Semi-Generalized Closed Set in the Closure Spaces

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Abstract. In this paper, we introduce the concept of semi generalized closed set (= $\mathcal{SG}-$ closed) sets, and semi generalized open (=$\mathcal{SG}-$ open) sets in the closure space .Furthermore, we study some of their properties. And investigate the relation between them .

Keywords: Closure spaces, $\mathcal{G}$ - closed sets, $\mathcal{G}$- open sets, $\mathcal{SG}-$ closed sets , $\mathcal{SG}$ – open sets.

1. introduction

The purpose of this paper is to introduce and study the concept of semi generalized closed sets in closure spaces . Closure spaces were introduced by E.Čech [1] in 1966 and then studied by many mathematicians, see e.g. [2], [3], [4] and [8]. Closure spaces are sets endowed with a grounded, extensive and monotone closure operator. Khampakdee [6],defined an study g-closed sets (2008), and then in[5], (2009) introduced the notion of semi open sets in closure spaces and showed their fundamental properties. The semi-open sets are used to define semi-open maps.

As a continuation of this work, we introduce and study in Section 3, a new class of sets namely $\mathcal{SG}-$ closed sets which is properly placed in between the class of semi-closed sets and the class of g-closed sets . In Section 4, the class of $\mathcal{SG}-$ open sets introduced and investigated. All definitions of the several concepts used throughout the sequel are explicitly stated in the following section.

2. preliminaries

Definition 2.1. [1] Let $\mathcal{K}$:P(M) →P(M) be a function identified on a power set P(M) of the set M, $\mathcal{K}$ will be the closure operator over M and the couple (M, $\mathcal{K}$) is called the closure space, if the following axioms are satisfied :

(1) $(\emptyset)=\emptyset$,

(2) $\mathcal{A} \subseteq \mathcal{K}(\mathcal{A})$ for each $\mathcal{A} \subseteq$ M,

(3) $\mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{K}(\mathcal{A}) \subseteq \mathcal{K}$ (B) for each $\mathcal{A}, \mathcal{B} \subseteq$ M.

Definition 2.2. [1] let $\mathcal{K}$ be the closure operator on $\mathcal{M}$ named idempotent when A⊆ M then $\mathcal{K}\,\mathcal{K}$ (A) $=\mathcal{K}$ (A), and we say $\mathcal{K}$ is called additive if $\mathcal{A}, \mathcal{B}$ are subset of $\mathcal{M}$ then $\mathcal{K}(\mathcal{A}) \cup \mathcal{K}(\mathcal{B}) = \mathcal{K}(\mathcal{A} \cup \mathcal{B})$.

Definition 2.3. A function int :P(M) →P(M) identified on the power set P(M) from the set M, the interior operator on M is called an interior operator that satisfies:

(1) int(M) = M,

(2) int(A) ⊆ A, for each A,B ⊆ M.

(3) A ⊆ B ⇒ int(A)⊆ int(B), for each A, B ⊆ M.

Definition 2.4. [8] Given the closure space ($\mathcal{M}, \mathcal{K}$) we identified the operator

$i\mathcal{K}$: P($\mathcal{M}$) → P($\mathcal{M}$) to be $i\mathcal{K}(\mathcal{A}) = \mathcal{M} - \mathcal{K}(\mathcal{M} - \mathcal{A})$.

Similarly, given a set $\mathcal{M}$ with an interior operator i, we identified an operator

$\mathcal{K}i$: P($\mathcal{M}$) → P($\mathcal{M}$) by $\mathcal{K}i\ (\mathcal{A}) = \mathcal{M} - i(\mathcal{M} - \mathcal{A})$.

Definition 2.5. [5] A subset A of $\mathcal{M}$ is said to be closed over the closure space ($\mathcal{M}$, $\mathcal{K}$) when $\mathcal{K}$ (A) = A. In addition, subset B of $\mathcal{M}$ is named open when its complement ($\mathcal{M}$\B) is closed. The empty set and the entire space where both open and closed Simultaneously.

Definition 2.6. [5] The closure space (N, E) named is subspace of ($\mathcal{M}$, $\mathcal{K}$) if N ⊆ M and E(A) = C(A) ∩ N for each subset A ⊆ N.

Proposition 2.7. [5] Let (N, E) be a closure subspace of ($\mathcal{M}$, $\mathcal{K}$). If G is an open set in X , then G ⊓N is an open set in (N, E).

