Nonsplit Neighbourhood Tree Domination Number In Connected Graphs

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**Abstract:** Let \( G = (V, E) \) be a connected graph. A subset \( D \) of \( V \) is called a dominating set of \( G \) if \( N[D] = V \). The minimum cardinality of a dominating set of \( G \) is called the domination number of \( G \) and is denoted by \( \gamma(G) \). A dominating set \( D \) of a graph \( G \) is called a tree dominating set (t-r set) if the induced subgraph \( (D) \) is a tree. The tree domination number \( \gamma_t(G) \) of \( G \) is the minimum cardinality of a tree dominating set. A tree dominating set \( D \) of a graph \( G \) is called a neighbourhood tree dominating set (ntd - set) if the induced subgraph \( (N[D]) \) is a tree. The neighbourhood tree domination number \( \gamma_{ntd}(G) \) of \( G \) is the minimum cardinality of a tree dominating set. A tree dominating set \( D \) of a graph \( G \) is called a nonsplit tree dominating set (nstd - set) if the induced subgraph \( (V - D) \) is connected. The nonsplit tree domination number \( \gamma_{nstd}(G) \) of \( G \) is the minimum cardinality of a nonsplit tree dominating set. A neighbourhood tree dominating set \( D \) of a graph \( G \) is called a nonsplit neighbourhood dominating set (nstd - set) if the induced subgraph \( (V(G) - D) \) is connected. The nonsplit neighbourhood tree domination number \( \gamma_{nstd}(G) \) of \( G \) is the minimum cardinality of a nonsplit neighbourhood tree dominating set of \( G \). In this paper, bounds for \( \gamma_{nstd}(G) \) and its exact values for some particular classes of graphs and cartesian product of some standard graphs are found.

**Keywords:** Domination number, connected domination number, tree domination number, neighbourhood tree domination number, nonsplit domination number.

**Mathematics Subject Classification:** 05C69

1. INTRODUCTION

The graphs considered here are nontrivial, finite and undirected. The order and size of \( G \) are denoted by \( n \) and \( m \) respectively. If \( D \subseteq V \), then \( N(D) = \bigcup_{v \in D} N(v) \) and \( N[D] = N(D) \cup D \) where \( N(v) \) is the set of vertices of \( G \) which are adjacent to \( v \). The concept of domination in graphs was introduced by Ore[13]. A subset \( D \) of \( V \) is called a dominating set of \( G \) if \( N[D] = V \). The minimum cardinality of a dominating set of \( G \) is called the domination number of \( G \) and is denoted by \( \gamma(G) \). Xuegang Chen, Liang Sun and Alice McRac [14] introduced the concept of tree domination in graphs. A dominating set \( D \) of \( G \) is called a tree dominating set, if the induced subgraph \( (D) \) is a tree. The minimum cardinality of a tree dominating set of \( G \) is the tree domination number of \( G \). Kulli and Janakiram [8, 9] introduced the concept of split and nonsplit domination in graphs.

A dominating set \( D \) of a graph \( G \) is called a nonsplit dominating set if the induced subgraph \( (V - D) \) is connected. The nonsplit domination number \( \gamma_{nstd}(G) \) of \( G \) is the minimum cardinality of a nonsplit dominating set. Muthammai and Chitiravalli [11, 12] defined the concept of split and nonsplit tree domination in graphs. A tree dominating set \( D \) of a graph \( G \) is called a nonsplit tree dominating set if the induced subgraph \( (V - D) \) is connected. The nonsplit tree domination number \( \gamma_{nstd}(G) \) of \( G \) is the minimum cardinality of a nonsplit tree dominating set.

V.R. Kulli introduced the concepts of split and nonsplit neighbourhood connected domination in graph.

A neighbourhood dominating set \( D \) of a graph \( G \) is called a nonsplit neighbourhood dominating set if the induced subgraph \( (V - D) \) is connected. The nonsplit neighbourhood domination number \( \gamma_{nnd}(G) \) of \( G \) is the minimum cardinality of a nonsplit neighbourhood dominating set.

