Hesitant Fuzzy Prime Ideal Of Ring.

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ABSTRACT. In this paper, we study hesitant fuzzy sets and some of its properties .We introduce the notions of hesitant fuzzy ideal, hesitant fuzzy prime ideal of a ring and hesitant fuzzy strongly prime ideal ,hesitant fuzzy 3- prime ideals .
And we give a characterization of hesitant fuzzy prime ideal ,also introduce relationships between hesitant prime ideal and strongly prime ,3-prime .
And some important basic operation on the hesitant fuzzy prime ideal of a ring .

KEYWORDS: HESITANT FUZZY SETS(HFS) , HESITANT FUZZY IDESL OF A RING (HFI (R)), HESITANT FUZZY PRIME IDEAL OF A RING (HFPI (R)).

1.INTRODUCTION.

Zadeh [14] in (1965) introduced the concept of fuzzy set(FS) in a set X as a mapping from X into [0,1] . Torra and Y.Narukawa [12] in (2009) proposed a new generalized type of fuzzy set called hesitant fuzzy set (HFS) and he defined the complement, union and intersection of HFSs. After that time , Xia and Xu [13] in (2011) gave some operational laws for HFSs, such as the addition and multiplication operations . Mohammad et.al. [1] in (2018) introduced the Hesitant Fuzzy Ideal, Hesitant Fuzzy Bi-Ideal, and Hesitant Fuzzy Interior Ideal in Γ-semigroup. In po-semigroups, the notions of hesitant fuzzy ideals, hesitant fuzzy prime ideals, hesitant fuzzy semiprime ideals, and hesitant 3-prime fuzzy ideals are introduced, along with some of their properties by M.Y .Abbasi ,A.F.Talee , et.al. [2] in 2018. Kim, Lim and Lee [3] in (2019) defined the Hesitant Fuzzy subgroupoid; Hesitant Fuzzy subgroup ; Hesitant Fuzzy subring. In 2020 Pairote Yiarayong [9] introduced a new concept of Hesitant Fuzzy bi-ideals and Hesitant Fuzzy interior ideals on ternary semigroups.

The remainder of the paper is organized as follows: in section two, we recall some definition along with some properties of hesitant fuzzy set and some results. In section three, hesitant fuzzy ideal of ring , hesitant fuzzy prime ideal , hesitant fuzzy strongly prime , 3- prime of ring and radical of prime ideal are presented. Finally ,we establish some results on an operations of hesitant fuzzy ideal of ring and Homomorphism on hesitant fuzzy prime ideal of ring introduced in section four .

2. DEFINITION AND PRELIMINARIES .

In this section , we will discuss the following definitions as well as some of the findings that will be expected in the following pages.

Definition 2.1 [12].

Let X be a reference set , a hesitant fuzzy set (in short , HFS) of X is a function h:X→ P[0,1] that returns a sub set of some values in [0,1] .
Where P[0,1] denotes the set of all sub set of [0,1] ,and expressed the HFS by a mathematical symbol: A={$< x, h_A(x) >: x \in X$}.
We will denote the set of all HFSs in X as HFS (X) .

Example 2.2 .

Let X={x_1, x_2, x_3} be a reference set , and $h_A(x_1) = \{0.5,0.7,0.9\}$, $h_A(x_2) = \{0.2,0.5,0.6\}$, $h_A(x_3) = \{0.4,0.7,0.8\}$,then we can express the HFS A as:-
A={$< x_1, \{0.5,0.7,0.9\} >, < x_2, \{0.2,0.5,0.6\} >, < x_3, \{0.4,0.7,0.8\} >$}.

Definition 2.3 [4,12,13] .

Let $h_1, h_2 \in$ HFS (X) Then , for each x ∈ X

1. we say that h_1 is a subset of h_2, denoted by $h_1 \subset h_2$, if $h_1(x) \subset h_2(x)$.
2. we say that h_1 is equal to h_2, denoted by $h_1 = h_2$, if $h_1(x) \subset h_2(x)$ and $h_2(x) \subset h_1(x)$.
3. The complement of h , $h^c(x)= \{1-\gamma/\gamma \in h(x)\}$.
4. Lower bound: $h^-(x) = \min \{h(x)\}$.
5. α −lower bound: $h_\alpha^-(x) = \{\gamma \in h(x)/\gamma \le \alpha\}$.
6. Upper bound: $h^+(x) = \max \{h(x)\}$.
7. α −upper bound: $h_\alpha^+(x) = \{\gamma \in h(x)/\gamma \ge \alpha\}$.
8. $(h_1 \cup h_2)(x) = h_1(x) \cup h_2(x) = \bigcup_{\gamma_1 \in h_1(x), \gamma_2 \in h_2(x)} \max\{\gamma_1, \gamma_2\}$.
9. $(h_1 \cap h_2)(x) = h_1(x) \cap h_2(x) = \bigcup_{\gamma_1 \in h_1(x), \gamma_2 \in h_2(x)} \min\{\gamma_1, \gamma_2\}$.
10. $h^\lambda(x) = \bigcup_{\gamma \in h(x)} \{\gamma^\lambda\} \cong \{\gamma^\lambda / \gamma \in h(x)\}$.
11. $\lambda h(x) = \bigcup_{\gamma \in h(x)} \{1-(1-\gamma)^\lambda\} \cong \{1-(1-\gamma)^\lambda / \gamma \in h(x)\}$.

