Eccentric Domination in Boolean Graph $BG_2(G)$ of a Graph G

M. Bhanumathi[1] and RM. Mariselvi[2]
[1]Principal (Retd.), Government Arts College for Women, Sivagangai
[2]Government Arts College for Women, Pudukkottai – 622001
(Affiliated to Bharathidasan University), Tamilnadu, India.
bhanu_ksp@yahoo.com[1], rmselvi0384@gmail.com[2]

Article History: Received: 11 January 2021; Revised: 12 February 2021; Accepted: 27 March 2021; Published online: 20 April 2021

Abstract: Let G be a simple (p, q) graph with vertex set V(G) and edge set E(G). $BG_2(G)$ is a graph with vertex set V(G) ∪ E(G) and two vertices are adjacent if and only if they correspond to two adjacent vertices of G, a vertex and an edge incident to it in G or two non-adjacent edges of G. In this paper, we studied eccentric domination number of Boolean graph $BG_2(G)$, obtained bounds of this parameter and determined its exact value for several classes of graphs.

Keywords: Domination number, eccentric domination number, Boolean graph.

1.Introduction

Let G be a finite simple, undirected graph on p vertices and q edges with vertex set V(G) and edge set E(G). For graph theoretic terminology refer to Harary[11], and Kulli[17].

The distance d(u, v) between two vertices u and v in G is the minimum length of a path joining them if any; otherwise d(u, v) = ∞. Let G be a connected graph and u be a vertex of G. The eccentricity e(v) of v is the distance to a vertex farthest from v. Thus, e(v) = max{d(u, v): u ∈ V}.The radius r(G) is the minimum eccentricity of the vertices, whereas the diameter diam(G) is the maximum eccentricity. For any connected graph G, r(G) ≤ diam(G) ≤ 2r(G). The vertex v is a central vertex if e(v) = r(G). The center C(G) is the set of all central vertices. The central sub graph ⟨C(G)⟩ of a graph G is the subgraph induced by the center. The vertex v is a peripheral vertex if e(v) = diam(G). The periphery P(G) is the set of all peripheral vertices. For a vertex v, each vertex at a distance e(v) from v is an eccentric vertex. Eccentric set of a vertex v is defined as E(v) = {u ∈ V(G) : d(u, v) = e(v)}. A graph is self-centered if every vertex is in the center. Thus, in a self-centered graph G all vertices have the same eccentricity, so r(G) = diam(G).

A vertex and an edge are said to cover each other if they are incident. A set of vertices which covers all the edges of a graph G is called a point cover for G, while a set of edges which covers all the vertices is a line cover. The smallest number of vertices in any point cover for G is called its point covering number or simply covering number and is denoted by $\alpha_0(G)$ or α_0. Similarly, α_1 is the smallest number of edges in any line cover of G and is called its line cover number. A set of vertices in G is independent if no two of them are adjacent. The largest number of vertices in such a set is called the point independence number of G and is denoted by $\beta_0(G)$ or β_0. A set of edges in a graph is independent if no two edges in the set are adjacent. By a matching in a graph G, we mean an independent set of edges in G. The edge independence number $\beta_1(G)$ of a graph G is a maximum cardinality of an independent set of edges. A perfect matching is a matching with every vertex of the graph is incident to exactly one edge of the matching. The graph G^+ is obtained from the graph G by attaching a pendant edge to each of the vertices of G.

The open neighborhood N(v) of a vertex v is the set of all vertices adjacent to v in G. N[v] = N(v) ∪ {v} is called the closed neighborhood of v. The second neighborhood $N_2(v)$ of a vertex v is the set of all vertices at distance two from v in G.

In 2007, Janakiraman, Bhanumathi and Muthammai defined the Boolean graph $BG_2(G)$ and studied its properties [12, 14, 15, 16]. Boolean graph $BG_2(G)$ is a graph with vertex set V(G) ∪ E(G) and edge set {E(T(G)) – E(L(G))} ∪ E($\overline{L(G)}$), where L(G) is the line graph of G and T(G) is the total graph of G. It is a graph with vertex set V(G) ∪ E(G) and two vertices are adjacent if and only if they correspond to two adjacent vertices of G, a vertex and an edge incident to it in G or two non-adjacent edges of G.

The concept of domination in graphs was introduced by Ore [18]. A set D ⊆ V(G) is said to be a dominating set of G, if every vertex in V(G) – D is adjacent to some vertex in D. D is said to be a minimal dominating set if D – {u} is not a dominating set for any u ∈ D. The domination number γ(G) of G is the minimum cardinality of a dominating set [10].

Janakiraman, Bhanumathi and Muthammai [13] introduced the concept of eccentric domination number of a graph. Eccentric domination in trees and various types of eccentric dominations were studied in [2, 3, 4, 5, 6, 7, 8, 9].

A set $D \subseteq V(G)$ is an eccentric dominating set if D is a dominating set of G and for every $v \in V - D$, there exists at least one eccentric point of v in D. The eccentric domination number $\gamma_{ed}(G)$ of a graph G equals the minimum cardinality of an eccentric dominating set. Obviously, $\gamma(G) \leq \gamma_{ed}(G)$.

Theorem 1.1[15]: Let G be a connected graph. Then, $\gamma(G) \leq \gamma(BG_2(G)) \leq \gamma(G) + 2$.

Theorem 1.2[16]: (i) Eccentricity of every line vertex is two in $BG_2(G)$ if $G \neq K_2$.

(ii) If $G = K_2$, $BG_2(G)$ is C_3.

Theorem 1.3[16]: Eccentricity of every point vertex in $BG_2(G)$ is 1, 2 or 3.

Theorem 1.4[16]: (i) Radius of $BG_2(G)$ is one if and only if $G = K_{1,n}$, $n \geq 1$.

(ii) $BG_2(G)$ is self-centered with radius two if and only if $G \neq K_{1,n}$ and $diam(G) \leq 2$.

