregation operator, beliefs, mathematical competencies, mathematics education

1. Introduction

In most of the real-world problems, within a social context, words are used to describe the system and the related issues. When there was a need to quantify the qualitative data, the introduction of fuzzy sets by L. A. Zadeh (1965) came as an alternative approach that is capable of quantifying linguistic expressions. The fuzzy sets characterise the membership value of an element belonging to a particular set. Using single values to represent the membership values is inadequate to fully describe the linguistic expression and also it has no place to express the hesitancy of the expert. In order to include the hesitancy in the expert's opinion several extensions of fuzzy sets have been introduced.

Atanassav (1983) came up with a special fuzzy set which included non-membership value along with the membership value [2]. From the membership and non-membership value of an element belonging to a set, the hesitancy degree of expert's opinion can be computed. Atanassav explained this scenario in uncertainty modelling convincingly using a substantial metaphor [2]. Let μ be the membership degree of the electorates which voted for a government and $\nu = 1 - \mu$ be the non-membership degree of the electorates which voted for a government. In this case, the information about the people who have not voted at all. Therefore, if the membership and non-membership could be defined then the information about the abstention could be inferred from the expression $\pi = 1 - \mu - \nu$. Hence it can be said that Intuitionistic fuzzy sets are efficient in grasping more information and thus, quantifying the uncertainty efficiently. This notion is further extended to interval valued intuitionistic fuzzy sets (IVIFS) by Atanasssav and Gargov [3] where intervals are used to represent the membership and non-membership degrees of an element belonging to a set.

Fuzzy Relational Maps are simple but efficient tools to study conceptual problems. It is basically a soft computing technique that is a combination of fuzzy logic and neural network techniques [23]. FRM are useful in analysing the complex cognitive problems that deals with very high uncertainty.

FRMs are similar to FCM in every way except the factors are divided into two disjoint sets. FRM is a bi-directed map where the concepts are represented by set of nodes and the interrelations between the concepts are represented by edges. The concept values and edge strength are quantified with fuzzy sets based on expert's opinion. The fixed points obtained in the iterative fuzzy inference process are used as weights and these weights are used to calculate the global weights. The weights obtained by this method are more reliable than the random weights as the interaction among the concepts is also taken into account [6], [10].

Hajek et al. proposed a new hybrid decision support method by integrating IVIFCM and TOPSIS method in [10] to deal with Multi Criteria Group Decision Making (MCGDM). In this paper, this hybrid model is adopted to study the influence between mathematics related beliefs and mathematical competencies. The factors of domain and range spaces are treated like criteria and alternatives in MCGDM problem. The ranking method gives the ideal about the most and the least influential factors. The advantages of this hybrid approach are: 1) The interval-valued intuitionistic fuzzy sets represent the uncertain and vague information with greater flexibility. 2) The IVIFRM model provides the interrelated interactions among the concepts [10].

It is widely believed that Galileo Galilei said: "Mathematics is the language in which God has written the universe". Human beings try to understand and operate the universe with the help of mathematics. So far, mathematics has been one of the successful means in empowering the humankind with necessary methods, tools and techniques to win over the universe defying even several laws of the nature. Mathematics and science are central to make an individual competent in understanding the world around better. In particular, Mathematics is generally recognized as a tool to explore the scientific and technological advancements. Recent researches on neurosciences and psychology have opened a new avenue that the world has had a turn around. An extensive study has been done on the significant factors of affective and cognitive domains.

There are several contributions from the research community that discuss on the influences among beliefs and mathematical performance. Evans emphasizes on aspects of emotion and cognition related to mathematics learning and beliefs about mathematics [5]. Malmivuori has developed a theoretical framework to explain the dynamic interaction of affect and cognition in relation to the learning processes involved in mathematics education [15]. Gomez-Chacon proposed a model for studying the interaction between cognition and affect in mathematics education [9]. Goldin conducted an extensive research on how affect influences mathematical problem solving [7]. Leder directed the world's focus on reconciling affective and cognitive approaches to research on mathematics learning [13]. All these contributions are useful in understanding the influence of beliefs on mathematics education and stimulated the attention to research on beliefs and mathematics learning around the world in recent years.

In this paper, an attempt is made to study the influence between mathematics related beliefs and mathematical competencies using IVIFRM-TOPSIS model that studies the interactions among the concepts and ranks the most influential factor.

2. Intuitionistic Fuzzy Sets – Preliminaries

The Intuitionistic fuzzy sets are one of the most used fuzzy extensions in many real-life applications. Let E be a fixed set. Atanassov Intuitionistic Fuzzy Set (IFS) [2] is of the form

$$A = \{\langle x, \ (\mu_a(x), \nu_a(x))\rangle | x \in E\}) \qquad (1)$$

where $\mu_a(x), \nu_a(x): X \rightarrow [0,1]$ are the degree of membership and non-membership of the element $x \in X$ respectively with a condition that for $x \in X$, $0 \leq \mu_a(x) + \nu_a(x) \leq 1$.

For every A (IFS) in X, $\pi_P(x) = 1 - \mu_a(x) - \nu_a(x)$, is called the degree of hesitation of $x \in E$ to A. A pair $A = (\mu_a, \nu_a)$ where $(\mu_a, \nu_a) \in [0,1]$ with $0 \leq \mu_a(x) + \nu_a(x) \leq 1$ is called an Intuitionistic Fuzzy Number (IFN) [2].

Atanassov and Gargov introduced interval-valued Intuitionistic fuzzy set (IVIFS) [3] in 1989. An interval-valued Intuitionistic fuzzy set A over E is defined as

$$A = \{\langle x, ([\mu_a^L(x), \mu_a^U(x)], [\nu_a^L(x), \nu_a^U(x)])\rangle | x \in E\} \quad (2)$$

Where $0 \leq \mu_a^L \leq \mu_a^U \leq 1$, $0 \leq \nu_a^L \leq \nu_a^U \leq 1$ and $0 \leq (\mu_a^U) + (\nu_a^U) \leq 1$ for all $x \in E$.