Definition 2.8. [6] The subset Q of closure space ($\mathcal{M}$, $\mathcal{K}$)which named is generalized closed set in the closure space (g- closed) if CQ⊆G whenever Q⊆G and G is open set over M. A subset A ⊑$\mathcal{M}$ is called a generalized –open (g-open) set if its complement is generalized closed set.

Definition 2.9. A subset A of a closure space ($\mathcal{M}$, $\mathcal{K}$) is said to be :

(1) a semi- open set [5] if there exists an open set G in ($\mathcal{M}$, $\mathcal{K}$) such that G ⊑ A ⊑ $\mathcal{K}$ G . A subset A ⊑X is called a semi-closed set if its complement is semi-open. And a semi-closed set if its complement is semi- closed set.

(2) a clopen set if A is open and closed set at the same time.

Lemma 2.10. Let A be a subset of a closure space ($\mathcal{M}$, $\mathcal{K}$). Then:

(1) the intersection of two g-open set is g-open set .

(2) if A is semi-closed set then A⊑ $i\mathcal{K}(\mathcal{K}$(A)).

(3) if A⊑B⊑$\mathcal{M}$ and A is g-open in B , B is g-open in $\mathcal{M}$ then A is g-open in $\mathcal{M}$.

Proposition 2.11. let (M, $\mathcal{C}$) be the closure space and ⊆ . Where a subset Q is open if and only if ⊆ $\mathcal{C}$ Anyplace is closed and ⊆ Q.

3. semi generalized closed set in the closure space

This section is dedicated to the introduction and discussion of the basic properties of the notion of a semi-generalized closed set.

Definition 3.1. [5] The subset Q of closure space ($\mathcal{M}$, $\mathcal{K}$) which named is semi-generalized closed set in the closure space ($sg - closed$) if $\mathcal{K}Q \subseteq G$ whenever $Q \subseteq G$ and G is $\mathcal{G}$- open set over M.

Example 3.2. Let $\mathcal{M}$ = {1, 2,3} then identified a closure operator $\mathcal{K}$ on M by

$\mathcal{K}\emptyset = \emptyset, \mathcal{K}\{2\} = \{2\}$,

$\mathcal{K}\{3\} = C\{2,3\} = \{2,3\}$,

$\mathcal{K}\{1\} = C\{2,1\} = C\{1,3\} = \mathcal{M}$,

$C\mathcal{M} = \mathcal{M}$.

Let $\mathcal{B}$={1,2}

$\mathcal{G}$-closed set on $\mathcal{M}$ is {1,2},{2},{2,3},$\mathcal{M}$, ∅

$\mathcal{G}$-open set on $\mathcal{M}$ is {1},{3},{1,3},$\mathcal{M}$, ∅

$\mathcal{B}$ is $\mathcal{SG}$-closed set .

Remark 3.3. Every $\mathcal{SG}$-closed set is $\mathcal{G}$-closed but not every $\mathcal{G}$-closed is $\mathcal{SG}$-closed as shown in the example bellow.

Example 3.4. If $\mathcal{M}$ ={1,2,3} then identified a closure operator $\mathcal{K}$ on M by