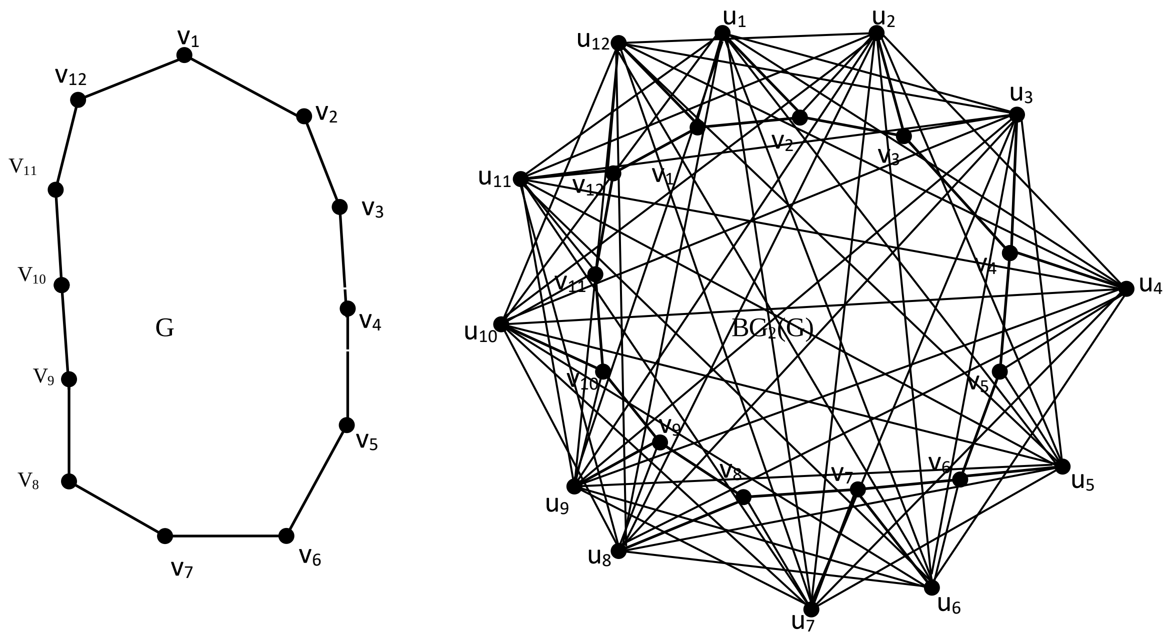
(iii) $BG_2(G)$ is bi-eccentric with diameter three if and only if $diam(G) \geq 3$.

2. Eccentric Domination in Boolean Graph $BG_2(G)$ of a Graph G

In this section, study of eccentric domination in Boolean graph $BG_2(G)$ is initiated and some bounds for $\gamma_{ed}(BG_2(G))$ are obtained. We have $\gamma(G) \leq \gamma_{ed}(G)$ for any graph G. Hence, $\gamma(BG_2(G)) \leq \gamma_{ed}(BG_2(G))$. Also, $\gamma(G) \leq \gamma(BG_2(G))$ by Theorem 1.1. Thus, $\gamma(G) \leq \gamma_{ed}(BG_2(G))$.

But $\gamma_{ed}(G) \leq \gamma_{ed}(BG_2(G))$ is not true.

Example 2.1



Here, $\gamma_{ed}(G) = 6$ and $\gamma_{ed}(BG_2(G)) = 5$.

Theorem 2.1 Let G be a graph without isolated vertices. Set of all point vertices is an eccentric dominating set of $BG_2(G)$; and hence $1 \leq \gamma_{ed}(BG_2(G)) \leq p$.

Proof: Distance from a line vertex to point vertices is one or two. Also, distance from a point vertex to line vertices is also one or two. So if G has more than one edge, then V(G) is an eccentric dominating set of $BG_2(G)$. Hence, $1 \leq \gamma_{ed}(BG_2(G)) \leq p$.

Corollary 2.1 The bounds are sharp, since $\gamma_{ed}(BG_2(G)) = 1$ if and only if $G = P_2$ and $\gamma_{ed}(BG_2(G)) = p$ if and only if $G = \overline{K_n}$.

Theorem 2.2 If G is unicentral tree of radius 2, then $\gamma_{ed}(BG_2(G)) \leq p - deg_G(u)$, where u is a central vertex.

Proof: If G is of radius two with unique central vertex u, then in, $BG_2(G)$, $r(BG_2(G)) = 2$ and $V - N_G(u)$ dominates all point vertices and line vertices of $BG_2(G)$. Each vertex of $V(BG_2(G)) - N_G(u)$ has their eccentric vertices in $V(G) - N_G(u)$ only. Therefore, $V(G) - N_G(u)$ is an eccentric dominating set of $BG_2(G)$. Hence, $\gamma_{ed}(BG_2(G)) \leq p - deg_G(u)$.

Theorem 2.3 For a bi-central tree T with radius 2, $\gamma_{ed}(BG_2(G)) \leq 4$.

Proof: Let u and v be the central vertices of G. In $BG_2(G)$, $N_G(u)$ and $N_G(v)$ are dominating set of $BG_2(G)$. Let x, y be the any two peripheral vertices at distance atmost 3 in $BG_2(G)$. S = {x, y, u, v} is an eccentric dominating set of $BG_2(G)$. Hence, $\gamma_{ed}(BG_2(G)) \le 4$.

Theorem 2.4 If G is a tree T, $\gamma_{ed}(BG_2(G)) \le p - \Delta(G) + 2$.

Proof: If G a has vertex v of maximum degree which is not a support, then $V(G) - N_G(u)$ is an eccentric dominating set of $BG_2(G)$. Hence, $\gamma_{ed}(BG_2(G)) \le p - \Delta(G)$. If G has a vertex v of maximum degree which is a support of pendant vertices, then in $BG_2(G)$, let $S = V(G) - N_G(u) \cup \{x, y\}$, where x, y are peripheral vertices of G. This S is an eccentric dominating set of $BG_2(G)$. Hence, $\gamma_{ed}(BG_2(G)) \le p - \Delta(G) + 2$.

Theorem 2.5 Let G be a tree, then $\gamma(BG_2(G)) \le \gamma_{ed}(BG_2(G)) \le \gamma(BG_2(G)) + 2$.

Proof: Let $S \subseteq V(BG_2(G))$ be a γ-set of $BG_2(G)$. Let $u, v \in V(G)$ such that u an v are peripheral vertices of G at distance = diam(G) to each other. Then u or v is an eccentric point of each vertices in G. Again u or v is an eccentric point of line vertices and point vertices in $BG_2(G)$. Therefore, $S = D \cup \{u, v\}$ is a γ_{ed}-set of $BG_2(G)$, where D is a dominating set of $BG_2(G)$. Hence, $\gamma_{ed}(BG_2(G)) \le \gamma(BG_2(G)) + 2$. Also, we know that $\gamma(G) \le \gamma_{ed}(G)$ for any graph G. Thus, $\gamma(BG_2(G)) \le \gamma_{ed}(BG_2(G)) \le \gamma(BG_2(G)) + 2$.