A pair $A = ([\mu_a^L, \mu_a^U], [\nu_a^L, \nu_a^U])$ is called interval-valued Intuitionistic fuzzy number (IVPFN) with $[\mu_a^L, \mu_a^U], [\nu_a^L, \nu_a^U] \in [0,1]$ and $0 \leq \mu_a^U + \nu_a^U \leq 1$, where $[\mu_a^L, \mu_a^U], [\nu_a^L, \nu_a^U]: X \to [0,1]$ are the degree of membership and non-membership of the element $x \in E$ respectively with a condition that for $x \in E$, $0 \leq (\mu_a^U) + (\nu_a^U) \leq 1$. The degree of hesitant membership is defined as

$$\pi_a(x) = [\pi_a^L(x), \pi_a^U(x)] = [(1 - \mu_a^U(x) - \nu_a^U(x)), (1 - \mu_a^L(x) - \nu_a^L(x))]$$

Some of the basic operations on interval-valued Intuitionistic fuzzy number are as follows [12].
Let $A_1 = ([\mu_1^L, \mu_1^U], [\nu_1^L, \nu_1^U])$ and $A_2 = ([\mu_2^L, \mu_2^U], [\nu_2^L, \nu_2^U])$ be two IVPFNs.

$$A_1 \oplus A_2 = \left(\begin{array}{c} \left[\begin{array}{c} \sqrt{\mu_1^L(x) + \mu_2^L(x) - \mu_1^L(x) \cdot \mu_2^L(x)}, \\ \sqrt{\mu_1^U(x) + \mu_2^U(x) - \mu_1^U(x) \cdot \mu_2^U(x)} \end{array} \right], \\ [(\nu_1^L(x) \cdot \nu_2^L(x)), (\nu_1^L(x) \cdot \nu_2^L(x))] \end{array} \right) \quad (3)$$

$$A_1 \otimes A_2 = \left(\begin{array}{c} [\mu_1^L(x) \cdot \mu_2^L(x), \mu_1^L(x) \cdot \mu_2^L(x)], \\ \left[\begin{array}{c} \sqrt{\nu_1^L(x) + \nu_2^L(x) - \nu_1^L(x) \cdot \nu_2^L(x)}, \\ \sqrt{\nu_1^U(x) + \nu_2^U(x) - \nu_1^U(x) \cdot \nu_2^U(x)}, \end{array} \right] \end{array} \right) \quad (4)$$

Xu introduced the following aggregation operators, defined on IVIFN in [24], [25] using operations of addition and multiplication. These operations were called interval-valued Intuitionistic fuzzy weighted average (IVIFWA) operator and interval-valued Intuitionistic fuzzy ordered weighted average (IVIFOWA) operator, respectively These aggregation operators are useful in aggregating the opinions of multiple experts.

Let $\{A_i\}$ be a collection of 'n' IVIFNs where $A_i = \langle [a_i, b_i], [c_i, d_i] \rangle$ $(i = 1,2, \dots \dots, n)$ is the i^{th} IVIFN. The IVIFWA operator with respect to a weighting vector 'w' is a map $IVIFWA: \Omega^n \to \Omega$ defined by

$$IVIFWA(A_1, A_2 \dots, A_n) = \oplus_{i=1}^n w_i A_i$$
$$= \left([(1 - \prod_{i=1}^n (1 - a_i)^{w_i}), (1 - \prod_{i=1}^n (1 - b_i)^{w_i})], [\prod_{i=1}^n {c_i}^{w_i}, \prod_{i=1}^n {d_i}^{w_i}] \right) \quad (5)$$

where $= (w_1, w_2, \dots \dots, w_n)^T$ $w_i \in [0,1]$, $\sum_{i=1}^n w_i = 1$.
Let $\{A_i\}$ be a collection of 'n' IVIFNs where $A_i = \langle [a_i, b_i], [c_i, d_i] \rangle$ $(i = 1,2, \dots \dots, n)$ is the i^{th} IVIFN. The IVIFWA operator with respect to a weighting vector 'w' is a map $IVIFOWA: \Omega^n \to \Omega$ defined by

$$IVIFOWA(A_1, A_2 \dots, A_n) = \oplus_{i=1}^n w_i A_{\sigma(i)}$$

$$= \left(\begin{array}{c} \left[\left(1 - \prod_{i=1}^n \left(1 - a_{\sigma(i)}\right)^{w_i}\right), \left(1 - \prod_{i=1}^n \left(1 - b_{\sigma(i)}\right)^{w_i}\right) \right], \\ \left[\prod_{i=1}^n {c_{\sigma(i)}}^{w_i}, \prod_{i=1}^n {d_{\sigma(i)}}^{w_i} \right] \end{array} \right) \quad (6)$$

where $(\sigma(1), \sigma(2), \dots \dots, \sigma(n))$ is a permutation of $(1, 2, \dots \dots, n)$ such that $A_{\sigma(i-1)} \geq A_{\sigma(i)}, i = 1, 2, \dots \dots, n$ and $\boldsymbol{w} = (w_1, w_2, \dots \dots, w_n)^T, w_i \in [0,1]$, $\sum_{i=1}^n w_i = 1$.

3. TOPSIS method based on IVIFS [10]

Technique for order of preference by similarity to ideal solution (TOPSIS) method is one of the effective multi-criteria group decision making methods. This method was initially proposed by Yoon (1981) and then extended by Boran (2009) to solve MCGDM problems. In the TOPSIS method based on IVIFS both criteria and alternatives are expressed in terms of IVIFS. The choice of each decision maker is represented with a decision matrix $R^{(k)} = \left(r_{ij}^{(k)}\right)_{m \times n}$ in terms of IVIFS. The decision matrices $R^{(k)}$ of different decision makers is aggregated using IVIFOWA operator.