$\mathcal{K}\emptyset = \emptyset$,
$\mathcal{K}\{1\} = \mathcal{K}\{1,2\} = C\{1,3\} = \mathcal{M}$,
$\mathcal{K}\{2\} = \mathcal{K}\{3\} = C\{2,3\} = \{2,3\}$,
$C\mathcal{M} = \mathcal{M}$.
$\mathcal{G}$-closed set on $\mathcal{M}$ is $\{1,2\},\{1,3\},\{3\},\{2\},\{2,3\},\mathcal{M}, \emptyset$
$\mathcal{G}$-open set on $\mathcal{M}$ is $\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\mathcal{M}, \emptyset$
then
a subset B = {1,2} is $\mathcal{G}$-closed but not
$\mathcal{SG}-$ closed. Since $\mathcal{B}\sqsubseteq\{1,2\}$ which is $\mathcal{G}$-open set but $\mathcal{K}(\mathcal{B})=\mathcal{M}\not\sqsubseteq\{1,2\}$.
Remark 3.5. from the above definition and example we have :
(1) every $\mathcal{SG}$-closed is $\mathcal{G}$-closed , however the convers is not true as shown in
(2) the union of two $\mathcal{SG}$-closed sets need not to be $\mathcal{SG}$-closed.
Example 3.6 let $\mathcal{M}$={1,2,3,4} then identified a closure operator $\mathcal{K}$ on M by
$\mathcal{K}\emptyset = \emptyset$,
$\mathcal{K}\{2,3\} = \mathcal{K}\{2,4\} = C\{2,3,4\} = \{2,3,4\}$,
$\mathcal{K}\{3\} = \mathcal{K}\{4\} = C\{3,4\} = \{3,4\}$,
$\mathcal{K}\{2\} = \mathcal{K}\{2\}$
$C\mathcal{M} = \mathcal{M}.\mathcal{K}$ for the other set equal to $\mathcal{M}$
{2},{3} is two $\mathcal{SG}$-closed sets but $\{2\} \cup \{3\}=\{2,3\}$ is not $\mathcal{SG}$-closed
Since {1,2,3} is $\mathcal{G}$-open set and $\{2,3\}\sqsubseteq\{1,2,3\}$
But $\{2,3\}=\{2,3,4\} \not\sqsubseteq\{1,2,3\}$
Not that {2,3} is $\mathcal{G}$-closed set.
Proposition 3.7. Let $\mathcal{B}$ be a $\mathcal{SG}-$ closed set in $\mathcal{M}$ .Then (B)\B does not contain any non-empty $\mathcal{G}$-closed set.
Proof. Assume that V is a $\mathcal{G}$-closed subset of $(\mathcal{B})\backslash\mathcal{B}$.
This implies that V ⊆ $(\mathcal{B})$ and V⊆$\mathcal{M}\backslash\mathcal{B}$.
Since $\mathcal{M}\backslash$V is a $\mathcal{G}$-open set , $\mathcal{B}$ is $\mathcal{SG}-$ closed and $\mathcal{K}(\mathcal{B})\subseteq \mathcal{M}\backslash$V .Therefore,
V ⊆$(\mathcal{B}) \sqcap (\mathcal{M} \backslash \mathcal{K}(\mathcal{B}))$=φ .Hence $\mathcal{K}(\mathcal{B})\backslash\mathcal{B}$ does not contain any non-empty $\mathcal{G}$-closed set.
Proposition 3.8. If $\mathcal{B}$ is $\mathcal{G}$-open and $\mathcal{SG}-$ closed sets in $\mathcal{M}$, then $\mathcal{B}$ is closed set.
Proof. Since $\mathcal{B}$ is $\mathcal{G}$-open and $\mathcal{SG}-$ closed, then $(\mathcal{B})\subseteq \mathcal{B}$, but $\mathcal{B} \subseteq\mathcal{K}(\mathcal{B})$ .Therefore $\mathcal{K}(\mathcal{B}) = \mathcal{B}$. Hence , $\mathcal{B}$ is closed.
Corollary 3.9. Let $\mathcal{B}$ be open and $\mathcal{SG}-$ closed sets in $\mathcal{M}$. Then $\mathcal{B}$ is semi-closed.
Proof: since $\mathcal{B}$ is open then it is $\mathcal{G}$-open and from Proposition 3.2. then $\mathcal{B}$ is closed set which is semi-closed set.
Proposition 3.10. Let H ⊆ $\mathcal{B}$ ⊆ $\mathcal{M}$. If H is $\mathcal{G}$-open in $\mathcal{M}$, and $\mathcal{B}$ is clopen in $\mathcal{M}$. Then H is $\mathcal{G}$-open in $\mathcal{B}$.
Proof. Let F be a closed subset of $\mathcal{B}$ such that F ⊆ H ,F closed set in $\mathcal{M}$, F ⊆ H then F is closed in $\mathcal{M}$, Since $\mathcal{B}$ is closed in $\mathcal{M}$, where H $\mathcal{G}$-open in $\mathcal{M}$, then F $\subseteq i_{\mathcal{K}}$(H), but $\mathcal{B}$ is $\mathcal{G}$-open in $\mathcal{M}$,F⊑H⊑B, F⊑B,F⊑ $i_{\mathcal{K}}$(B).F⊑ $i_{\mathcal{K}}$(H)⊓B , hence $i_{\mathcal{B}}$ (H) = $\mathcal{B}\cap i_{\mathcal{K}}$(H) holds. Hence, F = F ∩ $\mathcal{B}\subseteq i_{\mathcal{B}}$ (H) .Therefore, H is $\mathcal{G}$-open in $\mathcal{B}$.
Proposition 3.11. If V ⊆ $\mathcal{B}$ ⊆ $\mathcal{M}$,V is $\mathcal{SG}-$ closed in $\mathcal{B}$ and $\mathcal{B}$ is clopen, then V is $\mathcal{SG}-$ closed in $\mathcal{M}$.
Proof. Let H be a $\mathcal{G}$-open set in $\mathcal{M}$ and V ⊆ H . Then V ⊆ H ∩ $\mathcal{B}$ and H ∩ $\mathcal{B}$ is $\mathcal{G}$-open in $\mathcal{M}$. Hence by using Proposition 3.4, H ∩ $\mathcal{B}$ is $\mathcal{G}$-open in $\mathcal{B}$. Since V $\mathcal{SG}-$ closed in $\mathcal{B}$, then $\mathcal{K}_{\mathcal{B}}$ (V) ⊆ H ∩ $\mathcal{B}$. Since $\mathcal{B}$ is closed in $\mathcal{M}$, then $(\mathcal{B}) = \mathcal{B}$ Hence, we have (V) ⊆ H . This shows that V is $\mathcal{SG}-$ closed in $\mathcal{M}$.
Proposition 3.12. Let V ⊆ $\mathcal{B}$ ⊆ $\mathcal{M}$. If V is $-$ closed in $\mathcal{M}$ and $\mathcal{B}$ is open, then V $\mathcal{SG}-$ closed in $\mathcal{B}$.
Proof. If H is a $\mathcal{G}$-open set in $\mathcal{M}$ such that V ⊆H, V ⊆ $\mathcal{B}$ there for V ⊆ $\mathcal{B}$⊓H⊆ $\mathcal{B}$, and $\mathcal{B}$ is $\mathcal{G}$-open in $\mathcal{M}$, then by using Proposition 3.4 H is $\mathcal{G}$-open in $\mathcal{M}$. Where $\mathcal{B}$⊓H is $\mathcal{G}$-open in $\mathcal{M}$. Since, V is $\mathcal{SG}-$ closed in $\mathcal{M}$, $\mathcal{K}$(V)⊑ $\mathcal{K}$(B) but B is closed set (i.e. $\mathcal{K}$(B)= B) there for $\mathcal{K}$(V)⊑ B, $\mathcal{K}$(V)⊓B⊑H⊓B, then $\mathcal{K}$(V) ⊆ H . Hence $\mathcal{K}_{\mathcal{B}}$ (V) = $\mathcal{B}$ ∩$\mathcal{K}$(V) ⊆ H . Therefore V is $\mathcal{SG}-$ closed in $\mathcal{B}$.
Proposition 3.13. For a closure space ($\mathcal{M}$.) , if V ⊆ $\mathcal{B}$ ⊆ $\mathcal{M}$ and $\mathcal{B}$ is clopen in $\mathcal{M}$, then the following are equivalent:
(1) V is $\mathcal{SG}-$ closed in $\mathcal{B}$,
(2) V is $\mathcal{SG}-$ closed in $\mathcal{M}$.