Definition 2.4 [4] .

Let h ∈ HFS (X). Then h is called a hesitant fuzzy point(in short , HFP) with the support x ∈ X and the value λ , denoted by x_λ, if $x_\lambda : X \to P[0,1]$ is the mapping given by: for each y ∈ X,

$$x_\lambda(y) = \begin{cases} \lambda \subset [0,1] & \text{if } y = x \\ \emptyset & \text{otherwise} \end{cases}$$

We will denote the set of all HFPs in X as HFP (X).

Definition 2.5 [4] .

Let h ∈ HFS (X) and $x_\lambda \in$ HFP (X) . Then x_λ is said to be belong to h , denoted by $x_\lambda \in h$, if $\lambda \subseteq h(x)$.

Example 2.6 .

Suppose that X={a ,b} , and let $h_1 \in$ HFS (X) , given by:
$h_1(a) = \{0, 0.4, 0.7\}$, $h_1(b) = [0,0.6)$. And let $\lambda = \{0, 0.5\} \in P[0,1]$,then $a_\lambda(b) = \emptyset$ and $a_\lambda(a) = \{0,0.5\}$.Then

$$a_\lambda(y) = \begin{cases} \{0,0.5\} & \text{if } y = a \\ \emptyset & \text{otherwise} \end{cases}$$

Theorem 2.7 [4] .

Let h_1, h_2 $\in$ HFS (X) and $\{h_i / i \in I\} \subset$ HFS (X) .
(1) $h_1 \subset h_2$ if and only if $x_\lambda \in h_2$, for each $x_\lambda \in h_1$.
(2) $x_\lambda \in h_1 \cap h_2$ if and only if $x_\lambda \in h_1$ and $x_\lambda \in h_2$.
(3) If $x_\lambda \in h_1$ or $x_\lambda \in h_2$, then $x_\lambda \in h_1 \cup h_2$.
(4) $x_\lambda \in \bigcap_{i \in I} h_i$ if and only if $x_\lambda \in h_i$, for each $i \in I$.
(5) If $x_\lambda \in h_i$ for some $i \in I$, then $x_\lambda \in \bigcup_{i \in I} h_i$.

Definition 2.8 [4] .

Let X be a reference set and let $h_1, h_1 \in$ HFS (X).Then the hesitant fuzzy product h_1 and h_1,denoted by $h_1 \circ h_2$, is a HFS (X) defined by: for each $x \in X$,

1. $(h_1 \circ h_2)(x) = \begin{cases} \bigcup_{yz=x} [h_1(y) \cap h_2(z)] & \text{if } yz = x \\ \Phi & \text{if otherwise} \end{cases}$

Proposition 2.9 [3,4] .

Let $h_1, h_1 \in$ HFS (X) and $x_\alpha, y_\beta \in$ HFP (X).Then

1. $x_\alpha \circ y_\beta = (xy)_{\alpha \cap \beta}$
2. $x_\alpha + y_\beta = (x + y)_{\alpha \cap \beta}$
3. $x_\alpha - y_\beta = (x - y)_{\alpha \cap \beta}$
4. $x_\alpha y_\beta = (xy)_{\alpha \cap \beta}$
5. $h_1 \circ h_2 = \bigcup_{x_{\alpha \in h_1}, y_{\beta \in h_2}} x_\alpha \circ y_\beta$

Proof :- (1).

Let $t \in R$ and $t = ab$.

1. Then$(x_\alpha \circ y_\beta)(t) = \bigcup_{t=ab}[x_\alpha(a) \cap y_\beta(b)] = \alpha \cap \beta$.

$$(x_\alpha \circ y_\beta)(t) = \begin{cases} \bigcup_{t=ab}[x_\alpha(a) \cap y_\beta(b)] & \text{if } t = ab \\ \emptyset & \text{if otherwise} \end{cases} = \begin{cases} \alpha \cap \beta & \text{if } t = xy \\ \emptyset & \text{if otherwise} \end{cases}$$
$$= (xy)_{\alpha \cap \beta}$$

(2). Let $t \in R$ and $t = a - b$.

Thus $(x_\alpha - y_\beta)(t) = \bigcup_{t=a-b}[x_\alpha(a) \cap y_\beta(b)] = \alpha \cap \beta$

$$(x_\alpha - y_\beta)(t) = \begin{cases} \bigcup_{t=a-b}[x_\alpha(a) \cap y_\beta(b)] & \text{if } t = a - b \\ \emptyset & \text{if otherwise} \end{cases} = \begin{cases} \alpha \cap \beta & \text{if } t = x - y \\ \emptyset & \text{if otherwise} \end{cases}$$
$$= (x - y)_{\alpha \cap \beta}$$

3-HESITANT FUZZY PRIME IDEAL .

Definition 3.1 [4] .