The IVIFS Positive Ideal Solution (a^+) and Negative Ideal Solution (a^-) are calculated using equations (7) and (8)

$$a^+ = ([\mu^L_{a^+}(c), \mu^U_{a^+}(c)], [\nu^L_{a^+}(c), \nu^U_{a^+}(c)]) \quad (7)$$

$$a^- = ([\mu^L_{a^-}(c), \mu^U_{a^-}(c)], [\nu^L_{a^-}(c), \nu^U_{a^-}(c)]) \quad (8)$$

Let C_1 and C_2 be the set of benefit and cost criteria respectively. Then

$$\mu^L_{a^+}(c) = \left(\left(\max_i \mu^L_{a_i}(x) \middle| c \in C_1\right), \left(\min_i \mu^L_{a_i}(x) \middle| c \in C_2\right)\right)$$

$$\mu^L_{a^-}(c) = \left(\left(\max_i \mu^L_{a_i}(x) \middle| c \in C_1\right), \left(\min_i \mu^L_{a_i}(x) \middle| c \in C_2\right)\right)$$

$$\mu^U_{a^+}(c) = \left(\left(\max_i \mu^U_{a_i}(x) \middle| c \in C_1\right), \left(\min_i \mu^U_{a_i}(x) \middle| c \in C_2\right)\right)$$

$$\mu^U_{a^-}(c) = \left(\left(\max_i \mu^U_{a_i}(x) \middle| c \in C_1\right), \left(\min_i \mu^U_{a_i}(x) \middle| c \in C_2\right)\right)$$

$$\nu^L_{a^+}(c) = \left(\left(\max_i \nu^L_{a_i}(x) \middle| c \in C_1\right), \left(\min_i \nu^L_{a_i}(x) \middle| c \in C_2\right)\right)$$

$$\nu^L_{a^-}(c) = \left(\left(\max_i \nu^L_{a_i}(x) \middle| c \in C_1\right), \left(\min_i \nu^L_{a_i}(x) \middle| c \in C_2\right)\right)$$

$$\nu^U_{a^+}(c) = \left(\left(\max_i \nu^U_{a_i}(x) \middle| c \in C_1\right), \left(\min_i \nu^U_{a_i}(x) \middle| c \in C_2\right)\right)$$

$$\nu^U_{a^-}(c) = \left(\left(\max_i \nu^U_{a_i}(x) \middle| c \in C_1\right), \left(\min_i \nu^U_{a_i}(x) \middle| c \in C_2\right)\right)$$

The separation measures of each alternative a_i from IVIFS-PIS and IVIFS-NIS are calculated using the normalised Euclidean distance between the IVIFS are calculated using equations (9) and (10)

$$S_{i^+} = \sqrt{\frac{1}{2}\sum_{j=1}^{n}\begin{pmatrix} \left|\mu^L_{a_i}(c_j) - \mu^L_{a^+}(c_j)\right|^2 + \left|\mu^U_{a_i}(c_j) - \mu^U_{a^+}(c_j)\right|^2 \\ +\left|\nu^L_{a_i}(c_j) - \nu^L_{a^+}(c_j)\right|^2 + \left|\nu^U_{a_i}(c_j) - \nu^U_{a^+}(c_j)\right|^2 \end{pmatrix}} \quad (9)$$

$$S_{i^-} = \sqrt{\frac{1}{2}\sum_{j=1}^{n}\begin{pmatrix} \left|\mu^L_{a_i}(c_j) - \mu^L_{a^-}(c_j)\right|^2 + \left|\mu^U_{a_i}(c_j) - \mu^U_{a^-}(c_j)\right|^2 \\ +\left|\nu^L_{a_i}(c_j) - \nu^L_{a^-}(c_j)\right|^2 + \left|\nu^U_{a_i}(c_j) - \nu^U_{a^-}(c_j)\right|^2 \end{pmatrix}} \quad (10)$$

The relative closeness coefficient of the alternative (a_i) to the IVIFS-PIS (a^+) is calculated using equation (11).

$$CC_{i^+} = \frac{S_{i^-}}{S_{i^+} + S_{i^-}} \text{ with } 0 \leq CC_{i^+} \leq 1, i = 1, 2, \dots\dots, n \quad (11)$$

4. Interval Valued Intuitionistic Fuzzy Relational Maps Model

Fuzzy Relational Map (FRM) is an extension of Fuzzy Cognitive Map (FCM) that is constructed between two disjoint sets. FRMs are capable of modelling complex systems based on experts' opinion [21]. An FRM is a dynamical structure and captures the causal interactions between the concepts from two disjoint sets and the causal links between the concepts. This dynamical system of causal influence is usually represented by the adjacency matrix between the two disjoint sets of concepts. Both the concepts and edge weights of FRM are represented by Interval valued intuitionistic fuzzy sets.

Let $(d_1, d_2, \dots\dots, d_n)$ and $(r_1, r_2, \dots\dots, r_m)$ denote the nodes of the concepts in domain space and range space of an FRM respectively. In FRMs, an instantaneous state value d_i^{k+1}, where k denotes the index of iteration is calculated as follows.

$$d_i^{k+1} = \boldsymbol{f}\left(d_i^k + \sum_{j=1}^{n} r_j^k \cdot e_{ji}\right) \quad (12)$$

where $r_j^k = \boldsymbol{f}\left(\sum_{i=1}^{m} d_i^k \cdot {e_{ji}}^T\right)$ where $d_i^k, i = 1, \dots\dots, m$ and $r_j^k, j = 1, \dots\dots, n$ are the concepts in domain space and range space respectively, e_{ji} is the strength of the influence of the concept d_i^k on r_j^k and f is a non-linear activation function such as sigmoid or hyperbolic type of function.

