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Research Article

Equitable Coloring On Rooted Product Of Graphs

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Abstract - A finite and simple graph G is said to be equitably $k-$ colorable if its vertices can be partitioned into $k-$ classes $V_1, V_2, \dots, V_k$ such that each V_i is an independent set and $||V_i| - |V_j|| \leq 1$ holds for every i, j. The smallest integer k for which G is equitable chromatic number of G and denoted by $\chi_=(G)$. The equitable chromatic threshold of a graph G, denoted by $\chi_=^*(G)$, is the minimum t such that G is equitably $k-$ colorable for all $k \geq t$. This paper focuses on the equitable colorability of rooted product of graphs, in particular, exact values or upper bounds of $\chi_=(GoH)$ and $\chi_=^*(GoH)$ when G and H are cycles, paths, complete graphs and complete $n-$ partite graphs have been found.

Keywords: Equitable coloring, Equitable chromatic number, Equitable chromatic threshold, Rooted product, Cartesian Product.

Introduction

Let $G = (V, E)$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. All the definitions which are not discussed in this paper one may refer [1, 2]. All graph consider in this paper are simple, finite and undirected. Here $P_m, C_m, K_{1,m}, K_{k,m}, K_{x_1,x_2,\dots,x_r}$ and Q_r respectively denotes the path, the cycle, the star, the complete bipartite, the complete r partite and the hypercube graph on k, m, r vertices. Graph theory is one of the most interesting branches of mathematics, with wide applications in the domain of physical networks, organic molecules, ecosystems, sociological relationships, databases, or in the flow of control in a computer program. By a graph coloring, it means to assign a color to each vertex of the graph such that no two adjacent vertices have the same color. The minimum number of colors required for coloring of a graph G is called chromatic number and is denoted by $\chi(G)$. The vertices of the same color form a color class. Various types of graph coloring is there, one among them is equitable coloring. This coloring parameter was first introduced by Meyar [3] in 1973. If the set of vertices of a graph G can be partitioned into k classes $V_1, V_2, \dots, V_k$ such that each V_i is an independent set and the condition$||V_i| - |V_j|| \leq 1$ holds for every pair (i, j), then G is said to be equitably k-colorable. The smallest integer k for which G is equitably k-colorable is known as the equitable chromatic number of [7-10] G and is denoted by $\chi_=(G)$. It is obvious that $\chi(G) \leq \chi_=(G)$. Note that $\chi(G)$ and $\chi_=(G)$ can vary a lot. For example, $\chi(K_{1,n}) = 2 < 1 + \lceil \frac{n}{2} \rceil = \chi_=(K_{1,n})$ for $n \geq 3$. The equitable chromatic threshold of a graph G, denoted by $\chi_=^*(G)$, is the minimum t such that G is equitably $k-$ colorable for all $k \geq t$.

Applications of equitable coloring can be found in scheduling and timetabling. Consider, for example, a problem of constructing university timetables. As we know, we can model this problem as coloring the vertices of a graph G whose vertices correspond to classes, edges correspond to time conflicts between classes, and colors to hours. If the set of available rooms is restricted, then we may be forced to partition the vertex set into independent subsets of as near equal size as possible, since then the room usage is the highest. We can find another application of equitable coloring in transportation problems. Here, the vertices represent garbage collection routes and two such vertices are joined by an edge when the corresponding routes should not be run on the same day. The problem of assigning one of the six days of the work week to each route becomes the problem of 6-coloring of G.

In 1973, Meyer [3] formulated the following conjecture: Equitable Coloring Conjecture [3]. For any connected graph G, other than a complete graph or an odd cycle, $\chi_=(G) \leq \Delta(G)$. This conjecture has been verified for all graphs on six or fewer vertices. Lih and Wu [10] proved that the Equitable Coloring Conjecture is true for all bipartite graphs. Wang and Zhang [14] considered a broader class of graphs, namely r-partite graphs. They proved that Meyer's conjecture is true for complete graphs from this class. Also, the conjecture was confirmed for outerplanar graphs [12] and planar graphs with maximum degree at least 13 [13]. We also have a stronger conjecture: Equitable $\Delta -$ Coloring conjecture[7]. If G is a connected graph of degree Δ, other than a complete graph, an odd cycle or a complete bipartite graph $K_{2n+1,2n+1}$ for any $n \geq 1$, then G is equitably $\Delta -$ Colorable. The Equitable $\Delta -$ Coloring Conjecture holds for some classes of graphs, e.g., bipartite graphs [10], outerplanar graphs with $\Delta \geq 3$ [12] and planar graphs with $\Delta \geq 13$ [13]. The detailed survey of this type of coloring is found in Lih [15].

The following work makes provision for the preliminaries on equitable coloring of cartesian, weak tensor, strong tensor, corona products and follows the equitable coloring of rooted products. Using some specific properties we arrive at some exact values of equitable chromatic number and equitable chromatic threshold number.

Preliminaries

Before we go through the main results, we want several preliminary results.

Definition 2.1 [11] For graphs G and H, the Cartesian product of G and H is the graph $G \square H$ with vertex set $V(G \square H) = \{(x, y): x \in V(G), y \in V(H)\}$, and edge set $E(G \square H) = \{(x, u)(y, v): x = y \text{ with } uv \in E(H), \text{or } xy \in E(G) \text{ with } u = v\}$.

Theorem 2.2 [5] If G_1 and G_2 are equitably $k-$colorable, then $G_1 \square G_2$ is equitably $k-$colorable.