Corollary 2.5 Let G be a tree, then $\gamma(G) \le \gamma_{ed}(BG_2(G)) \le \gamma(G) + 4$.

Proof: Proof follows from Theorem 1.1 and Theorem 2.5.

Theorem 2.6 If G is of radius one and diameter two, then $\gamma_{ed}(BG_2(G)) \le 2 + \delta(G)$.

Proof: diam(G) = 2. Let $u \in V(G)$ with $deg_G u = \delta(G)$ and e(u) = 2 in G and let $uv = e \in E(G)$. In $BG_2(G)$, $diam(BG_2(G)) = 2$, $r(BG_2(G)) \le 2$ by Theorem 1.4. Consider $S = \{u, e\} \cup \{N(u)\}$. S dominates all the point vertices and $u \in S$ is eccentric to point vertices in V – S also. All the edges incident with u and elements of N(u) are dominated by u and vertices of N(u) in $BG_2(G)$. If an edge e_1 is in $N_2(u)$ in G, then it is dominated by e in $BG_2(G)$. Also, all line vertices not in S is eccentric to some vertices of S in $BG_2(G)$. Therefore, S is an eccentric dominating set for $BG_2(G)$. Hence, $\gamma_{ed}(BG_2(G)) \le 2 + \delta(G)$ and S is a connected eccentric dominating set for $BG_2(G)$.

Theorem 2.7 If $G \ne K_{1,n}$ is of radius one with a unique central vertex u, then $\gamma_{ed}(BG_2(G)) = 3$.

Proof: Let G be a graph with radius one with a unique central vertex u. In $BG_2(G)$, u dominates all point vertices and line vertices incident with u in G. Let $e = uv \in E(G)$. Now in $BG_2(G)$, Consider $S = \{u, v, e\} \subseteq V(BG_2(G))$. S is a dominating set of $BG_2(G)$. $BG_2(G)$ is two self-centered by Theorem 1.4. In $BG_2(G)$, the line vertex e is eccentric to all point vertices except u and v; u is eccentric to all line vertices which are not incident with u in G; v is eccentric to all line vertices which are not incident with v in G. Therefore, S is a minimum eccentric dominating set of $BG_2(G)$. Hence, $\gamma_{ed}(BG_2(G)) = 3$.

Theorem 2.8 If G is of radius two and diameter three and if G has a pendant vertex v of eccentricity three, then $\gamma_{ed}(BG_2(G)) \le \Delta(G) + 2$.

Proof: If G has a pendant vertex v of eccentricity three, then its support vertex u is of eccentricity two. In $BG_2(G)$, $N_G(u)$ dominates all point and line vertices. Therefore, $S = N_G(u) \cup \{v, e\}$, where uv = e is an eccentric dominating set of $BG_2(G)$. Hence, $BG_2(G) \le \Delta(G) + 2$.

Theorem 2.9 If G is a graph with radius two, diameter three, then $\gamma_{ed}(BG_2(G)) \le p - \Delta(G) + 2$.

Proof: Let $u \in V(G)$ with deg u = $\Delta(G)$. Since radius of G is two and diameter three, all the point vertices in $BG_2(G)$ has their eccentric vertices atmost at distance three from u. Also, eccentricity of line vertices in $BG_2(G)$ is two by Theorem 1.4. All the edges incident with u are dominated by u in $BG_2(G)$ and are also eccentric to a point vertex w, where $w \in N_2(u)$. Suppose e_1 is an edge in $\langle N(u)\rangle$, then e_1 is not dominated by (V – N(u)). Hence the following cases arise:

Case(i): If all the edges in $\langle N(u)\rangle$ are adjacent or incident at a vertex v, then $(V - N(u)) \cup \{v\}$ is an eccentric dominating set of $BG_2(G)$.

Case(ii): If all the edges in $\langle N(u)\rangle$ form a C_3, then $(V - N(u)) \cup \{v, e\}$ where, $v \in N(u)$ and e is an edge in $\langle N(u)\rangle$ form an eccentric dominating set of $BG_2(G)$.

Case(iii): If $\langle N(u)\rangle$ has atleast two non-adjacent edges e_1, e_2, then $(V - N(u)) \cup \{e_1, e_2\}$ form an eccentric dominating set of $BG_2(G)$.

Hence in all cases, $\gamma_{ed}(BG_2(G)) \le p - \Delta(G) + 2$.

Theorem 2.10 If G is a graph with radius greater than two, then $\gamma_{ed}(BG_2(G)) \le p - \Delta(G) + 1$.

Proof: In this case, $BG_2(G)$ is bi-eccentric with diameter 3 by Theorem1.4. Let $u \in V(G)$ such that deg u = $\Delta(G)$. Let $v \in V(G)$ such that v is an eccentric vertex of u. Let $e = vw \in E(G)$. Vertices in $V(BG_2(G)) - N_G(u)$ has their eccentric vertices in $V - N_G(u)$. Then $(V - N_G(u)) \cup \{e\}$ is an eccentric dominating set of $BG_2(G)$.

Hence, $\gamma_{ed}(BG_2(G)) \le p - \Delta(G) + 1$.

Theorem 2.11 If $G \ne K_{1,n}$, r(G) = 1, diam(G) = 2 and G has a pendant vertex, then $\gamma_{ed}(BG_2(G)) = 3 = \gamma_c(BG_2(G))$.

Proof: $G \neq K_{1,n}$. Consider a pendant vertex $u \in V(G)$ and let $v \in V(G)$ be its adjacent vertex in G, $e = uv \in E(G)$, v is a central vertex of G. Now in $BG_2(G)$, $S = \{u, v, e\}$ is an eccentric dominating set. Thus, $\gamma_{ed}(BG_2(G)) = 3 = \gamma_c(BG_2(G))$.

Theorem 2.12 Let G be a connected graph with $p \geq 3$. Then, $\gamma_{ed}(BG_2(G)) \leq \gamma_{ed}(G) + 2$.