If (R ,+, .) be a ring and $h \in$ HFS (R).Then h is a hesitant fuzzy subring (in short ,HFR) if and only if , for any $x, y \in R$

1. $h(x - y) \supseteq h(x) \cap h(y)$
2. $h(xy) \supseteq h(x) \cap h(y)$

We will denote the set of all HFR_S as HFR (R) .

Definition 3.2 .

If (R ,+, .) be a ring and $h \in HFR(R)$, $h \neq \emptyset$, then h is said to be a hesitant fuzzy ideal (in short , HFI) of R , if and only if ,for any x ,y ∈ R.

1. $h(x-y) \supseteq h(x) \cap h(y)$
2. $h(xy) \supseteq h(x) \cup h(y)$

We will denote the set of all HFIs of R as HFI (R) .

Example 3.3.

Let (Z_4 ,+,.) be a ring where Z_4 ={0,1,2,3} and the mapping h: $Z_4 \rightarrow P[0,1]$ defined as follows: h(0)=[0.2,0.8) , h(1)= (0.3,0.7)=h(3) , h(2)= [0.2 ,0.5].Then we can easily that h ∈ HFI (R).

Theorem 3.4 .

Let $h \in HFR(R)$, then $h \in HFI(R)$ if and only if :

1. For all $x_\alpha, y_\beta \in h$, $x_\alpha - y_\beta \in h$
2. For all $x_\alpha \in HFP(R)$, $y_\beta \in h$, $x_\alpha y_\beta \in h$

Proof:-

Suppose that $h \in HFI(R)$, and x_α , $y_\beta \in h$, so that $\alpha \subseteq h(x)$, $\beta \subseteq h(y)$

So $h(x-y) \supseteq h(x) \cap h(y) \supseteq \alpha \cap \beta$, then $x_\alpha - y_\beta = (x-y)_{\alpha\cap\beta} \in h$

Thus $x_\alpha - y_\beta \in h$.

Now let $x_\alpha \in HFP(R)$, $y_\beta \in h$. Then $h(xy) \supseteq h(x) \cup h(y) \supseteq h(x) \supseteq \alpha \supseteq \alpha \cap \beta$.

And $h(xy) \supseteq h(x) \cup h(y) \supseteq h(y) \supseteq \beta \supseteq \alpha \cap \beta$

Hence $x_\alpha y_\beta = (xy)_{\alpha\cap\beta} \in h$. So $x_\alpha y_\beta \in h$.

Suppose that the conditions are met , $x, y \in R$.

Let $t = h(x) \cup h(y)$, and x_t , $y_t \in h$. Such that $x_t - y_t \in h$.

So that $(x-y)_t \in h$, it is follows $t \subseteq h(x-y)$.

Thus $h(x) \cup h(y) \subseteq h(x-y)$,then $h(x) \cap h(y) \subseteq h(x) \cup h(y) \subseteq h(x-y)$

Hence $h(x-y) \supseteq h(x) \cap h(y)$

Now

Let $x_t \in HFP(R)$, $y_t \in h$,such that $(xy)_t \in h$.

So $t \subseteq h(xy)$ this implies $h(x) \cup h(y) \subseteq h(xy)$.

Hence $h \in HFI(R)$ .This completes the proof.

Proposition 3.5 .

Let h_1 and h_2 be two HFI of R . Then $h_1 \cap h_2 \in HFI(R)$.

Proof:-

Let $x_\alpha, y_\beta \in h_1 \cap h_2$ implies $x_\alpha, y_\beta \in h_1$ and $x_\alpha, y_\beta \in h_2$ since h_1 , h_2 be two HFI of a ring R . Then $x_\alpha - y_\beta \in h_1$ and $x_\alpha - y_\beta \in h_2$, so $x_\alpha - y_\beta \in h_1 \cap h_2$.

Also $x_\alpha y_\beta \in h_1$ and $x_\alpha y_\beta \in h_2$,it is follows $x_\alpha y_\beta \in h_1 \cap h_2$

Thus $h_1 \cap h_2 \in HFI(R)$.

Theorem 3.6 .

Let $\{h_i / i \in I\}$ be a family of a HFI of R , then $\bigcap_{i\in I} h_i \in HFI(R)$.

Proof:-

Let x_α , $y_\beta \in \bigcap_{i\in I} h_i$, so that $x_\alpha, y_\beta \in h_i$, for all $i \in I$.[From definition 2.7 point (2)]

Since $h_i \in HFI(R)$, thus $x_\alpha - y_\beta \in h_i$, for all $i \in I$
Hence $x_\alpha - y_\beta \in \bigcap_{i\in I} h_i$. Also $x_\alpha y_\beta \in h_i$, for all $i \in I$,then $x_\alpha y_\beta \in \bigcap_{i\in I} h_i$
Thus $\bigcap_{i\in I} h_i \in HFI(R)$.

Definition 3.7 .

Let $h \in HFI(R)$. A hesitant fuzzy set $\sqrt{h} : R \to P[0,1]$,defined as
$\sqrt{h} = \bigcup_{n\in N}\{h(x^n)\}$, is called a hesitant fuzzy nil radical of h .

Theorem 3.8.

If h is a HFI of a ring R , then so is $\sqrt{h}$.