Proof. (1)⇒(2). Let V be $\mathcal{SG}-$ closed in $\mathcal{B}$. Then by Proposition 3.4, V is $\mathcal{SG}-$closed in $\mathcal{M}$.
(2)⇒(1). If V is $\mathcal{SG}-$ closed in $\mathcal{M}$, then by Proposition 3.6, V is $\mathcal{SG}-$ closed in $\mathcal{B}$.

4. semi generalized - open sets in the closure spaces

The aim of this section is to introduce the concept of a semi- generalized open set and study some of their properties.

Definition 4.1. A subset $\mathcal{B}$ of a closure space ($\mathcal{M}$.) is called a semi-generalized open (= $\mathcal{SG}-$ open) set if $\mathcal{M}\backslash\mathcal{B}$ is $\mathcal{SG}-$ closed .

Proposition 4.2. A subset $\mathcal{B}$ of a closure space ($\mathcal{M}$.) is $\mathcal{SG}-$ open if and only if F ⊆ $_{\mathcal{K}}(\mathcal{B})$ whenever F is $\mathcal{G}$-closed and F⊆$\mathcal{B}$.

Proof. Suppose that $\mathcal{B}$ is $\mathcal{SG}-$ open in $\mathcal{M}$, F is $\mathcal{G}$-closed and F ⊆ $\mathcal{B}$. Then $\mathcal{M}$\F is $\mathcal{G}$-open and $\mathcal{M}\backslash\mathcal{B}$⊆$\mathcal{M}$\F. Since, $\mathcal{M}\backslash\mathcal{B}$ is $\mathcal{SG}-$ closed, then ($\mathcal{M}\backslash\mathcal{B}$) ⊆$\mathcal{M}$\F. but, ($\mathcal{M}\backslash\mathcal{B}$) =$\mathcal{M}\backslash$ $i_{\mathcal{K}}(\mathcal{B})$⊆$\mathcal{M}$\F. Hence F ⊆ ($\mathcal{B}$).