The membership value of the concepts or edge strength in conventional FRM is represented by real numbers. The single values may be insufficient to quantify the qualitative data and take in the uncertainty. In such situations, interval valued Intuitionistic fuzzy sets are more appropriate than single values of membership as they take into account the membership, non-membership and hesitancy degrees of elements. Applying the addition and multiplication operators for IVIFS from (3) and (4), the inference in conventional FRM defined by (10) can be reformulated as follows:

$$d_i^{k+1} = \{[\mu_P^L(d), \mu_P^U(d)], [\nu_P^L(x), \nu_P^U(d)]\}_{\boldsymbol{i}}^{\boldsymbol{k+1}}$$

$$= \boldsymbol{f}\left(\{[\mu_P^L(d), \mu_P^U(d)], [\nu_P^L(d), \nu_P^U(d)]\}_i^k \oplus \begin{pmatrix} \oplus_{j=1}^n \{[\mu_P^L(r), \mu_P^U(r)], [\nu_P^L(r), \nu_P^U(r)]\}_j^k \\ \otimes\{[\mu_P^L(w), \mu_P^U(w)], [\nu_P^L(w), \nu_P^U(w)]\}_{ji} \end{pmatrix}\right) \quad (13)$$

Where $\{[\mu_P^L(r), \mu_P^U(r)], [\nu_P^L(r), \nu_P^U(r)]\}_j^k = \boldsymbol{f}\left(\begin{pmatrix} \oplus_{j=1}^n \{[\mu_P^L(r), \mu_P^U(r)], [\nu_P^L(r), \nu_P^U(r)]\}_j^k \\ \otimes\{[\mu_P^L(w), \mu_P^U(w)], [\nu_P^L(w), \nu_P^U(w)]\}_{ji}^T \end{pmatrix}\right)$

5. IVIFRM-TOPSIS – A fuzzy hybrid approach

In order to explore the influence of mathematics related beliefs on mathematical competencies Multi Criteria Group Decision Making (MCGDM) approach is adopted in this study. Integrating two efficient method to find the influence of beliefs on improving mathematical competencies and rank the most influential beliefs give a clear assessment of the study. The IVIFRM model is capable of representing the imprecise knowledge of experts involved in decision making and also it takes into account the interactions among the criteria. The TOPSIS method is a ranking method that is employed to study the order of influence based on the Euclidean distance measures. The integrated IVIFRM-TOPSIS method is used to model the causal relationship between mathematics related beliefs on mathematical competencies. The algorithm for construction and analysis of IVIFRM-TOPSIS is described in the following steps.

Step 1: The factors that constitute the nodes of the domain and range IVIFRM are chosen with the help of experts and they are given in Table-1.

Step 2: The relationship between nodes is obtained from decision makers (DM) on their domain knowledge. Using linguistic evaluations that describe the causal relations between concepts of graph-based Intuitionistic FRM model is constructed.

Step 3: Let $c_1, c_2, \dots\dots, c_n$ be 'n' elements of the domain space and let $a_1, a_2, \dots\dots, a_m$ be the 'm' elements of range space. Let 'K' be the number of DMs. From the information provided by the DMs, an IVIF decision matrix, $R^{(k)} = \left(r_{ij}^{(k)}\right)_{m\times n}$ for each kth DM is constructed where $r_{ij}^{(k)} \in [0,1]$.

Step 4: The weighted aggregated decision matrix $R \otimes W$ is calculated as follows.

Step 4a: The weight of the kth DM is calculated using equation (14) where $\sum_{k=1}^{K} \lambda_k = 1$

$$\lambda_k = \frac{\mu_k + \pi_k\left(\frac{\mu_k}{\mu_k+\nu_k}\right)}{\sum_{k=1}^{K} \mu_k + \pi_k\left(\frac{\mu_k}{\mu_k+\nu_k}\right)} \quad (14)$$

Step 4b: The aggregated decision matrix R is obtained from the decision matrices $R^{(k)}$ provided by each DM using the DM's weights obtained in Step 4a and IVIFOWA operator. The IVIFS addition and multiplication operators defined in (3) and (4) are used. The resulting weighted aggregated IVIF matrix is with values that are interval valued Intuitionistic fuzzy number (IVIFN).

Step 5: The IVIFN values obtained in the above step is taken as the edge strength of causal relation between the variables of domain space and range space of FRM model. These causal values of the edges constitute the adjacency matrix of the IVIFRM.

Step 6: Using the Initial state vector of the concepts from Table-5 and the edge strength from the relational matrix (Table-4), the IVIFRM were simulated using the formula given in equation (13) until the steady state is reached. Sigmoid functions were used as activation functions. The resultant steady-state vector values of the concepts are taken to be the weights $W^* = (w_1^*, w_2^*, \dots\dots, w_n^*)$.

Step 7: The global weights of the criteria $W' = (w_1', w_2', \dots\dots, w_n')$ are calculated by combining the local weights (W) and steady-state weights (W^*) of criteria using equation (15).

$$W' = W_j \oplus \left(W_j \otimes W_j^*\right) \quad (15)$$

Step 8: The weighted aggregated decision matrix $R' = R \otimes W' = \left(r_{ij}'\right)_{m\times n}$ is calculated.

Step 9: The IVIFS-PIS (a^+) and NIS (a^-) are obtained from the decision matrix using equation (7) and (8)
Step 10: The separation measures (S_{i^+} and S_{i^-}) are computed using the normalised Euclidean distance between IVIFs using equation (9) and (10)
Step 11: The relative closeness coefficient (CC_{i^+}) of the alternative (a_i) to the IVIFS-PIS (a^+) is calculated using equation (11)
Step 12: From the values of relative coefficient the alternatives are ranked and the most influential factor and the order of influence is assessed.

6. Description of the Problem

Mathematics education is considered to be the foundation of scientific and technological knowledge. Therefore, mathematics is considered not only as a core component of the curriculum but also as a critical filter to many educational and career opportunities [13]. Developing mathematical competencies during early years of education is highly advocated as it has a long-lasting effect on their learning in future [11]. Harris also quotes in [11] that there is a connection between being competent in early math and success in school. Development of early math skills could be the strongest predictor of later success in both reading and math [11]. Mathematical competencies play a major role in a child's growth and help them to have a sense of world around them. Possessing mathematical competencies is strongly related to increased levels of knowledge, understanding and intelligence. Hence it becomes pertinent to impart education in a manner that children develop mathematical competencies in the early years and that in turn would make them competent to face the world with greater confidence.