Proof :-

Suppose $h \in HFI(R)$,and for any $x, y \in R$,we have
$\sqrt{h}(x-y) = \bigcup_{n\in N}\{h(x-y)^n\}$,we prove the result by induction.
Clearly the result is true for n=1.
So $\sqrt{h}(x-y) = \bigcup_{n=1}\{h(x-y)^1\} = \bigcup_{n=1}\{h(x-y)\} \supseteq \bigcup_{n=1}\{h(x) \cap h(y)\}$
$=\bigcup_{n=1}\{h(x)\} \cap \bigcup_{n=1}\{h(y)\} = \sqrt{h}(x) \cap \sqrt{h}(y)$
Assume the result is true for n = r
So $\sqrt{h}(x-y) = \bigcup_{n=r}\{h(x-y)^r\} \supseteq \bigcup_{n=r}\{h(x^r) \cap h(y^r)\}$
$=\bigcup_{n=r}\{h(x^r)\} \cap \bigcup_{n=r}\{h(y^r)\}=\sqrt{h}(x) \cap \sqrt{h}(y)$
Now $\sqrt{h}(x-y) = \bigcup_{n=r+1}\{h(x-y)^{r+1}\}=\bigcup_{n=r+1}\{ h((x-y)^r(x-y)^1)\}$
Since $h \in HFI(R)$
So $\bigcup_{n=r+1}\{ h((x-y)^r(x-y)^1)\} \supseteq \{\bigcup_{n=r}\{h((x-y)^r)\} \cup \{\bigcup_{n=1} h((x-y)^1)\} \supseteq$
$\{\bigcup_{n=r}\{h(x^r)\} \cap \bigcup_{n=r}\{h(y^r)\}\} \cup \{\bigcup_{n=1}\{h(x)\} \cap \bigcup_{n=1}\{h(y)\}\}$
$= \{\sqrt{h}(x) \cap \sqrt{h}(y)\} \cup \{\sqrt{h}(x) \cap \sqrt{h}(y)\} = \sqrt{h}(x) \cap \sqrt{h}(y)$.
Now we find.
$\sqrt{h}(xy) = \bigcup_{n\in N}\{h((xy)^n)\} = \bigcup_{n\in N}\{h(x^n y^n)\} \supseteq \bigcup_{n\in N}\{h(x^n) \cup h(y^n)\}$
$=[\bigcup_{n\in N}\{h(x^n)\}] \cup [\bigcup_{n\in N}\{h(y^n)\}] =\sqrt{h}(x) \cup \sqrt{h}(y)$
Thus $\sqrt{h} \in HFI(R)$.

Definition 3.9.

An hesitant fuzzy ideal h of a ring R , is called to be a hesitant fuzzy prime ideal (in short , HFPI) if for any two hesitant fuzzy points x_α , $y_\beta \in HFP(R)$, $x_\alpha \circ y_\beta \in h$ implies either $x_\alpha \in h$ or $y_\beta \in h$.

Will denote the set of all $HFPI_S$ in R as HFPI (R).

Theorem 3.10.

Let $h \in HFI(R)$ is $h \in HFPI(R)$ if and only if for all $x , y \in R$,$h(x) \cup h(y) \supseteq h(xy)$.

Proof:-

Suppose h is $\in HFPI(R)$.
If possible , let $h(x_\circ) \cup h(y_\circ) \subset h(x_\circ y_\circ)$ for some $x_\circ , y_\circ \in R$.
Put $t = h(x_\circ y_\circ)$, then $h(x_\circ) \cup h(y_\circ) \subset t$ and $(x_\circ y_\circ)_t \in h$.
So $h(x_\circ) \subset t$ this implies $(x_\circ)_t \notin h$ and $h(y_\circ) \subset t$ this implies $(y_\circ)_t \notin h$.
This is a contradiction. Therefore , for all $x , y \in R$, $h(x) \cup h(y) \supseteq h(xy)$.
Suppose the condition hold .Now

Let x_t , y_t $\in$ HFP (R), Such that $x_t \circ y_t \in h$,then $(xy)_t \in h$ and let $x_t \notin h$ and $y_t \notin h$. Put $t = h(xy)$.
If $x_t \notin h$, then $h(x) \subset t$, so that $h(x) \subset h(xy)$.
If $y_t \notin h$, then $h(y) \subset t$, so that $h(y) \subset h(xy)$
So $h(x) \cup h(y) \subset h(xy)$, this is a contradiction
Thus $x_t \circ y_t \in h$ implies either $x_t \in h$ or $y_t \in h$.
Thus h $\in$ HFPI (R).

Remark 3.11 : If h $\in$ HFPI (R) , then for any $x, y \in R$, $h(xy) = h(x) \cup h(y)$.

Proposition 3.12

Every HFPI (R) is HFI (R).

Proof:

Assume that h $\in$HFPI (R) .
So h$\in$ HFR (R) and $h(xy) = h(x) \cup h(y)$,then $h(xy) \supseteq h(x) \cup h(y)$.
Thus h$\in$ HFI (R) .

The converse of Theorem (3.12) may not to be true , as seen in the following counter example .

Example 3.13.