Proof: Let $D \subseteq V(G)$ be an eccentric dominating set of G with cardinality $\gamma_{ed}(G)$. Let $u \in D$ be such that u is adjacent to $v \in V(G)$, $e = uv \in E(G)$. Consider $S = D \cup \{v, e\} \subseteq V(BG_2(G))$. The vertex v dominates incident edges in G and the edge e dominates non adjacent edges in G. All point vertices in $V(BG_2(G)) – S$ have their eccentric vertices in S. Also, the line vertices of $V(BG_2(G)) – S$ have u or v as eccentric vertices, since eccentricity of every line vertex is two in $BG_2(G)$. Therefore, S is an eccentric dominating set of $BG_2(G)$. Hence, $\gamma_{ed}(BG_2(G)) \leq \gamma_{ed}(G) + 2$.

Remark 2.1 $\gamma_{ed}(G) \leq \gamma_{ed}(BG_2(G))$ is not true. Refer Example 2.1.

Theorem 2.13 Let G be a graph with diam(G) = 2. If there exists a vertex $v \in V(G)$ such that $\langle N_2(v)\rangle$ is totally disconnected, then $\gamma_{ed}(BG_2(G)) \leq \Delta(G) + 2$.

Proof: Let $v \in V(G)$ be such that $\langle N_2(v)\rangle$ is totally disconnected Let $S = N(v) \cup \{u, w\}$, where $u, w \in N_2(v)$. Since $\langle N_2(v)\rangle$ is totally disconnected, all the edges of G are incident with vertices of S. Therefore, vertices of $BG_2(G) - S$ are adjacent to atleast one vertex in S. Also, the vertices of $V(BG_2(G)) - S$ has u, w as eccentric vertices. Hence, $\gamma_{ed}(BG_2(G)) \leq |S| = |N(v)| + 2 \leq \Delta(G) + 2$.

Theorem 2.14 Let G be a connected graph. Then line independent set of G is an eccentric dominating set for $BG_2(G)$ if and only if G is a graph with $p \geq 6$ and G has a perfect matching with diam(G) ≤ 2.

Proof: Let D be a line independent set of G. If D is an eccentric dominating set for $BG_2(G)$, it dominates every point vertices of $BG_2(G)$, that is D is a line cover of G. D is independent and cover all vertices of G implies that D is a perfect matching. If $p \geq 3$ and $\beta_0(G) \geq 3$, then every edge in E(G) – D has atleast one edge in D, which is not adjacent to e in G. Thus D dominates all line vertices of $BG_2(G)$ also. Hence, D is a dominating set of $BG_2(G)$. Therefore, G must be a graph with even number of vertices and has a perfect matching. Also, eccentricity of every line vertex in $BG_2(G)$ is two and if diam(G) ≥ 3, then eccentricity of point vertex is three in $BG_2(G)$. Hence, D is an eccentric dominating set implies that G is a graph with $p \geq 6$ and G has a perfect matching with diam(G) ≤ 2.

Conversely, let G has a perfect matching with diam(G) ≤ 2 and $p \geq 6$. This implies, G cannot be $K_{1,n}$. Let D be a perfect matching of G. D dominates all point and line vertices of $BG_2(G)$. Since diam(G) ≤ 2 and $G \neq K_{1,n}$, line vertices of $BG_2(G)$ is of eccentricity two. Therefore, $BG_2(G)$ is a 2 self-centered graph. In $BG_2(G)$, every edge in E(G) – D is adjacent with some edge in D. Hence, in $BG_2(G)$, every line vertex has eccentric vertex in D. Every point vertices of V(G) is non incident with some edge of D in G. Therefore, point vertex in $BG_2(G)$ has eccentric vertex in D. Hence, D is an eccentric dominating set of $BG_2(G)$.

Remark 2.1 If p = 4 and G has a perfect matching, then D cannot be a dominating set of $BG_2(G)$.

Theorem 2.15 Let G be a connected graph. Maximal independent set of G is an eccentric dominating set of $BG_2(G)$ if and only if G satisfies any one of the following (i) $G = K_{1,n}$, $n \geq 3$ (ii) G is bipartite and if $v \in V(G) – D$ such that $e_G(v) = 2$ then v is not adjacent to atleast one element of D, if $v \in V(G) – D$ such that $e_G(v) \geq 3$ then there exists $w \in S$ such that d(v, w) ≥ 3.

Proof: Let G be a connected graph. Let D be the maximal independent set of G. So, $D \subseteq V(G)$ such that D is independent. Since, D is maximal independent it is a dominating set of G. So, D dominates the point vertices in $BG_2(G)$. Now, to dominate the line vertices of $BG_2(G)$, D must be a point cover of G also. D is maximal independent implies that V(G) – D is a point cover of G. Also, D is a point cover of G implies that V(G) – D has no edges and so it is independent. Thus both D and V(G) – D are independent. Therefore, G is bipartite. When $p > 3$ and $G \neq K_n$, every line vertex of $BG_2(G)$ has eccentric vertices in D. But point vertices which are not in D need not have eccentric vertices in D. D has eccentric vertices of other point vertices if D satisfies condition (ii) only. Hence the theorem is proved. On the otherhand, if all the conditions are satisfied, then any maximal independent set of G is an eccentric dominating set of $BG_2(G)$.

Theorem 2.16 G is a connected (p, q) graph with $p \geq 4$. Set of all line vertices is an eccentric dominating set of $BG_2(G)$ if and only if diameter of G is 1 or 2.

Proof: Eccentricity of line vertices in $BG_2(G)$ is always two and eccentricity of point vertex is 1, 2 or 3. Hence, E(G) is an eccentric dominating set only when diam(G) ≤ 2 by Theorem 1.3.

Converse: **Case(i):** r(G) = d(G) = 1. That is $G = K_n$. In this case, E(G) is an eccentric dominating set of $BG_2(G)$.

Case(ii): r(G) = 1, d(G) = 2. If $G = K_{1,n}$, $BG_2(G)$ is of radius one and E(G) is an eccentric dominating set of $BG_2(G)$. When $G \neq K_{1,n}$, $BG_2(G)$ is two self centered. For a point vertex u, a line vertex e which is not incident with u in G is an eccentric vertex in $BG_2(G)$. So, E(G) is an eccentric dominating set of $BG_2(G)$.