Mathematical competencies comprise skills such as mathematising, reasoning, devising strategies, representation, communication, using symbolic/formal/technical language and operations [22]. Every kind of intelligence has one or more mathematical competencies beneath. Having acquired different kinds mathematical competencies enable an individual to have an edge over others in any field. Niss states that the concept of mathematical competence focuses on the enactment of mathematics rather than on the subject matter [18]. Ross Turner claims that mathematical competencies can be thought of as a set of individual characteristics or qualities possessed to a greater or lesser extent by each person [22]. Several recent researchers have identified that the more one possesses and can activate these competencies, the better they will be able to make effective use of mathematical knowledge to solve contextualised problems. The educational use of mathematical competencies is that they can be used as means to design curriculum and implementation of teaching and learning practices.

The major difference between average students and high achievers in mathematics is their belief system. Teaching and learning process consists of an interaction between persons for the purpose of developing and sharing meanings. [4]. The specific means that shape this interaction is basically beliefs that creates a cognitive bonding and yield the expected learning. All knowledge is basically 'a belief' [8]. McLeod in [16] describes that the beliefs about mathematics, mathematics teaching, self, and about contextual factors relevant to mathematics learning are the influential factors in developing mathematical competencies. Almost all the students have a self-talk when they have to deal with mathematical problems. They make statements such as: mathematics is a mathematics is a difficult subject (beliefs about the subject), my mathematics teachers are very boring (about teachers), I feel happy to learn new concepts (about self), I am good at Algebra (self-confidence) and so on. The statements are nothing but, the expressions of the beliefs one holds strong in their mind [19], [20]. Several researches have confirmed the fact that there is a strong relation between personal beliefs and mathematics learning. As Fishbein (1987) points out: there is a world of stabilized beliefs which profoundly influence the reception and the practice of mathematical and scientific knowledge which are not just remnants of primitive reasoning, but productive components of every other type of reasoning [8]. Stumper in [21] notes that it is not possible to establish causality between specific beliefs and behaviour in dealing with problems, but confirms that a beliefs system works as an explanatory model.

In this paper, a study is carried out to identify the causal influence of beliefs on mathematical competence. A list of 12 belief clusters related to mathematics is taken from [14] and 8 mathematical competencies from [17] are considered to be the factors of domain and range spaces of FRM. The experts agreed unanimously to work with these factors [Table-1] and provided the relational map between them in terms of linguistic expressions.

Beliefs that influence mathematics learning	**Mathematical competencies**
B_1: Previous experience with Mathematics	M_1: Mathematical thinking
B_2: Feelings about School	M_2: Mathematical problem handling
B_3: Feelings about Mathematics	M_3: Mathematical modelling
B_4: Effort in Mathematics	M_4: Mathematical reasoning
B_5: Non-School influences on Motivation	M_5: Mathematical representation
B_6: Self-Confidence in Mathematics	M_6: Mathematical symbols and formula
B_7: Natural ability in Mathematics	M_7: Mathematical communication
B_8: Goal orientation and effort	M_8: Mathematical aids and tools
B_9: Study habits in Mathematics	
B_{10}: Mathematics content	
B_{11}: Assessment Practices	
B_{12}: Students' expectation of teachers	

Table 1: Factors of Domain and Range spaces of FRM

6.1. Analysis of the problem using IVIFRM-TOPSIS

The mathematics related beliefs $D = (B_1, B_2, \dots\dots, B_{12})$ and mathematical competencies $R = (M_1, M_2, \dots\dots, M_8)$ are taken to be the elements of domain and range spaces respectively. The beliefs and competencies are treated as alternatives and criteria of the decision matrix accordingly. The Decision Makers (DM_1, DM_2, DM_3) are teachers from three different levels of education participated in this decision-making process using FRM model. Each of them was asked to construct an FRM using linguistic expressions independently (Table-3).

Linguistic terms to assess the alternatives		Linguistic terms to assess the Criteria		Linguistic terms to assess the importance of DMs	
Linguistic term	**IVIFS**	**Linguistic term**	**IVIFS**	**Linguistic term**	**IFS**
Very strong (VS)	([0.75, 0.85], [0.00, 0.10])	Very important (VI)	([0.90, 0.90], [0.10, 0.10])	Very important (VI)	(0.90, 0.10)
Strong (S)	([0.51, 0.70], [0.15, 0.25])	Important (I)	([0.40, 0.76], [0.00, 0.21])	Important (I)	(0.75, 0.20)
Medium (M)	([0.35, 0.50], [0.30, 0.45])	Medium (M)	([0.15, 0.51], [0.25, 0.46])	Medium (M)	(0.50, 0.45)
Weak (W)	([0.16, 0.30], [0.46, 0.60])	Unimportant (U)	([0.00, 0.36], [0.40, 0.61])	Unimportant (U)	(0.35, 0.60)
Very weak (VW)	([0.00, 0.15], [0.65, 0.80])	Very unimportant (VU)	([0.10, 0.10], [0.90, 0.90])	Very unimportant (VU)	(0.10, 0.90)

Table 2: Linguistic terms

$R^{(1)}$								$R^{(2)}$								$R^{(3)}$						
M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_1	M_2	M_3	M_4	M_5	M_6	M_7
M	M	W	W	S	VS	S	S	S	S	S	M	S	VS	S	S	S	M	S	W	S	VS	M
M	M	M	M	M	S	M	S	S	S	S	M	S	S	M	S	S	M	S	M	S	S	M
S	VS	VS	VS	S	VS	S	S	S	VS	VS	S	VS	VS	S	VS	S	VS	S	S	S	S	S
S	S	S	S	S	VS	S	S	S	VS	VS	S	S	S	VS	VS	S	S	S	S	S	VS	VS
M	M	W	W	M	M	M	M	S	S	S	M	S	S	S	S	S	M	S	W	S	M	S
M	VS	VS	VS	VS	VS	VS	VS	VS	VS	VS	S	VS	VS	S	VS	VS	VS	VS	VS	VS	VS	S
M	VS	VS	VS	S	VS	S	S	VS	VS	VS	S	VS	S	VS	VS	S	VS	VS	VS	S	VS	VS