The Cartesian product of two graphs \( G_1 \) and \( G_2 \) is the graph, denoted by \( G_1 \times G_2 \) with \( V(G_1 \times G_2) = V(G_1) \times V(G_2) \) (where \( x \) denotes the Cartesian product of sets) and two vertices \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \) in \( V(G_1 \times G_2) \) are adjacent in \( G_1 \times G_2 \) whenever \( [u_1 = v_1 \text{ and } (u_2, v_2) \in E(G_2)] \) or \( [u_2 = v_2 \text{ and } (u_1, v_1) \in E(G_1)] \).

In this paper, bounds for \( \gamma_{nstd}(G) \) and its exact values for some particular classes of graphs and cartesian product of some standard graphs are found.
2. PRIOR RESULTS

Theorem 2.1: [2] For any graph G, \( \kappa(G) \leq \delta(G) \).

Theorem 2.2: [14] For any connected graph G with \( n \geq 3 \), \( \gamma_{nt}(G) \leq n - 2 \).

Theorem 2.3: [14] For any connected graph G with \( \gamma_t(G) = n - 2 \) iff G \( \cong P_n \) (or) \( C_n \).

Theorem 2.4: [11] For any connected graph G, \( \gamma(G) \leq \gamma_{nt}(G) \).

Theorem 2.5: [11] For any connected graph G with n vertices, \( \gamma_{ntd}(G) = \gamma(G) \) if and only if G \( \cong H + K_1 \), where H is a connected graph with \( (n - 1) \) vertices.

Theorem 2.6: [11] For any graph G, \( \gamma(G) \leq \gamma_{ntd}(G) \).

Theorem 2.7: [11] For any cycle \( C_n \) on n vertices, \( \gamma_{ntd}(C_n) = n - 2, n \geq 3 \).

Theorem 2.8: [9] For any connected graph G, \( \gamma_{tad}(G) \leq n - 1 \). Further equality holds if and only if G is a star.

3. MAIN RESULTS

In this section, non-split neighbourhood tree domination number is defined and studied.

3.1. Non-split Neighbourhood Tree Domination Number in Connected Graphs

**Definition 3.1.1:**

A neighbourhood tree dominating set \( D \) of G is called a non-split neighbourhood tree dominating set, if the induced subgraph \( \langle V(G) - D \rangle \) is connected. The non-split neighbourhood tree domination number \( \gamma_{nsntr}(G) \) of G is the minimum cardinality of a non-split neighbourhood tree dominating set of G.

Not all connected graphs have a non-split neighbourhood tree dominating set. For example, the Path \( P_n \) (\( n > 5 \)) has a neighbourhood tree dominating set, but no non-split neighbourhood tree dominating set.

If the non-split neighbourhood tree domination number does not exist for a given connected graph G, then \( \gamma_{nsntr}(G) \) is defined to be zero.

**Example 3.1.1:**

![Figure 3.1](image1)

In the graph given in Figure 3.1, \( D = \{v_4, v_5, v_7\} \) is a minimum non-split neighbourhood tree dominating set and the induced subgraph \( \langle N(D) \rangle \cong P_4 = \{v_3, v_2, v_6, v_1\} \) is a tree and \( \langle V(G) - D \rangle \) is connected and \( \gamma_{nsntr}(G) = 3 \).

**Remark 3.1.1:**

Since \( \langle V(G) - D \rangle \) is connected for any \( \gamma_{nsntr} \) - set D of a connected graph G, \( \left| V(G) - D \right| \geq 1 \).

**Example 3.1.2**

![Figure 3.2](image2)

In the graph G given in Figure 3.2, \( D = \{v_2, v_4, v_5, v_7\} \) is a minimum dominating set and the induced subgraph \( \langle N(D) \rangle \) is a tree, but \( \langle V(G) - D \rangle \) is disconnected.

**Remark 3.1.2:**

Every non-split neighbourhood tree dominating set is a dominating set and also a neighbourhood tree dominating set. Therefore, \( \gamma(G) \leq \gamma_{ntd}(G) \leq \gamma_{nsntr}(G) \). Therefore, for any nontrivial connected graph G, \( \gamma_{ntd}(G) = \min \{\gamma_{ntd}(G), \gamma_{nsntr}(G)\} \).