Case(iii): r(G) = d(G) = 2. In this case also $BG_2(G)$ is 2 self-centered and E(G) is an eccentric dominating set of $BG_2(G)$.

Theorem 2.17 Let $G \neq K_{1,n}$ be a graph with $p \geq 3$. Then $\gamma_{ed}(BG_2(G)) = 2$ if and only if G satisfies any one of the following: (i) $K_{1,2}$ (ii) K_2 (iii) $K_1 \cup K_2$.

Proof: Assume that $\gamma_{ed}(BG_2(G)) = 2$.

Case(i): Let $D = \{u, v\} \subseteq V(G)$ is an eccentric dominating set for $BG_2(G)$.

D is a dominating set for $BG_2(G)$. Therefore, all point vertices are adjacent to u or v or both in G and all the edges in G are incident with u or v and the vertex u and v are non adjacent in G. Hence, D is a point cover of G. Suppose $d(u, v) \geq 3$, $D = \{u, v\}$ cannot be a point cover, so $d(u, v) \leq 2$. If $d(u, v) = 1$, the line vertex $e = uv$ cannot be dominated by D in $BG_2(G)$. Hence, d(u, v) must be two in G. Let uwv be a path in G. Since D is a point cover, all the edges must be incident with u or v. But the vertex w is adjacent to both u and v and hence w has eccentric vertex in D if $e(w) = 1$ in G. Hence, D is an eccentric dominating set only when $d(u, v) = 2$ and w is a centre of G and G is of radius one. If there exists vertex x not adjacent to u and not adjacent to v and adjacent to w, then x is not dominated by D in $BG_2(G)$. Hence, the only possibility is $G = K_{1,2}$.

Case(ii): $D = \{u, e\} \subseteq V(BG_2(G))$, $u \in V(G)$, $e \in E(G)$ is an eccentric dominating set for $BG_2(G)$.

Subcase(i): $D = \{u, e\}$, e is incident with u in G. Let $e = uv \in E(G)$. D is an eccentric dominating set in $BG_2(G)$. This implies that eccentricity of v in $BG_2(G)$ is one. Thus, v is of eccentricity one in G. If there exists any other edges incident with v in G, then they cannot be dominated by D in $BG_2(G)$. Hence, $G = K_2$ only.

Subcase(ii): $D = \{u, e\}$, e is not incident with u. Let $e = xy \in E(G)$.

(i) Suppose u is adjacent to any one of x and y say x. In this case, eccentricity of x in G must be one. Hence, $r(G) = 1$ and there exists no other edges incident with x. Hence $G = K_{1,2}$. If u is adjacent to both x and y, the vertex y has no eccentric vertex in D. So, this case is not possible.

(ii) Suppose u is not adjacent to both x and y. Suppose u is not isolated there exists e_1 incident with u. Let $e_1 = uu_1 \in E(G)$. Then e_1 is adjacent to both u and e in $BG_2(G)$, so e_1 has no eccentric vertex in D. So, this is not possible. So u must be isolated and there exists no other edges. Hence $G = K_2 \cup K_1$.

Case(iii): $D = \{e_1, e_2\} \subseteq V(BG_2(G))$, $e_1, e_2 \in E(G)$ is an eccentric dominating set for $BG_2(G)$.

Subcase(i): $D = \{e_1, e_2\}$, e_1 and e_2 are adjacent in G. D is an eccentric dominating set for $BG_2(G)$. D is a dominating set for $BG_2(G)$. Therefore, all point vertices incident with e_1 or e_2 or both in G and e_1, e_2 are adjacent in G. Hence, G must be $K_{1,2}$.

Subcase(i): $D = \{e_1, e_2\}$, e_1 and e_2 are non adjacent in G. Let $e_1 = uv$, $e_2 = xy \in E(G)$. D is a dominating set of $BG_2(G)$ implies that there exists no other point vertices and hence, no non adjacent edges. If there exists an edge e adjacent to both e_1 and e_2 in G, then in $BG_2(G)$ the corresponding line vertex cannot be dominated by D in $BG_2(G)$. Hence, $G = 2K_2$, and in this case D is a dominating set, but point vertices has no eccentric vertices. Hence, this case is also not possible.

This proves the theorem.

3. Eccentric Domination number of $BG_2(G)$ for some particular graphs

In this section, the exact value of $\gamma_{ed}(BG_2(G))$ for some particular classes of graphs are determined.

Theorem 3.1 For a non-trival path P_n on n vertices, where $n \geq 3$.

(i) $\gamma_{ed}(BG_2(P_n)) = (n / 3) + 1$, if $n = 3k$, $k > 1$.

(ii) $\gamma_{ed}(BG_2(P_n)) = \lfloor n / 3 \rfloor + 1$, if $n = 3k + 1$

(iii) $\gamma_{ed}(BG_2(P_n)) = \lceil n / 3 \rceil + 1$, if $n = 3k + 2$

Proof: Let $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and $e_i = v_iv_{i+1}$, $1 \leq i \leq n - 1$. Let $u_i \in V(BG_2(P_n))$ be the vertex corresponding to e_i in $BG_2(P_n)$. Then $v_1, v_2, v_3, \dots, v_n, u_1, u_2, u_3, \dots, u_{n-1} \in V(BG_2(P_n))$. Thus $|V(BG_2(P_n))| = 2n - 1$.

Case(i): $n = 3k$.

Let $S = \{u_1, v_3, v_6 \dots, v_{n-3}, v_n\}$. S is a minimal eccentric dominating set of $BG_2(P_n)$. $|S| = \lfloor n / 3 \rfloor + 1$. Therefore, $\gamma_{ed}(BG_2(P_n)) \leq (n / 3) + 1$. We have $\gamma(BG_2(G)) \leq \gamma_{ed}(BG_2(G))$. Therefore, $\gamma_{ed}(BG_2(P_n)) \geq \gamma(BG_2(G)) = (n / 3) + 1$. Hence, $\gamma_{ed}(BG_2(P_n)) = (n / 3)+ 1$.

Case(ii): $n = 3k + 1$.