M	M	W	W	S	VS	S	S	VS	VS	VS	S	VS	VS	M	VS	VS	M	S	W	VS	VS	M
S	S	S	S	S	VS	S	S	S	S	VS	M	S	S	S	VS	S	S	S	S	S	VS	S
M	M	W	W	M	M	M	M	S	VS	VS	S	S	S	S	VS	S	W	VS	W	S	M	S
S	S	S	S	S	VS	S	S	S	S	S	S	S	S	S	S	S	S	S	S	S	VS	S
S	S	S	S	S	VS	S	S	M	M	VS	S	S	VS	VS	VS	M	S	VS	S	S	VS	VS

Table 3: Relational Matrix Provided by the DMs

The relational matrices $R^{(1)}$, $R^{(2)}$, and $R^{(3)}$ are provided by the three DMs respectively. These relational matrices of FRM model, constructed with the linguistic terms provided by the experts, is quantified with values from IVIFS. The importance of each DM is expressed in linguistic terms as follows: DM_1: strong, DM_2: strong and DM_3: medium. The weight of each DM is calculated and the weight information of the DMs is given by $w = (0.38, 0.38, 0.25)$. These FRMs provided by the individual decision makers are aggregated by applying the IVIFOWA operator method and it is given in Table-4.

	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8
B_1	([0.48, 0.66], [0.19, 0.31])	([0.42 0.59], [0.25, 0.39])	([0.00, 0.36], [0.40, 0.61])	([0.24, 0.39], [0.41, 0.56])	([0.51, 0.70], [0.15, 0.25])	([0.75 0.85], [0.00, 0.10])	([0.48, 0.66], [0.19, 0.31])	([0.48, 0.66], [0.19, 0.31])
B_2	([0.48, 0.66], [0.19, 0.31])	([0.42 0.59], [0.25, 0.39])	([0.48, 0.66], [0.19, 0.31])	([0.35, 0.50], [0.30, 0.45])	([0.48, 0.66], [0.19, 0.31])	([0.51, 0.70], [0.15, 0.25])	([0.35 0.50], [0.30, 0.45])	([0.51, 0.70], [0.15, 0.25])
B_3	([0.51, 0.70], [0.15, 0.25])	([0.75 0.85], [0.00, 0.10])	([0.71 0.82], [0.00, 0.15])	([0.62 0.85], [0.00, 0.22])	([0.62 0.77], [0.00, 0.20])	([0.71 0.82], [0.00, 0.14])	([0.51, 0.70], [0.15, 0.25])	([0.62 0.77], [0.00, 0.20])
B_4	([0.51, 0.70], [0.15, 0.25])	([0.62 0.77], [0.00, 0.20])	([0.62 0.77], [0.00, 0.22])	([0.51 0.82], [0.15, 0.25])	([0.51, 0.77], [0.15, 0.25])	([0.71 0.82], [0.00, 0.14])	([0.71 0.82], [0.00, 0.14])	([0.62 0.77], [0.00, 0.20])
B_5	([0.48, 0.66], [0.19, 0.31])	([0.42 0.59], [0.25, 0.39])	([0.44, 0.63], [0.23, 0.34])	([0.24, 0.39], [0.41, 0.56])	([0.48, 0.74], [0.19, 0.31])	([0.42 0.69], [0.25, 0.39])	([0.48, 0.66], [0.19, 0.31])	([0.52 0.66], [0.00, 0.34])
B_6	([0.48, 0.66], [0.19, 0.31])	([0.75 0.85], [0.00, 0.10])	([0.75 0.85], [0.00, 0.10])	([0.71, 0.82], [0.00, 0.14])	([0.75 0.85], [0.00, 0.10])	([0.71 0.82], [0.00, 0.14])	([0.71 0.82], [0.00, 0.14])	([0.75 0.85], [0.00, 0.10])
B_7	([0.69, 0.80], [0.00, 0.25])	([0.75 0.85], [0.00, 0.10])	([0.75 0.85], [0.00, 0.10])	([0.71, 0.82], [0.00, 0.14])	([0.62 0.77], [0.00, 0.20])	([0.71 0.82], [0.00, 0.14])	([0.71 0.82], [0.00, 0.14])	([0.62 0.77], [0.00, 0.20])
B_8	([0.60, 0.74], [0.00, 0.17])	([0.55, 0.69], [0.00, 0.31])	([0.57, 0.72], [0.00, 0.27])	([0.32, 0.49], [0.34, 0.48])	([0.71, 0.82], [0.00, 0.14])	([0.75 0.85], [0.00, 0.10])	([0.42, 0.59], [0.25, 0.39])	([0.62 0.77], [0.00, 0.20])
B_9	([0.51, 0.70], [0.15, 0.25])	([0.51, 0.70], [0.15, 0.25])	([0.62 0.77], [0.00, 0.20])	([0.48, 0.66], [0.19, 0.31])	([0.51, 0.70], [0.15, 0.25])	([0.71 0.82], [0.00, 0.14])	([0.51, 0.70], [0.15, 0.25])	([0.62 0.77], [0.00, 0.20])
B_{10}	([0.48, 0.66], [0.19,	([0.52, 0.66, [0.00,	([0.67 0.78], [0.00,	([0.32, 0.49], [0.34,	([0.48, 0.66], [0.19,	([0.42 0.59], [0.25,	([0.48, 0.66], [0.19,	([0.55, 0.69], [0.00,