These are illustrated below.
Example 3.1.3:

In Figure 3.3(a), $D_1 = \{v_1, v_5\}$ is a minimum nonsplit neighbourhood tree dominating set.

In Figure 3.3(b), $D_2 = \{v_1\}$ is a minimum nonsplit neighbourhood tree dominating set.

In Figure 3.3(c), $D_3 = \{v_2, v_3, v_4\}$ is a minimum nonsplit neighbourhood tree dominating set.

Example 3.1.4:

In Figure 3.4, $D_1 = \{v_3, v_4, v_7, v_9\}$ is a minimum nonsplit neighbourhood tree dominating set.

$D_2 = \{v_1, v_2, v_5, v_6, v_7, v_8, v_9\}$ is a nonsplit neighbourhood tree dominating set and $\gamma(G) = 2, \gamma_{ntr}(G) = 3, \gamma_{sntr}(G) = 4, \gamma_{nsntr}(G) = 7$. Here, $\gamma(G) < \gamma_{ntr}(G), \gamma(G) < \gamma_{nsntr}(G), \gamma_{ntr}(G) < \gamma_{nsntr}(G)$.

Example 3.1.5:

In Figure 3.5, $H$ is a spanning subgraph of a connected graph $G$. $D_1 = \{v_3, v_5\}$ is a minimum non-split neighbourhood tree dominating set of $G$ and $\gamma_{ntr}(G) = 2$. The set $D_2 = \{v_3, v_5, v_6, v_7\}$ is a non-split neighbourhood tree dominating set of $H$ and $\gamma_{ntr}(H) = 4$.

Therefore, $\gamma_{ntr}(G) < \gamma_{ntr}(H)$. 
Example 3.1.6:

![Diagram](image)

In Figure 3.6., H is a spanning subgraph of G and \{v_3, v_4\} is a minimum nonsplit neighbourhood tree dominating set of G, \(\gamma_{nnsnt}(G) = 2\). The set \(\{v_1, v_4\}\) is a minimum nonsplit neighbourhood tree dominating set of H and \(\gamma_{nnsnt}(H) = 2\). Therefore, \(\gamma_{nnsnt}(G) = \gamma_{nnsnt}(H)\).

In the following, the exact values of \(\gamma_{nnsnt}(G)\) for some standard graphs are given.

(a) For any path \(P_n\) on \(n\) vertices, \(\gamma_{nnsnt}(P_n) = n - 2\), \(n \geq 4\).

(b) If G is a spider, then \(\gamma_{nnsnt}(G) = n + 1\).

(c) If G is a wounded spider, then \(\gamma_{nnsnt}(G) = p + 1\), where \(p\) is the number of pendant vertices which are adjacent to nonwounded legs.

(d) For any triangular cactus graph \(T_p\) whose blocks are \(p\) triangles with \(p \geq 1\), \(\gamma_{nnsnt}(T_p) = \frac{p}{2}\) where \(p > 2\) and \(p\) is odd.

(e) If \(S_{m,n}\), \((1 \leq m \leq n)\) is a double star, then \(\gamma_{nnsnt}(S_{m,n}) = m + n\).

Theorem 3.1.1:

If T is a tree which is not a star, then \(\gamma_{nnsnt}(T) \leq n - 2\).

Proof:

Suppose T is not a star. Then T has two adjacent cut vertices u and v, such that \(\deg u, \deg v \geq 2\). This implies that \(D = \{V - \{u, v\}\}\) is a nonsplit neighbourhood tree dominating set of T. Therefore, \(\gamma_{nnsnt}(T) \leq |D| = |V(T) - \{u, v\}| = n - 2\).

3.2. Nonsplit Neighbourhood Tree Domination Number of Cartesian product of Graphs

In this section, nonsplit neighbourhood tree domination numbers of \(P_2 \times C_n, P_3 \times C_n, P_2 \times P_n, P_3 \times P_n\) are found.

Theorem 3.2.1:

For the graph \(P_2 \times P_n\) (\(n \geq 5, n\) is odd), \(\gamma_{nnsnt}(P_2 \times P_n) = \left\lceil \frac{n}{2} \right\rceil\).