Let $S = \{u_1, v_3, v_6 \dots, v_{n-1}\}$. S is a minimal eccentric dominating set of $BG_2(P_n)$. $|S| = \lfloor n / 3 \rfloor + 1$. Therefore, $\gamma_{ed}(BG_2(P_n)) \leq \lfloor n / 3 \rfloor + 1$. We have $\gamma(BG_2(G)) \leq \gamma_{ed}(BG_2(G))$. Therefore, $\gamma_{ed}(BG_2(P_n)) \geq \gamma(BG_2(G)) = \lfloor n / 3 \rfloor + 1$. Hence, $\gamma_{ed}(BG_2(P_n)) = \lfloor n / 3 \rfloor + 1$.

Case(iii): $n = 3k + 2$

Let $S = \{u_1, v_3, v_6 \dots, v_{n-2}, v_n\}$. S is a minimal eccentric dominating set of $BG_2(P_n)$. $|S| = \lceil n / 3 \rceil + 1$ Therefore, $\gamma_{ed}(BG_2(P_n)) \leq \lceil n / 3 \rceil + 1$. We have $\gamma(BG_2(G)) \leq \gamma_{ed}(BG_2(G))$. Therefore, $\gamma_{ed}(BG_2(P_n)) \geq \gamma(BG_2(G)) = \lceil n / 3 \rceil + 1$. Hence, $\gamma_{ed}(BG_2(P_n)) = \lceil n / 3 \rceil + 1$.

Remark 3.1: When $G = P_2$. $S = \{v_1\}$ is a minimum eccentric dominating set of $BG_2(G)$. Hence, $\gamma_{ed}(BG_2(P_2)) = 1$. When $G = P_3$. $S = \{v_1, v_1\}$ is a minimum eccentric dominating set of $BG_2(G)$. Hence, $\gamma_{ed}(BG_2(P_2)) = 2$.

Theorem 3.2 For $n \geq 5$, (i) $\gamma_{ed}(BG_2(C_n)) = (n / 3) + 1$, $n = 3k$.

(ii) $\gamma_{ed}(BG_2(C_n)) = \lceil n / 3 \rceil + 1$, $n = 3k + 1$ or $n = 3k + 2$.

Proof: Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $e_i = v_iv_{i+1}$, $1 \le i \le n - 1$ and $e_n = v_nv_1$. Let u_i be the vertex corresponding to e_i in $BG_2(C_n)$. Then $v_1, v_2, v_3, \ldots, v_n, u_1, u_2, u_3, \ldots, u_n \in V(BG_2(C_n))$. Thus $|V(BG_2(C_n))| = 2n$.

Case(i): $n = 3k$

Let $S = \{v_1, v_4, v_7 \ldots, v_{n-2}, u_{n-1}\}$. S is an eccentric dominating set of $BG_2(C_n)$. $|S| = (n / 3) + 1$. Therefore, $\gamma_{ed}(BG_2(C_n)) \le (n / 3) + 1$. We have $\gamma(BG_2(G)) \le \gamma_{ed}(BG_2(G))$. Therefore, $\gamma_{ed}(BG_2(C_n)) \ge \gamma(BG_2(G)) = (n / 3) + 1$. Hence, $\gamma_{ed}(BG_2(C_n)) = (n / 3) + 1$.

Case(ii): $n = 3k + 1$ or $n = 3k + 2$.

Let $S = \{v_1, v_4, v_7, \ldots, v_{n-1}, u_{n-1}\}$. S is an eccentric dominating set of $BG_2(C_n)$. $|S| = \lceil n / 3 \rceil + 1$. Therefore, $\gamma_{ed}(BG_2(C_n)) \le \lceil n / 3 \rceil + 1$. We have $\gamma(BG_2(G)) \le \gamma_{ed}(BG_2(G))$. Therefore, $\gamma_{ed}(BG_2(C_n)) \ge \gamma(BG_2(G)) = \lceil n / 3 \rceil + 1$. Hence, $\gamma_{ed}(BG_2(C_n)) = \lceil n / 3 \rceil + 1$.

Remark 3.2 When $G = C_3, C_4$. $S = \{v_1, v_2, v_3\}$ is a minimum eccentric dominating set of $BG_2(G)$. Hence, $\gamma_{ed}(BG_2(C_3)) = \gamma_{ed}(BG_2(C_4)) = 3$. When $G = C_5$. $S = \{v_1, v_3, v_5\}$ is a minimum eccentric dominating set of $BG_2(G)$. Hence, $\gamma_{ed}(BG_2(C_5)) = 3$.

Theorem 3.3 $\gamma_{ed}(BG_2(K_n)) = 3$, $n \ge 3$

Proof: Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of K_n and let u_{ij}, $i < j$, $i, j = 1, 2, 3, \ldots, n$ be the added vertices corresponding the edges e_{ij} of K_n to obtain $BG_2(K_n)$. Thus $V(BG_2(K_n)) = \{v_1, v_2, v_3, \ldots, v_n\} \bigcup_{i<j} \{u_{ij}\}$, $i, j = 1, 2, 3 \ldots, n$. The graph $BG_2(K_n)$ has $n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$ vertices. Eccentricity of every point vertex and line vertex of $BG_2(K_n)$ is two. Therefore it is a self-centered graph. Let $S = \{v_1, v_2, u_{12}\}$, $v_1, v_2 \in V(G)$ and $u_{12} \in E(G)$. S dominates all point vertices and line vertices and is also an eccentric dominating set of $BG_2(K_n)$. Hence, $\gamma_{ed}(BG_2(K_n)) = 3$.

Remark 3.3 When $G = K_2$, $S = \{v_1\}$ is a minimum eccentric dominating set of $BG_2(G)$. Hence, $\gamma_{ed}(BG_2(K_n)) = 1$.

Theorem 3.4 $\gamma_{ed}(BG_2(K_{1,n})) = 3$, $n \ge 3$.

Proof: $V' = V(BG_2(K_{1,n}))$. Let $S = \{v, v_1, v_n\}$, where v is the central vertex of G and v_1, v_2, are pendant vertices. The central vertex dominates all point vertices and line vertices in $V' - S$ and v_1, v_n are eccentric vertices of $V' - S$. Hence, $\gamma_{ed}(BG_2(K_{1,n})) = 3$.

Remark 3.4 When $G = K_{1,2}$. $S = \{e_1, e_2\}$ is a minimum eccentric dominating set of $BG_2(G)$. Hence, $\gamma_{ed}(BG_2(K_{1,2})) = 2$

Remark 3.5 Let $S = \{v, v_1, e_1\}$, where $e_1 = vv_1 \in E(G)$. S is also an eccentric dominating set of $BG_2(K_{1,n})$.