	0.31])	0.34])	0.19])	0.48])	0.31])	0.39])	0.31])	0.31])
B_{11}	([0.51, 0.70], [0.15, 0.25])	([0.51, 0.70], [0.15, 0.25])	([0.51, 0.70], [0.15, 0.25])	([0.51, 0.70], [0.15, 0.25])	([0.51, 0.70], [0.15, 0.25])	([0.71 0.82], [0.00, 0.14])	([0.51, 0.70], [0.15, 0.25])	([0.51, 0.70], [0.15, 0.25])
B_{12}	([0.39, 0.59], [0.25, 0.59])	([0.47, 0.66], [0.19, 0.31])	([0.71, 0.82], [0.00, 0.14])	([0.51, 0.70], [0.15, 0.25])	([0.51, 0.70], [0.15, 0.25])	([0.75 0.85], [0.00, 0.10])	([0.71 0.82], [0.00, 0.14])	([0.62 0.77], [0.00, 0.20])

Table 4: Aggregated matrix of IVIFRM

M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8
([0.50, 0.80], [0.05, 0.10])	([0.55, 0.75], [0.10, 0.20])	([0.30, 0.50], [0.15, 0.30])	([0.40, 0.75], [0.05, 0.25])	([0.45, 0.60], [0.15, 0.25])	([0.45, 0.65], [0.05, 0.15])	([0.40, 0.55], [0.00, 0.15])	([0.60, 0.70], [0.10, 0.20])

Table 5: Input criteria weights

M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8
([1.00, 1.00], [0.05, 0.09])	([1.00, 1.00], [0.09, 0.13])	([1.00, 1.00], [0.05, 0.09])	([1.00, 1.00], [0.11, 0.15])	([1.00, 1.00], [0.05, 0.14])	([0.75 0.85], [0.00, 0.10])	([1.00, 1.00], [0.00, 0.11])	([1.00, 1.00], [0.09, 0.13])

Table 6: Steady-state weights

The local weights of the criteria $W = (w_1, w_2, \dots \dots, w_8)$ (Table-8) were calculated from the DMs opinions in (Table-7) represented by IVIFSs from (Table-2). The steady-state weights $W^* = (w_1^*, w_2^*, \dots \dots, w_n^*)$ (Table-6) are calculated. From the local weights and steady-state weights the global weights $W' = (w_1', w_2', \dots \dots, w_n')$ (Table-9) are computed. Using the global weights, the weighted aggregated IVIFS decision matrix R' (Table-10) are constructed. The IVIFS-PIS (a^+) and IVIFS-NIS (a^-) (Table-11) are obtained. The separation measures (S_{i^+}) and (S_{i^-}) are computed from this the relative closeness coefficient (CC_{i^+}) and the corresponding ranking is obtained (Table-12).

	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8
DM_1	VI	I	M	VI	I	M	I	I
DM_2	VI	VI	M	VI	M	M	I	I
DM_3	VI	I	I	M	I	I	I	I

Table 7: The importance of criteria

M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8
([0.80, 0.90], [0.00, 0.10])	([0.61, 0.83], [0.00, 0.15])	([0.22, 0.59], [0.20, 0.37])	([0.72, 0.85], [0.00, 0.14])	([0.32, 0.69], [0.14, 0.27])	([0.22, 0.59], [0.20, 0.37])	([0.40, 0.76], [0.10, 0.20])	([0.40, 0.76], [0.10, 0.20])

Table 8: Aggregated local weights

M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8
([0.96, 0.99], [0.00, 0.02])	([0.85, 0.97], [0.00, 0.04])	([0.39, 0.83], [0.06, 0.17])	([0.92, 0.98], [0.00, 0.04])	([0.54, 0.90], [0.03, 0.10])	([0.39, 0.83], [0.05, 0.16])	([0.64, 0.94], [0.01, 0.06])	([0.64, 0.94], [0.02, 0.06])