Proof:

Let \(G \cong P_2 \times P_n\) and let \(V(G) = \bigcup_{i=1}^{n} \{v_{ij}, v_{ij+1}\} \) where \(\{v_{ij}, v_{ij+1}\} \cong P_2, i = 1, 2\) and \(\{v_{1j}, v_{2j}, ..., v_{nj}\} \cong P_2, j = 1, 2, ..., n\) and \(P_2^i\) is the \(i\)th copy of \(P_2\) and \(P_2^j\) is the \(j\)th copy of \(P_2\) in \(G\).

Let \(D = \bigcup_{i=1}^{\lfloor \frac{n-3}{4} \rfloor + 1} \{v_{4i-1,1}\} \cup \bigcup_{i=1}^{\lfloor \frac{n-1}{4} \rfloor + 1} \{v_{4i-3,2}\}\). Then \(D \subseteq V(G)\). Here, \(v_{n1}\) and \(v_{n2}\) are adjacent to \(v_{12}\) and \(v_{n1}\) and \(v_{n-1,2}\) are adjacent to \(v_{22}\) and \(v_{2n+1,2}\) is adjacent to \(v_{2n+1,1}\) (\(i \geq 1\)).

Therefore, \(D\) is a dominating set of \(G\) and \(\langle D \rangle \subseteq P_{3n-1,1}\). Since \(\langle D \rangle\) is a tree and \(\langle V(G) - D \rangle\) is connected, \(D\) is a nonsplit neighbourhood tree dominating set of \(G\) and is minimum.

Hence \(\gamma_{nnsnt}(G) = |D| = \left\lceil \frac{n}{2} \right\rceil\).

Remark 3.2.1:

\(\gamma_{nnsnt}(P_2 \times P_3) = 2\), the set \(\{v_{31}, v_{12}\}\) is a minimum nonsplit neighbourhood tree dominating set of \(P_2 \times P_3\), where \(v_{21}, v_{22}\) are the vertices of degree 3 in \(P_2 \times P_3\).
Example 3.2.1:

In the graph $P_2 \times P_3$ given in Figure 3.7, minimum non-split neighbourhood tree dominating set is $D = \{v_{11}, v_{32}\}$, where $(N(D)) \equiv P_4$ and $\gamma_{nssntr}(P_2 \times P_3) = 2$.

Theorem 3.2.2:

For the graph $P_3 \times P_n$ $(n \geq 3)$, $\gamma_{nssntr}(P_3 \times P_n) = n$.

Proof:

Let $G \cong P_3 \times P_n$ and let $V(G) = \bigcup_{i=1}^{n} \{v_{i1}, v_{i2}, v_{i3}\}$ where $\langle \{v_{i1}, v_{i2}, v_{i3}\} \rangle \equiv P_{i3}$, $i = 1, 2, 3$ and $\langle \{v_{i1}, v_{i2}, \ldots, v_{in}\} \rangle \equiv P_{i}$, $j = 1, 2, \ldots, n$ and $P_{i}$ is the $i$th copy of $P_3$ and $P_{i}$ is the $j$th copy of $P_n$ in $G$.

Let $D = \bigcup_{i=1}^{n} \{v_{2i,3}\} \cup \bigcup_{i=1}^{n} \{v_{2i-1,1}\}$. Then $D \subseteq V(G)$. Here, $v_{2i,2}$ is adjacent to $v_{2i,3}$ $(i \geq 1)$ and $v_{2i-1,2}$ is adjacent to $v_{2i-1,1}$ $(i \geq 1)$. Therefore, $D$ is a dominating set of $G$ and $\gamma_{nssntr}(G) = P_2 \circ P_1$. Since $\langle N(D) \rangle$ is a tree and $\langle V - D \rangle$ is connected, $D$ is a non-split neighbourhood tree dominating set of $G$ and is minimum.

Hence $\gamma_{nssntr}(G) = |D| = n$.

Example 3.2.2:

In the graph $P_2 \times C_n$ $(n = 3)$, $\gamma_{nssntr}(P_2 \times C_n) = 2$.