Conversely, Suppose that F ⊆ ($\mathcal{B}$) whenever F⊆$\mathcal{B}$ and F is $\mathcal{G}$-closed. If H is a $\mathcal{G}$-open set in $\mathcal{M}$ containing $\mathcal{M}\backslash\mathcal{B}$, then $\mathcal{M}$\H is a $\mathcal{G}$-closed set contained in $\mathcal{B}$. Hence by hypothesis, $\mathcal{M}$\H ⊆ ($\mathcal{B}$), then by taking the complements ,we have, $\mathcal{K}(\mathcal{M}\backslash\mathcal{B})$ ⊆ H .Therefor $\mathcal{M}\backslash\mathcal{B}$ is $\mathcal{SG}-$ closed in $\mathcal{M}$ and hence $\mathcal{B}$ is $\mathcal{SG}-$ open in $\mathcal{M}$.

Remark 4.3. The intersection of two $\mathcal{SG}-$ open sets need not to be $\mathcal{SG}-$ open.

Example 4.4. If $\mathcal{M}$ ={1,2,3,4} then identified a closure operator $\mathcal{K}$ on M by
$\mathcal{K}\emptyset = \emptyset,\ \ \mathcal{K}\{3\} = \mathcal{K}\{4\} = C\{3{,}4\} = \{3{,}4\},$
$\mathcal{K}\{2{,}3{,}4\} = \mathcal{K}\{2{,}3\} = C\{2{,}4\} = \{2{,}3{,}4\}, \mathcal{K}\{2\} = \mathcal{K}\{2\}$
$C\mathcal{M} = \mathcal{M}.\mathcal{K}$ for the other set equal to $\mathcal{M}$
then the sets {1,3,4} and {1,2,4} are $\mathcal{SG}-$ open
sets but their intersection {1,4} is not $\mathcal{SG}-$ open.
Note: {4} is $\mathcal{G}$-closed , {4}⊑{1,4} , {4}⋢ $i_{\mathcal{K}}$({1,4}) since $i_{\mathcal{K}}$({1,4})={1}.

Proposition 4.5. If $\mathcal{B}$ is $\mathcal{SG}-$ open in $\mathcal{M}$, then H=$\mathcal{M}$, whenever H is $\mathcal{G}$-open and ($\mathcal{B}$)U($\mathcal{M} \backslash \mathcal{B}$) ⊆H .
Proof. Assume that H is $\mathcal{G}$-open and ($\mathcal{B}$)) ∪ ($\mathcal{M}\backslash\mathcal{B}$) ⊆ H. Hence $\mathcal{M}$\H⊆($\mathcal{M}\backslash\mathcal{B}$) ∩$\mathcal{B}$ = $\mathcal{K}(\mathcal{M}\backslash\mathcal{B})\backslash(\mathcal{M}\backslash\mathcal{B})$. Since, $\mathcal{M}$\H is $\mathcal{G}$-closed and $\mathcal{M}\backslash\mathcal{B}$ is $\mathcal{SG}-$ closed, then by Proposition 3.7, $\mathcal{M}$\H=∅ and hence, H=$\mathcal{M}$.
Proposition 4.6. If B is $\mathcal{SG}-$ closed, then ($\mathcal{B}$)\ $\mathcal{B}$ is $\mathcal{SG}-$ open.
Proof . Suppose that $\mathcal{B}$ is $\mathcal{SG}-$ closed. Then by Proposition3.7, ($\mathcal{B}$)\$\mathcal{B}$ does not contain any non-empty $\mathcal{G}$-closed set. Therefore, ($\mathcal{B}$)\$\mathcal{B}$ is $\mathcal{SG}-$ open.
Proposition 4.7. For each p ∈ $\mathcal{M}$, then either {p} is $\mathcal{G}$-closed or $\mathcal{M}$\{p} is $\mathcal{SG}-$ closed.
Proof. If {p} is not $\mathcal{G}$-closed, then the only $\mathcal{G}$-open set containing $\mathcal{M}$\{p} is $\mathcal{M}$, hence, $\mathcal{K}$($\mathcal{M}$\ {p})⊆ $\mathcal{M}$ is contained in $\mathcal{M}$ and therefore, $\mathcal{M}$\{p} is $\mathcal{SG}-$ closed.

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