Theorem 3.5 $\gamma_{ed}(BG_2(K_{m,n})) = 3$, $m, n \ge 2$.

Proof: When $G = K_{m,n}$. $V(G) = V_1 \cup V_2$. $|V_1| = m$ and $|V_2| = n$. $E(G) = \{e_{ij} / 1 \le i \le m, 1 \le j \le n\}$ where $e_{ij} = u_iv_j$ for all $1 \le i \le m$, $1 \le j \le n$. Thus $V(BG_2(K_{m,n})) = (V_1 \cup V_2) \cup \{e_{ij} / 1 \le i \le m, 1 \le j \le n\}$. Let $S = \{u, v, e\}$, $u \in V_1$, $v \in V_2$ and $uv = e \in E(G)$. The vertex u dominates all point vertices of V_2 and line vertices which are edges incident with u in G. The vertex v dominates all point vertices of V_1 and line vertices which are edges incident with v in G. The line vertex e dominates all line vertices which are edges not incident with both u and v. The vertex u is an eccentric vertex of V_1 and non incident edges of G and the vertex v is an eccentric vertex of V_2 and non incident edges of G. Therefore, S is an minimum eccentric dominating set of $BG_2(K_{m,n})$. Hence, $\gamma_{ed}(BG_2(K_{m,n})) = 3$.

Theorem 3.6 $\gamma_{ed}(BG_2(W_n)) = 3$, where $W_n = K_1 + P_n$.

Proof: Let $S = \{u, v, e\}$, where u and v are adjacent vertices and v is the central vertex. $uv = e \in E(G)$. u and v dominates all point vertices and incident edges in $BG_2(W_n)$ and e dominates non adjacent edges in $BG_2(W_n)$. The vertex u is a eccentric vertex of non adjacent point vertices and non incident line vertices and the vertex e is a eccentric vertex of non adjacent line vertices and non incident point vertices in G. Therefore, eccentricity of every point vertex and line vertex of $BG_2(W_n)$ is two. This implies, it is a self-centered graph. S is an eccentric dominating set of $BG_2(W_n)$. Also, S is a minimum eccentric dominating set of $BG_2(W_n)$. Hence, $\gamma_{ed}(BG_2(W_n)) = 3$.

Theorem 3.7 $\gamma_{ed}(BG_2(F_n)) = 3$, where $F_n = K_1 + P_n$.

Proof: Let $v_1, v_2, v_3, \ldots, v_n$, v (v is the central vertex of F_n) be the vertices of F_n and let $e_j = vv_j$, $j = 1, 2, \ldots, n$, and $v_iv_j = e_{ij}$ ($j = i + 1$, $i = 1, 2, 3, \ldots, n$) be the edges of F_n. Let $v_1, v_2, \ldots, v_n, v, u_1, u_2, \ldots, u_n, e_{12}, e_{23}, \ldots, e_{n-1,n}$ be the corresponding vertices of $BG_2(F_n)$. Thus $V((BG_2(F_n))$ has 3n vertices. $S = \{v, v_1, e_1\}$ is the eccentric dominating set of $BG_2(F_n)$. Eccentricity of every point vertex and line vertex of $BG_2(F_n)$ is two. Therefore it is a self-centered graph. The vertex v_1 is an eccentric vertex of e_j, $j > 1$ in $BG_2(F_n)$. The vertex v is the eccentric vertex of the line vertex e_{12} in $BG_2(F_n)$. For other e_{ij}'s v is the eccentric vertex in $BG_2(F_n)$. For point vertex v_i, $i > 1$, line vertex e is an eccentric point. Therefore, S is a minimum eccentric dominating set of $BG_2(F_n)$. Hence, $\gamma_{ed}(BG_2(F_n)) = 3$.

Theorem 3.8 $\gamma_{ed}(BG_2(P_n^+)) = n$.

Proof: Let G = P_n^+ be a graph obtained from P_n by attaching exactly one pendant edge at each vertex of P_n. Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices and $e_{12}, e_{23}, e_{34}, \ldots, e_{n-1,n}$ be the edges in P_n, where $e_{i,i+1} = v_iv_{i+1}$, i = 1, 2, 3, …, n-1. Let u_i be the pendant vertex attached to v_i in P_n^+, i = 1, 2, 3, …, n. Then $v_1, v_2, v_3, \ldots, v_n, u_1, u_2, u_3, \ldots, u_n, e_{11}, e_{22}, e_{33}, \ldots, e_{nn}, e_{12}, e_{23}, e_{34}, \ldots, e_{n-1,n} \in V(BG_2(P_n^+))$. Thus $|V(BG_2(P_n^+))| = 4n - 1$. Let S = $\{u_1, u_n, v_2, v_3, \ldots, v_{n-1}\}$. u_1 and u_n are two peripheral vertices $BG_2(G)$. S is an eccentric dominating set of $BG_2(P_n^+)$. |S| = n. Thus, $\gamma_{ed}(BG_2(P_n^+)) \leq n$. Also, $\gamma(BG_2(G)) \leq \gamma_{ed}(BG_2(G))$. $\gamma(G) = n$ and $\gamma(BG_2(G)) \geq \gamma(G)$. Hence, $\gamma_{ed}(BG_2(P_n^+)) = n$.

Theorem 3.9 $\gamma_{ed}(BG_2(C_n^+)) = n$.

Proof: Let G = C_n^+ be a graph obtained from C_n by attaching exactly one pendant edge at each vertex of C_n. Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices and $e_{12}, e_{23}, e_{34}, \ldots, e_{n1}$ be the edges in C_n, where $e_{i,i+1} = v_iv_{i+1}$, $1 \leq i \leq n-1$ and $e_{n1} = v_nv_1$. Let u_i be the pendant vertex attached to v_i in C_n^+, i = 1, 2, …, n, where $e_i = u_iv_i$, $1 \leq i \leq n$. Then $v_1, v_2, v_3, \ldots, v_n, u_1, u_2, u_3, \ldots, u_n, e_1, e_2, e_3, \ldots, e_n, e_{12}, e_{23}, e_{34}, \ldots, e_{n1} \in V(BG_2(C_n^+))$. Thus $|V(BG_2(C_n^+))| = 4n$. $u_i, v_i \in V(BG_2(C_n^+))$ has all u_j's and v_j's, $i \neq j$ as eccentric vertices and $e_{ii} \in V(BG_2(C_n^+))$ has all u_j's and v_j's, $i \neq j$ as eccentric vertices. $e_{ij} \in V(BG_2(C_n^+))$ has u_r or u_s as eccentric vertices. Let S = $\{v_1, v_2, v_3, \ldots, v_n\}$. S is an eccentric dominating set of $BG_2(C_n^+)$. |S| = n. Therefore, $\gamma_{ed}(BG_2(C_n^+)) \leq n$. $\gamma(BG_2(G)) \geq \gamma(G) = n$. This implies that $\gamma_{ed}(BG_2(G)) \geq n$. Hence, $\gamma_{ed}(BG_2(C_n^+)) = n$.