Theorem 3.2.3:

For the graph $P_2 \times C_n$ $(n \geq 3)$, $\gamma_{nssntr}(P_2 \times C_n) = 2$.

Proof:

Let $G \cong P_2 \times C_n$ and let $V(G) = \bigcup_{i=1}^{n} \{v_{i1}, v_{i2}\}$, where $\langle \{v_{i1}, v_{i2}\} \rangle \equiv P_{i2}$, $i = 1, 2$ and $\langle \{v_{i1}, v_{i2}, \ldots, v_{jn}\} \rangle \equiv C_{n}$, $j = 1, 2, \ldots, n$ and $P_{i}$ is the $i$th copy of $P_2$ and $C_{n}$ is the $j$th copy of $C_n$ in $G$.

Let $D = \{v_{31}, v_{22}\}$. Then $D \subseteq V(G)$. Here, $v_{11}, v_{21}$ are adjacent to $v_{31}$ and $v_{12}, v_{22}$ are adjacent to $v_{22}$. Therefore, $D$ is a dominating set of $G$ and $\gamma_{nssntr}(G) = P_2 \circ P_1$. Since $\langle N(D) \rangle$ is a tree and $\langle V(G) - D \rangle$ is connected, $D$ is a non-split neighbourhood tree dominating set of $G$ and $\gamma_{nssntr}(G) \leq |D| = 2$.

Let $D'$ be a non-split neighbourhood tree dominating set of $P_2 \times C_n$. Since $\gamma(P_3 \times C_3) = \left\lfloor \frac{3n}{2} \right\rfloor = 2$ and $\gamma_{nssntr}(G) \geq \gamma(G)$ and $\gamma_{nssntr}(G) \geq \gamma_{nssntr}(G)$. Therefore, $\gamma_{nssntr}(G) = 2$. 

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Example 3.2.3:

In the graph $P_2 \times C_3$ given in Figure 3.9, minimum nonsplit neighbourhood tree dominating set is $D = \{v_{31}, v_{22}\}$, where $(N(D)) \equiv P_4$ and $\gamma_{nsntr}(P_2 \times C_3) = 2$.

Remark 3.2.2:

For $n \geq 4$, $\gamma_{ns}(P_2 \times C_n) = 0$, since there exists no nonsplit neighbourhood tree dominating set of $P_2 \times C_n$. Let $D$ be a dominating set of $P_2 \times C_n$. If $D$ contains two vertices, then either $(N(D))$ is not a tree or $(N(D))$ contains a cycle. If $D$ contains at least three vertices, then $(N(D))$ contains a cycle.

Theorem 3.2.4:

For the graph $P_3 \times C_n$ ($n = 3$), $\gamma_{nsntr}(P_3 \times C_n) = 3$.

Proof:

Let $G \equiv P_3 \times C_n$, $n \geq 4$ and let $V(G) = \bigcup_{i=1}^{n} \{v_{ij}, v_{i2}, v_{i3}\}$ such that $(\{v_{ij}, v_{i2}, v_{i3}\}) \equiv P_3^i$, $i = 1, 2, 3$ and $(\{v_{ij}, v_{j2}, ..., v_{mn}\}) \equiv C_n^j$, $j = 1, 2, ..., n$ where $P_3^i$ is the $i^{th}$ copy of $P_3$ and $C_n^j$ is the $j^{th}$ copy of $C_n$ in $G$.

Let $D = \{v_{31}, v_{12}, v_{33}\}$. Then $D \subseteq V(G)$. Here, $v_{22}$ is adjacent to $v_{21}$ and $v_{11}$, $v_{21}$, $v_{22}$ are adjacent to $v_{31}$ and $v_{32}$, $v_{13}$, $v_{23}$ are adjacent to $v_{33}$. Therefore, $D$ is a dominating set of $G$ and $(N(D))$ is a connected graph obtained from $P_3$ by attaching a pendant edge at $v_{22}$. Since $(N(D))$ is a tree and $(V(G) - D)$ is connected, $D$ is a nonsplit neighbourhood tree dominating set of $G$ and $\gamma_{nsntr}(G) \leq |D| = 3$.