Table 9: Global weights

	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8
B_1	([0.46, 0.66], [0.19, 0.32])	([0.35, 0.57], [0.25, 0.41])	([0.18, 0.53], [0.27, 0.46])	([0.22, 0.38], [0.41, 0.57])	([0.28, 0.64], [0.18, 0.32])	([0.30, 0.71], [0.05, 0.25])	([0.31, 0.63], [0.20, 0.35])	([0.31, 0.63], [0.21, 0.35])
B_2	([0.46, 0.66], [0.19, 0.32])	([0.35, 0.57], [0.25, 0.41])	([0.19, 0.55], [0.24, 0.43])	([0.32, 0.49], [0.30, 0.47])	([0.26, 0.60], [0.22, 0.38])	([0.20, 0.59], [0.19, 0.37])	([0.23, 0.48], [0.30, 0.48])	([0.33, 0.66], [0.16, 0.29])
B_3	([0.49, 0.70], [0.15, 0.26])	([0.64, 0.83], [0.00, 0.13])	([0.28, 0.69], [0.06, 0.30])	([0.57, 0.83], [0.00, 0.25])	([0.33, 0.70], [0.03, 0.28])	([0.28, 0.69], [0.05, 0.28])	([0.33, 0.66], [0.16, 0.29])	([0.40, 0.73], [0.02, 0.24])
B_4	([0.49, 0.70], [0.15, 0.26])	([0.53, 0.75], [0.00, 0.23])	([0.25, 0.64], [0.06, 0.35])	([0.47, 0.81], [0.15, 0.28])	([0.28, 0.70], [0.18, 0.32])	([0.28, 0.71], [0.05, 0.28])	([0.46, 0.78], [0.01, 0.19])	([0.40, 0.73], [0.02, 0.24])
B_5	([0.46, 0.66], [0.19, 0.32])	([0.35, 0.57], [0.25, 0.41])	([0.18, 0.53], [0.27, 0.46])	([0.22, 0.38], [0.41, 0.57])	([0.26, 0.67], [0.22, 0.38])	([0.17, 0.57], [0.28, 0.49])	([0.31, 0.63], [0.20, 0.35])	([0.33, 0.62], [0.02, 0.38])
B_6	([0.46, 0.66], [0.19, 0.32])	([0.64, 0.83], [0.00, 0.13])	([0.30, 0.71], [0.06, 0.25])	([0.65, 0.81], [0.00, 0.17])	([0.40, 0.77], [0.03, 0.19])	([0.28, 0.69], [0.05, 0.28])	([0.46, 0.78], [0.01, 0.19])	([0.48, 0.81], [0.02, 0.15])
B_7	([0.57, 0.73], [0.00, 0.26])	([0.64, 0.83], [0.00, 0.13])	([0.30, 0.71], [0.06, 0.25])	([0.65, 0.81], [0.00, 0.17])	([0.33, 0.70], [0.03, 0.28])	([0.28, 0.69], [0.05, 0.28])	([0.46, 0.78], [0.01, 0.19])	([0.40, 0.73], [0.02, 0.24])
B_8	([0.66, 0.79], [0.00, 0.19])	([0.46, 0.67], [0.00, 0.33])	([0.22, 0.60], [0.06, 0.40])	([0.29, 0.48], [0.34, 0.50])	([0.38, 0.75], [0.03, 0.23])	([0.30, 0.71], [0.05, 0.25])	([0.27, 0.56], [0.26, 0.42])	([0.40, 0.73], [0.02, 0.24])
B_9	([0.49, 0.70], [0.15, 0.26])	([0.43, 0.68], [0.15, 0.28])	([0.25, 0.64], [0.06, 0.33])	([0.44, 0.65], [0.19, 0.33])	([0.28, 0.64], [0.18, 0.32])	([0.28, 0.69], [0.05, 0.28])	([0.33, 0.66], [0.16, 0.29])	([0.40, 0.73], [0.02, 0.24])
B_{10}	([0.46, 0.66], [0.19, 0.32])	([0.44, 0.64], [0.00, 0.37])	([0.26, 0.65], [0.06, 0.33])	([0.29, 0.48], [0.34, 0.50])	([0.26, 0.60], [0.22, 0.38])	([0.17, 0.49], [0.28, 0.49])	([0.31, 0.63], [0.20, 0.35])	([0.35, 0.65], [0.02, 0.35])
B_{11}	([0.49, 0.70], [0.15, 0.26])	([0.43, 0.68], [0.15, 0.28])	([0.20, 0.59], [0.20, 0.38])	([0.47, 0.69], [0.15, 0.28])	([0.28, 0.64], [0.18, 0.32])	([0.28, 0.69], [0.05, 0.28])	([0.33, 0.66], [0.16, 0.29])	([0.33, 0.66], [0.16, 0.29])
B_{12}	([0.38, 0.59], [0.25, 0.40])	([0.40, 0.64], [0.19, 0.34])	([0.28, 0.69], [0.06, 0.29])	([0.47, 0.69], [0.15, 0.28])	([0.28, 0.64], [0.18, 0.32])	([0.30, 0.71], [0.05, 0.25])	([0.46, 0.78], [0.01, 0.19])	([0.40, 0.73], [0.02, 0.24])

Table 10: Aggregated weighted IVIFS decision matrix

	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8
a^+	([0.66, 0.79],	([0.64, 0.83],	([0.30, 0.71],	([0.65, 0.83],	([0.40, 0.77],	([0.30, 0.71],	([0.46, 0.78],	([0.48, 0.81],

	[0.00, 0.19])	[0.00, 0.13])	[0.06, 0.25])	[0.00, 0.17])	[0.03, 0.19])	[0.05, 0.25])	[0.01, 0.19])	[0.02, 0.15])
a^-	([0.38, 0.59], [0.25, 0.40])	([0.35, 0.57], [0.25, 0.41])	([0.18, 0.53], [0.27, 0.46])	([0.22, 0.38], [0.41, 0.57])	([0.26, 0.60], [0.22, 0.38])	([0.17, 0.49], [0.28, 0.49])	([0.23, 0.48], [0.30, 0.48])	([0.31, 0.62], [0.21, 0.38])

Table 11: IVIFS-PIS and IVIFS-PIS

Beliefs	S_{i^+}	S_{i^-}	CC_{i^+}	Rank
B_1	0.9117	0.3643	0.2855	10
B_2	0.8684	0.2422	0.2181	11
B_3	0.3596	0.8490	0.7025	3
B_4	0.4088	0.7925	0.6597	4
B_5	0.9314	0.2506	0.2120	12
B_6	0.3115	0.9694	0.7568	2
B_7	0.2610	0.9645	0.7870	1
B_8	0.6804	0.6177	0.4758	8
B_9	0.5289	0.6091	0.5352	6
B_{10}	0.7847	0.3893	0.3316	9
B_{11}	0.5526	0.5953	0.5186	7
B_{12}	0.5789	0.7140	0.5522	5

Table 12: Separation measures, relative closeness coefficients and rank

6.2. Results and Discussion

From the steady-state values of IVIFRM, the beliefs are very important. From the TOPSIS method the influence mathematical beliefs on mathematical competencies. are ranked as follows: $B_7 \succ B_6 \succ B_3 \succ B_4 \succ B_{12} \succ B_9 \succ B_{11} \succ B_8 \succ B_{10} \succ B_1 \succ B_2 \succ B_5$. From rankings it can be inferred that B_7 (Natural ability in Mathematics) is the most influential, B_6 (Self-confidence in Mathematics) is the next most influential and B_5 (Non-school influences on motivation) is the least influential belief clusters.

7. Conclusion

The intuitionistic fuzzy set, an extension of ordinary fuzzy sets, is more sophisticated as they include the non-membership values besides the membership values. The presence of non-membership value enables the measurement of hesitancy of the decision maker and thus the information provided by the expert is complete to a certain to an extent. Further the use of interval-valued intuitionistic fuzzy sets allows the expert to choose the membership and non-membership values of an element belonging to a set as a continuous range between two points belonging to the unit interval [0,1]. The adaptation of interval-valued intuitionistic fuzzy sets makes room for more information which may be missed out otherwise. The rankings obtained in this approach is rather reliable than the conventional TOPSIS as the interaction among the factors of criteria is taken in by integrating the TOPSIS method with IVIFRM model.

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