Let (Z_4 ,+,.) be a ring where Z_4 ={0,1,2,3} and the mapping $h: Z_4 \to P[0,1]$ defined as follows: h(0) = [0.2,0.8] , h(1) = (0.3,0.7) = h(3) , h(2) = [0.2 ,0.7].
Then we can easily that h $\in$ HFI (R).
But $h(2.2) = h(0) = [0.2,0.8] \nsubseteq h(2) \cup h(2) = [0.2,0.7]$.
So h $\notin$ HFPI (R).

Proposition 3.14

If R is a ring and h is any hesitant fuzzy prime ideal of R ,then $h_E = \{x \in R: h(x) \supseteq E\}$,where $E \subseteq P[0,1]$ is a prime ideal of R.

Proof :

Suppose h is hesitant fuzzy prime ideal of R. and
Let a , b $\in$ R such that $ab \in h_E$. So $h(ab) \supseteq E$,then $(ab)_E \in h$.
Since h is hesitant fuzzy prime ideal of R and .Then
$a_E \in h$, so that $E \subseteq h(a)$, hence $a \in h_E$ or $b_E \in h$, so that
$E \subseteq h(b)$, hence $b \in h_E$.
Thus $ab \in h_E$ implies either $a \in h_E$ or $b \in h_E$. Hence h_E is a prime ideal of R.
The converse of Theorem (3.12) may not to be true , as seen in the following counter example .

Example 3.15.

Let (R , +, .) where R = {0,1,2} be a ring knowledge as follows :

.	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Let h be a hesitant fuzzy set of R such that : h= $\begin{cases}[0,1] & \text{if } x = 0 \\ \emptyset & \text{otherwise}\end{cases}$

And let $E = [0.2,0.6]$

You can easily prove that $h \in$ HFI (R) .

Now we find , $h(12) = h(0) = [0,1] \not\subset h(1) \cup h(2) = \emptyset$

So h is not hesitant prime .

But $h_E = \{0\}$, hence h_E of h is prime ideal of R.

Definition 3.16.

A HFI (R) is called a hesitant fuzzy strongly prime ideal (in short ,HFSPI) if for any $x, y \in R$, $h(xy) = h(x)$ or $h(xy) = h(y)$.

Will denote the set of all HFSPIs in R as HFSPI (R).

Example 3.17 .

Let $(Z_2, +, .)$ be a ring and h(0)=[0.1,0.8] , h(1)=[0.3 ,0.6]

Then we easily see that h is HFI in R .

Hence h(0.0)=h(0) , h(0.1)=h(0) , h(1.0)=h(0) , h(1.1)=h(1)

Thus h $\in$ HFSPI (R).

Theorem 3.18 .

Every a hesitant strongly fuzzy prime ideal is a hesitant fuzzy prime ideal .

Proof :-

Suppose h is hesitant strongly fuzzy prime ideal and x_α , $y_\beta \in$ HFP (R),

Such that $x_\alpha \circ y_\beta \in h$ this implies $(xy)_{\alpha\cap\beta} \in h$, let $\delta = \alpha \cap \beta$

So $(xy)_\delta \in h$, then $\delta \subseteq h(xy)$.

Since $h(xy) = h(x)$ so that $\delta \subseteq h(x)$, it is follows $x_\delta \in h$.

or $h(xy) = h(y)$ so that $\delta \subseteq h(y)$, it is follows $y_\delta \in h$.

So $(xy)_\delta \in h$ implies either $x_\delta \in h$ or $y_\delta \in h$.

Thus h $\in$ HFPI (R).

The converse of Theorem (3.12) may not to be true , as seen in the following counter example .

Example 3.19.

Let $(R, +, .)$ where $R = \{a, b, c\}$ be a ring knowledge as follows :

.	a	b	c
a	a	a	a
b	**a**	b	a
c	a	a	c

+	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

Let h be a hesitant fuzzy set of R such that : h= $\begin{cases}[0,1] & \text{if } x = a, b \\ \{0\} & \text{if } x = c\end{cases}$

You can easily prove that h$\in$ HFR (R) .

Now we must prove h$\in$ HFPI (R)

$h(aa) = h(a) = h(a) \cup h(a) = [0,1]$

$h(ab) = h(a) = h(a) \cup h(b) = [0,1]$

$h(ac) = h(a) = h(a) \cup h(c) = [0,1]$

$h(bb) = h(b) = h(b) \cup h(b) = [0,1]$
$h(bc) = h(a) = h(b) \cup h(c) = [0,1]$
$h(cc) = h(c) = h(c) \cup h(c) = \{0\}$
Thus $h \in$ HFPI (R)
But $h(bc) = h(a)$, hence $a \neq b$ and $a \neq c$.
So $h \notin$ HFSPI (R).

Proposition 3.20.
Let $h \in$ HFSPI (R) and $h_* = \{x \in R : h(x) = h(0)\}$.Then h_* is a strongly prime ideal of R.
Proof :-
Suppose $h \in$ HFSPI (R), since $0 \in R$, then $h(0) = h(0)$, this means $0 \in h_*$.
Thus $h_* \neq \emptyset$.
Let $a, b \in R$ and $ab \in h_*$.
Thus $h(ab) = h(0)$, since $h \in$ HFSPI (R) ,then
$h(ab) = h(a) = h(0)$, then $h(a) = h(0)$ implies that $a \in h_*$
or
$h(ab) = h(b) = h(0)$, then $h(a) = h(0)$ implies that $b \in h_*$.
So $ab \in h_*$ ethier $a \in h_*$ or $b \in h_*$.
Hence h_* is an strongly prime ideal of R.