Theorem 3.10 If G is a wounded spider with atleast one non-wounded leg, then $\gamma_{ed}(BG_2(G)) = s + 2$, where s is the number of support vertices which are adjacent to non-wounded legs.

Proof: Let G be a wounded spider. Let u be the vertex of maximum degree $\Delta(G)$, and S be the set of support vertices which are adjacent to non-wounded legs. In $BG_2(G)$, the set S form a dominating set of $BG_2(G)$. But it is not an eccentric dominating set. Adding any one peripheral vertex of G, form a minimum eccentric dominating set of $BG_2(G)$. Hence, $\gamma_{ed}(BG_2(G)) = s + 2$.

Theorem 3.11 If G is a spider such that length of each leg is two, then $\gamma_{ed}(BG_2(G)) = \Delta(G) + 1$.

Proof: Let G be a spider and u be a vertex of maximum degree $\Delta(G)$. u is the central vertex. In $BG_2(G)$, $N_G(u)$ dominates all point vertices and line vertices. Adding any one peripheral vertex of G with $N_G(u)$, form a minimum eccentric dominating set of $BG_2(G)$. Hence, $\gamma_{ed}(BG_2(G)) = \Delta(G) + 1$.

References:

1. Bhanumathi,M., A study on some structural properties of graphs and some new graph operation on graphs, Thesis, Bharathidasan University, 2004.
2. Bhanumathi.M., On Connected Eccentric Domination in Graphs, Elixir Dis. Math. 90(2016) 37639-37643. ISSN: 2229-712x. IF: 6.865.
3. Bhanumath.M. and S.Muthammai., On Eccentric Domination in Trees, International Journals of Engineering science, Advanced Computing and Bio-Technology Vol:2, No.1, pp 38-46, 2011. ISSN: 2249-5584(Print), ISSN: 2249-5592(Online)
4. Bhanumath.M. and S.Muthammai., Further Results On Eccentric Domination graphs, International Journals of Engineering science, Advanced Computing and Bio-Technology Volume:3, Issue 4, pp 185-190, 2012. ISSN: 2249-5584(Print), ISSN: 2249-5592(Online)
5. Bhanumathi.M. and John Flavia.J., Total eccentric domination in Graphs, International Journals of Engineering science, Advanced Computing and Bio-Technology Vol:3, No.2, April - June 2014 pp 49-65.
6. Bhanumathi.M. and M.Kavitha., On Connected Eccentric Domination in trees, International Journals of Engineering science, Advanced Computing and Bio-Technology Volume:8, No. 3, July-September 2017, pp. 133-142. ISSN: 2249-5584(Print), ISSN: 2249-5592(Online). SJIF: 3.376.
7. Bhanumathi.M. and R.Niroja., Eccentric Domination in Splitting graphs of some graphs, International Journals of Engineering science, Advanced Computing and Bio-Technology Volume: 11, No. 2(2016), pp. 179-188. ISSN: 0973 – 4554.
8. Bhanumathi.M. and R.Niroja., Eccentric Domination and Restrained Domination in Circulant graphs, International Journals of Engineering science, Advanced Computing and Bio-Technology Volume: 9, No. 1, January - March 2018, pp. 1-11. ISSN: 2249-5584(Print), ISSN: 2249-5592(Online). SJIF: 3.376(2017).
9. Bhanumathi.M. and R.Niroja., Isolated Eccentric Domination in Graphs, International Journal of Advanced Research trends in Engineering and Technology(IJARTET) Vol. 5, Special issue 12, Apirl 2018, pp. 951 - 955. ISSN(P): 2394-3777, ISSN(E): 2394-3785.
10. Cockayne,E.J. and Hedetniemi,S.T., Towards a Theory of Domination in Graphs, Network, 7 : 247-261.1977
11. Harary,F., Graph Theory, Addition-Wesley Publishing Company Reading, Mass (1972).

12. Janakiraman,T.N., Bhanumathi,M. and Muthammai,S., Point-set Domination of the Boolean Graph $BG_2(G)$. Proceeding of the National Conference on Mathematical Techniques and Application (NCMTA 2007) Jan 5 and 6, 2007, S.R.M University, Chennai. pp.191-206.
13. Janakiraman,T.N., Bhanumathi,M. and Muthammai,S., Eccentric Domination in Graphs, International Journals of Engineering science, Advanced Computing and BioTechnology Vol:1, No.2, pp1-16, 2010.
14. Janakiraman,T.N., Bhanumathi,M. and Muthammai,S., On the Boolean Graph $BG_2(G)$ of a Graph G. International Journal of Engineering Science, Advanced Computing and Bio-Technology Vol.3, No.2, April-June 2012, pp:93-107
15. Janakiraman,T.N., Bhanumathi,M. and Muthammai,S., Domination Parameters of Boolean Graph $BG_2(G)$ and its Complement, International Journal of Engineering Science, Advanced Computing and Bio-Technology Vol.3, No.3, July-September 2012, pp.115-135.
16. Janakiraman,T.N., Bhanumathi,M. and Muthammai,S., Eccentricity properties of the Boolean graphs $BG_2(G)$ and $BG_3(G)$, International Journal of Engineering Science, Advanced Computing and Bio-Technology, Volume :4 , Issue :2, Page : 32-42
17. Kulli,V.R., Theory of Domination in Graphs, Vishwa International Publication, Gulbarga, India.
18. Ore.O., Theory of graphs, Amer. Math. Soc. Colloq. Publ., 38, Providence(1962). International Publication, Gulbarga, India.