Let $D'$ be a nonsplit neighbourhood tree dominating set of $P_3 \times C_n$. Since $\gamma(P_3 \times C_3) = \left\lceil \frac{3n}{4} \right\rceil = 3$ and $\gamma_{nsntr}(G) \geq \gamma(G)$ and $\gamma_{nsntr}(G) \geq \gamma_{ns}(G)$. Therefore, $\gamma_{ns}(G) = 3$.

Example 3.2.4:

In the graph $P_3 \times C_3$ given in Figure 3.10, minimum nonsplit neighbourhood tree dominating set is $D = \{v_{21}, v_{32}, v_{13}\}$, where $(N(D)) \equiv P_6$ and $\gamma_{nsntr}(P_3 \times C_3) = 3$.

Remark 3.2.3:

For $n \geq 4$, $\gamma_{ns}(P_3 \times C_n) = 0$, since there exists no nonsplit neighbourhood tree dominating set of $P_3 \times C_n$. If a dominating set $D$ of $P_3 \times C_n$ contains at least three vertices, then the induced subgraph $(N(D))$ contains a cycle.

Theorem 3.2.5:

For the graph $P_4 \times C_n$ ($n = 3$), $\gamma_{nsntr}(P_4 \times C_n) = 4$. 
Proof:

Let \( G \equiv P_4 \times C_n \), \( n \geq 6 \) and let \( V(G) = \bigcup_{i=1}^{n} \{v_{1i}, v_{2i}, v_{3i}, v_{4i}\} \) such that \( \langle \{v_{1i}, v_{2i}, v_{3i}, v_{4i}\} \rangle \equiv P_i \), \( i = 1, 2, 3, 4 \) and \( \langle \{v_{1j}, v_{2j}, \ldots, v_{nj}\} \rangle \equiv C_n \), \( j = 1, 2, \ldots, n \), where \( P_i \) is the \( i \)th copy of \( P_4 \) and \( C_n \) is the \( j \)th copy of \( C_n \) in \( G \).

Let \( D = \{v_{31}, v_{22}, v_{13}, v_{34}\} \). Then \( D \subseteq V(G) \). Here, \( v_{11}, v_{21}, v_{32} \) are adjacent to \( v_{31} \), and \( v_{12}, v_{23}, v_{33} \) are adjacent to \( v_{32} \). Therefore, \( D \) is a dominating set of \( G \) and \( \langle N(D) \rangle \rangle \equiv P_8 \). Since \( \langle N(D) \rangle \) is a tree and \( \langle V(G) \rangle \rangle \) is connected, \( D \) is a neighbourhood tree dominating set of \( G \) and \( \gamma_{nt}(G) \leq |D| = 4 \).

Let \( D' \) be a non-split dominating set of \( P_3 \times C_n \).

Since \( \gamma(P_4 \times C_3) = 3n + 1 = 4 \) and \( \gamma_{nt}(G) \geq \gamma(G) \) and \( \gamma_{nt}(G) \geq \gamma_{nt}(G) \). Therefore, \( \gamma_{nt}(G) = 4 \).

**Example 3.2.5:**

In the graph \( P_4 \times C_3 \) given in Figure 3.11, minimum non-split neighbourhood tree dominating set is \( D = \{v_{31}, v_{22}, v_{13}, v_{34}\} \), where \( \langle N(D) \rangle \rangle \equiv P_8 \), and \( \gamma_{nt}(P_4 \times C_3) = 4 \).

**Remark 3.2.4:**

For \( n \geq 4 \), \( \gamma_{nt}(P_4 \times C_n) = 0 \), since there exists no neighbourhood tree dominating set of \( P_4 \times C_n \). The graph \( P_4 \times C_3 \) can be divided into two blocks \( P_2 \times C_4 \) and \( P_2 \times C_4 \). \( \gamma_{nt}(P_2 \times C_4) = 0 \). If a dominating set \( D \) of \( P_4 \times C_3 \) contains three vertices, then \( \langle N(D) \rangle \rangle \) contains a cycle.

**Remark 3.2.5:**

For \( n \geq 2 \), \( \gamma_{nt}(P_n \times C_3) = n \).

**Reference**