Definition 3.21:
A hesitant fuzzy ideal of a ring R is called hesitant 3-prime of ring if for any $x, y, z \in R$.
1. $h(xyz) \subseteq h(xy) \cup h(xz)$
2. $h(xyz) \subseteq h(yx) \cup h(yz)$
3. $h(xyz) \subseteq h(zx) \cup h(zy)$

Remark 3.22 : if h is hesitant fuzzy 3-prime ideal ,then for any $x, y. z \in R$
$h(xyz) = h(xy) \cup h(xz)$
$= h(yx) \cup h(yz)$
$= h(zx) \cup h(zy)$

Theorem 3.23.
Let R be a ring and $h \in$ HFI (R) . If h is prime ,then h is 3-prime .
Proof. Let $h \in$ HFPI (R) and for any $x, y, z \in R$.
$h(xyz) = h\big((xy)z\big) = h(xy) \cup h(z) \subseteq h(xy) \cup h(xz)$ [$h \in$ HFI (R)]
So $h(xy) \cup h(xz) = h(x) \cup h(y) \cup h(z) = h(xyz)$.
In the same way ,we find
$h(xyz) = h(x(yz)) = h(x) \cup h(yz) \subseteq h(yx) \cup h(yz)$ [$h \in$ HFI(R)]
So $h(yx) \cup h(yz) = h(x) \cup h(y) \cup h(z) = h(xyz)$
Since $h \in$ HFPI (R) , we have $h(xyz) = h(yzx)$.
$h(xyz) = h(yzx) = h(y(zx)) \subseteq h(y) \cup h(zx) \subseteq h(zy) \cup h(zx)$ [$h \in$ HFI(R)]
So $h(zx) \cup h(zy) = h(x) \cup h(y) \cup h(z) = h(xyz)$.
Hence h is hesitant 3-prime ideal .

In general the 3-prime ideal hesitant fuzzy need not necessarily hesitant prime ideal fuzzy as shown in the following example.

Example 3.24.

Let $(R, +, .)$ where $R = \{0,1,2\}$ be a ring knowledge as follows :

.	0	1	2
0	0	0	0
1	**0**	1	0
2	0	0	2

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Let h be a hesitant fuzzy set of R such that : h= $\begin{cases} [0,1] & \text{if } x = 0 \\ \emptyset & \text{otherwise} \end{cases}$

You can easily prove that $h \in HFI(R)$.

For any x, y and $z \in R$, let one of x, y and z be 0 , then we get .

$h(xyz) = h(0) = [0,1] \subseteq h(xy) \cup h(xz)$.

$h(xyz) = h(0) = [0,1] \subseteq h(yx) \cup h(yz)$.

$h(xyz) = h(0) = [0,1] \subseteq h(zx) \cup h(zy)$.

In case x, y and z be different from 0 , then $h(xyz)$has the following cases:

$h(111) = h(1) \subseteq h(11) \cup h(11)$

$h(112) = h(0) \subseteq h(11) \cup h(12) = h(12) \cup h(21)$

$h(222) = h(2) \subseteq h(22) \cup h(22)$

$h(221) = h(0) \subseteq h(22) \cup h(21) = h(21) \cup h(12)$

Hence h is a hesitant fuzzy 3-prime ideal of R

But $h(12) = [0,1] \not\subset h(1) \cup h(2) = \emptyset$

Therefore, h is not hesitant prime .

Theorem 3.25.

Let R be a ring with an identity e and h any hesitant fuzzy set of R ,then h is 3-prime if and only if h is prime .

Proof :

Assume that h is hesitant prime ideal of R ,and $x, y, z \in R$.

So $h(xyz) = h\big((xy)z\big) = h(xy) \cup h(z) \subseteq h(xy) \cup h(xz) = h(x) \cup h(y) \cup h(z)$
$= h(xyz)$.

So $h(xyz) = h(xy) \cup h(xz)$.

In the same way ,we get

$h(xyz) = h(x(yz)) = h(x) \cup h(yz) \subseteq h(xy) \cup h(yz) = h(x) \cup h(y) \cup h(z)$
$=h(xyz)$.

So $h(xyz) = h(xy) \cup h(yz) = h(yx) \cup h(yz)$.

Since h is prime ,we have $h(xyz) = h(yzx)$

Hence $h(xyz) = h(yzx) = h\big(y(zx)\big) = h(y) \cup h(zx) \subseteq h(zy) \cup h(zx)$
$= h(x) \cup h(y) \cup h(z) = h(xyz)$.

So $h(xyz) = h(zy) \cup h(zx)$

Thus h is hesitant 3-prime ideal.

Now suppose that h is hesitant 3-prime ideal of R ,and $x, y \in R$.

Since R be a ring with an identity e .

So $h(xy) = h(xye) = h(xe) \cup h(ye) = h(x) \cup h(y)$.
Hence h is hesitant prime ideal.

Proposition 3.26 .

If $h \in$ HFPI (R) ,then $h(x_1 x_2 \dots \dots x_n) = h(x_1) \cup h(x_2) \dots \dots h(x_n)$,
for all $x_1, x_2 \dots x_n \in R$.

Proof :-

We will do the demonstration by induction on n.
∴The proposition is evident for $n = 2$, it is follows $h(x_1 x_2) = h(x_1) \cup h(x_2)$.
∴ Let us suppose that it is true for $n = r$, it is follows $h(x_1 x_2 \dots x_r) = h(x_1) \cup h(x_2) \dots \cup h(x_r)$.
Now we must prove is true for $n = r + 1$.
$h(x_1 x_2 \dots \dots x_{n+1}) = h\big((x_1 x_2 \dots \dots x_n)(x_{n+1})\big) = h(x_1 x_2 \dots \dots x_r) \cup h(x_{r+1})$
$= h(x_1) \cup h(x_2) \dots . . \cup h(x_r) \cup h(x_{r+1})$.
This implies $h(x_1 x_2 \dots \dots x_{r+1}) = h(x_1) \cup h(x_2) \dots . . \cup h(x_n) \cup h(x_{r+1})$.
With this the proof was completed .

Theorem 3.27 .

Let $h \in$ HFPI (R) and $\theta \subset [0,1]$,then the set $h^\theta = \{x \in R: h(x) \subseteq \theta\}$ is prime ideal of R.

Proof:-

Suppose that $h \in$ HFPI (R) and $\theta \subset [0,1]$, let $x, y \in R$ and $xy \in h^\theta$
,then $h(xy) \subseteq \theta$.
Since $h \in$ HFPI (R) it is follows $h(xy) = h(x) \cup h(y) \subseteq \theta$
So $h(x) \subseteq \theta$, then $x \in h^\theta$ and $h(y) \subseteq \theta$, then $y \in h^\theta$
Hence h^θ is prime ideal of R .

Proposition 3.28 .

Let R be a ring and $h_1 \in$ HFPI (R). If h_2 is a hesitant fuzzy subring of R, then $h_1 \cap h_2 \in$ HFPI (R).

Proof:-

Suppose $h_1 \in$ HFPI (R) and $h_2 \in$ HFR (R) .
Let $x , y \in R$.Then $(h_1 \cap h_2)(xy) = h_1(xy) \cap h_2(xy) \subseteq \{h_1(x) \cup h_1(y)\} \cap \{h_2(x) \cap h_2(y)\} \subseteq$
$\{h_1(x) \cap h_2(x)\} \cup \{h_1(y) \cap h_2(y)\} = (h_1 \cap h_2)(x) \cup (h_1 \cap h_2)(y)$
So $(h_1 \cap h_2)(xy) \subseteq (h_1 \cap h_2)(x) \cup (h_1 \cap h_2)(y)$.
Hence $h_1 \cap h_2 \in$ HFPI (R) .

Proposition 3.29.

Every hesitant fuzzy prime ideal , then $\sqrt{h}$ is hesitant fuzzy prime ideal.

Proof :- suppose that $h \in$ HFPI (R) , and $x , y \in R$.
Now $\sqrt{h}(xy) = \cup_{n\in N}(h(xy)^n) = \cup_{n\in N} h(x^n y^n) = \cup_{n\in N}\{h(x^n) \cup h(y^n)\}$
$=\{\cup_{n\in N} h(x^n)\} \cup \{\cup_{n\in N} h(y^n)\} = \sqrt{h}(x) \cup \sqrt{h}(y)$.
Thus $\sqrt{h}(xy) = \sqrt{h}(x) \cup \sqrt{h}(y)$.
Hence $\sqrt{h} \in$ HFPI (R) .

4-Some results on an operations of Hesitant fuzzy prime ideal of ring.

We will show some of the properties that satisfy the HFPI (R) in this section.

Proposition 4.1 .

Let h_1 , $h_2 \in$ HFPI (R) . Then $h_1 \cap h_2 \in$ HFPI (R)

Proof :-

Suppose h_1 , $h_2 \in$ HFPI (R) and $(xy)_t \in h_1 \cap h_2$.

Then $(xy)_t \in h_1$ and $(xy)_t \in h_2$.

Since $h_1 \in$ HFPI (R) and $(xy)_t \in h_1$,we have $x_t \in h_1$ or $y_t \in h_1$.

Again , since $h_2 \in$ HFPI (R) and $(xy)_t \in h_2$,we have $x_t \in h_2$ or $y_t \in h_2$.

Thus ,either $x_t \in h_1 \cap h_2$ or $y_t \in h_1 \cap h_2$.

So $h_1 \cap h_2 \in$ HFPI (R) ◊

Proposition 4.2 .

Every hesitant fuzzy prime ideal ,then h^λ is hesitant fuzzy prime ideal.

Proof:- Suppose $h \in$ HFPI (R), and $x, y \in R$.

$h^\lambda(xy) = \{\gamma^\lambda : \gamma \in h(xy)\} = \{\gamma^\lambda : \gamma \in h(x) \cup h(y)\} = \{\gamma^\lambda : \gamma \in h(x) \vee \gamma \in h(y)\}$

$= \{\gamma^\lambda : \gamma \in h(x)\} \cup \{\gamma^\lambda : \gamma \in h(y)\} = h^\lambda(x) \cup h^\lambda(y)$

So $h^\lambda(xy) = h^\lambda(x) \cup h^\lambda(y)$.

Hence $h^\lambda \in$ HFPI (R).

Theorem 4.3 .

Let a non-constant hesitant fuzzy ideal h : R⟶ P[0,1] is hesitant fuzzy prime ideal of R then $h_\alpha^- \in$ HFPI (R) , $\alpha \in [0,1]$.

Proof . Suppose $h \in$ HFPI (R) and x , y ∈ R .

Hence $h_\alpha^-(xy) = \{\gamma \in h(xy) : \gamma \le \alpha\} = \{\gamma \in h(x) \cup h(y) : \gamma \le \alpha\}$

$= \{\gamma \in h(x) \vee \gamma \in h(y) : \gamma \le \alpha\} = \{\gamma \in h(x) : \gamma \le \alpha\} \cup \{\gamma \in h(y) : \gamma \le \alpha\}$

$= h_\alpha^-(x) \cup h_\alpha^-(y)$ .Then $h_\alpha^-(xy) = h_\alpha^-(x) \cup h_\alpha^-(y)$.

Thus $h_\alpha^- \in$ HFPI (R) .

Proposition 4.4 .

Let $h \in$ HFPI (R) . Then $h^c \in$ HFPI (R)

Proof:- Suppose $h \in$ HFPI (R) , x , y ∈ R

$h^c(xy) = \{1-\gamma / \gamma \in h(xy)\} = \{1-\gamma/\gamma \in h(x) \cup h(y)\}$

$= \{1-\gamma/ \gamma \in h(x) \vee \gamma \in h(y)\} = \{1-\gamma/\gamma \in h(x)\} \cup \{1-\gamma/\gamma \in h(y)$

$= \{1-\gamma/\gamma \in h(x)\} \cup \{1-\gamma/\gamma \in h(y)\} = h^c(x) \cup h^c(y)$

Thus $h^c(xy) = h^c(x) \cup h^c(y)$.Thus $h^c \in$ HFPI (R) .

Theorem 4.5.

Let $f: R \to R^\star$ be a homomorphism of rings .If f is onto and $h_R \in$ HFPI (R) , then $f(h_R) \in$ HFPI $(R^\star)$.

Proof:- Let $x_\alpha, y_\beta \in HFP(R^\star)$ such that $x_\alpha y_\beta \in f(h_R)$ since f onto homomorphism

So that there exists a_α,$b_\beta \in$ HFP(R) such that $f(a_\alpha) = x_\alpha$, $f(b_\beta) = y_\beta$

Thus $f(a_\alpha) f(b_\beta) \in f(h_R)$ this implies $f(a_\alpha b_\beta) \in f(h_R)$ which means

$a_\alpha b_\beta \in h_R$ implies either $a_\alpha \in h_R$ or $b_\beta \in h_R$.

If $a_\alpha \in h_R$ this implies $f(a_\alpha) \in f(h_R)$, then $x_\alpha \in f(h_R)$.

Or

If $b_\alpha \in h_R$ this implies $f(b_\alpha) \in f(h_R)$, then $y_\alpha \in f(h_R)$.

Thus $x_\alpha y_\beta \in f(h_R)$ implies either $x_\alpha \in f(h_R)$ or $y_\alpha \in f(h_R)$.

Theorem 4.6.

Let $f: R \to R^{\star}$ be a homomorphism of rings . If $h_{R^{\star}} \in HFPI(R^{\star})$, then $f^{-1}(h_{R^{\star}}) \in HFPI(R)$.

Proof:- Let $x_{\alpha}, y_{\beta} \in HFP(R)$ and $x_{\alpha}y_{\beta} \in f^{-1}(h_{R^{\star}})$.

This implies that $f(x_{\alpha}y_{\beta}) \in f\,f^{-1}(h_{R^{\star}}) = h_{R^{\star}}$ so that $f(x_{\alpha})f(y_{\beta}) \in h_{R^{\star}}$

which means $(f(x))_{\alpha}(f(y))_{\beta} \in h_{R^{\star}}$.

Since $h_{R^{\star}}$is hesitant fuzzy prime ideal of a ring $R^{\star}$.

Implies either $(f(x))_{\alpha} \in h_{R^{\star}}$, then $f(x_{\alpha}) \in h_{R^{\star}}$ so $x_{\alpha} \in f^{-1}(h_{R^{\star}})$ or $(f(y))_{\alpha} \in h_{R^{\star}}$,then $f(y_{\alpha}) \in h_{R^{\star}}$ so $y_{\alpha} \in f^{-1}(h_{R^{\star}})$

Thus $x_{\alpha}y_{\beta} \in f^{-1}(h_{R^{\star}})$implies either $x_{\alpha} \in f^{-1}(h_{R^{\star}})$ or $y_{\alpha} \in f^{-1}(h_{R^{\star}})$

Thus $f^{-1}(h_{R^{\star}}) \in HFPI(R)$.

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