Theorem 2.3 [4] Let G and H be two graphs with $V(G) = V(H)$ such that $E(G) \subseteq E(H)$.Then $\chi_{=}(G) \leq \chi_{=}(H)$.

Theorem 2.4 [4] $\chi(G_1 \square G_2) = \max\{\chi(G_1), \chi(G_2)\}$.

Theorem 2.5 [4] $\chi_{=}(G_1 \square G_2) = \max\{\chi(G_1), \chi(G_2)\}$.

Corollary 1 [4] $\chi_{=}^{*}(G_1 \square G_2) \leq \max\{\chi_{=}^{*}(G_1), \chi_{=}^{*}(G_2)\}$.

Definition 2.6 [4] The strong tensor product of graphs G and H is the graph $G \times H$ with vertex set $V(G) \times V(H)$ and edge set $\{(x, y)(x', y'): \{(x, y), (x', y')\} \in E(G \square H) \cup E(G \times H)\}$.

Theorem 2.7 [4] Let G_1, G_2 be graphs with atleast one edge each. Then

$$\chi_{=}(G_1 \boxtimes G_2) \geq \max\{\chi(G_1), \chi(G_2)\} + 2.$$

Theorem 2.8 [17] The weak tensor product of graphs G and H is the graph $G \times H$ with vertex set $V(G) \times V(H)$ and edge set $\{(x, y)(x', y'): xx' \in E(G) \text{ and } yy' \in E(H)\}$.

Theorem 2.9 [18] $\chi(K_m \times K_n) = \min(m, n)$.

Theorem 2.10 [5] $\chi_{=}(K_m \times K_n) = \min(m, n)$.

$$\chi_{=}(C_m \times C_n) = \chi_{=}^{*}(C_m \times C_n) = \begin{cases} 2 \text{ if mn is even} \\ 3 \text{ otherwise} \end{cases}$$

Theorem 2.11 [16] The corona of two graphs G and H is the graph GoH formed one copy of G and $|V(G)|$ copies of H, where the i^{th} vertex in the i^{th} copy of H.

Result 1 Let $G_1 = K_{3,3}$ and $G_2 = K_{1,1,2}$. Then $\chi_{=}(G_1 \square G_2) = 4$.

Graph products are interesting and useful in many situations. For example, Sabbidussi [21] showed that any graph has the unique decomposition into prime factors under the Cartesian product. Feigenbaum and Sch affer [19] showed that the strong tensor product admits a polynomial algorithm for decomposing a given connected graph into its factors. An analogous result with respect to weak tensor product is due to Imrich [20]. The complexity of many problems, also equitable coloring, that deal with very large and complicated graphs is reduced greatly if one is able to fully characterize the properties of less complicated prime factors.

Equitable coloring of rooted products

Here we discuss about the equitable coloring on the rooted product of graphs. A graph in which one vertex is fixed as a root vertex to distinguish it from other vertices is called a rooted graph. Let G be a graph with n vertices and H be a sequence of n rooted graphs $H_1, H_2, \dots, H_n$. The rooted product graph $G(H)$ is obtained from the graphs $G, H_1, H_2, \dots, H_n$ by identifying the root vertex of H_i with the i^{th} vertex of G [6].

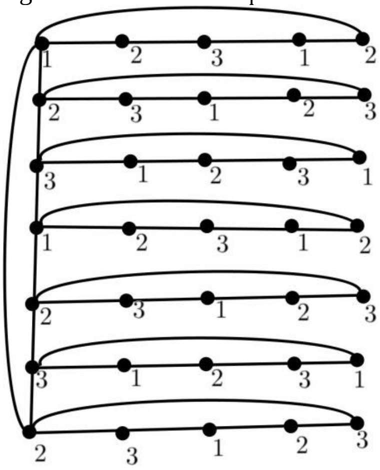

Figure 1: C_7 o C_5

Theorem 3.1 Let G_1 and G_2 be any two graphs. Then $\chi_{=}(G_1 o G_2) = \text{Max}\{\chi(G_1), \chi(G_2)\}$.

Proof. Since $V(G_1 o G_2) = V(G_1 \square G_2)$. $E(G_1 o G_2) \subseteq E(G_1 \square G_2)$. It follows that $\chi_{=}(G_1 o G_2) \leq \chi_{=}(G_1 \square G_2)$. Hence

$$\chi_{=}(G_1 o G_2) \leq \chi_{=}(G_1 \square G_2) = Max\{\chi(G_1), \chi(G_2)\}. \quad (1)$$

Also since $G_1, G_2 \leq G_1 o G_2$. $\chi_{=}(G_1), \chi_{=}(G_2) \leq \chi_{=}(G_1 o G_2)$. Hence

$$max(\chi_=(G_1), \chi_=(G_2)) \leq \chi_=(G_1 o G_2) \quad (2)$$

From (1) and (2) $\chi_=(G_1 o G_2) = Max\{\chi(G_1), \chi(G_2)\}$.

Note: In view of the Theorem 2.1, $\chi_=(G_1 o G_2) \leq \chi_=(G_1 \Box G_2)$.

The equality need not hold for every pair of graphs. Here is an example where equality fails. Let $G_1 = K_{3,3}$ and $G_2 = K_{1,1,2}$. We have $\chi_=(G_1) = 2$ and $\chi_=(G_2) = 3$ but $\chi_=(G_1 \Box G_2) = 4$, $\chi_=(G_1 o G_2) = 3$. Hence $\chi_=(G_1 o G_2) < \chi_=(G_1 \Box G_2)$.

Theorem 3.2 If G_1 and G_2 are equitably $k-$ colorable then $G_1 o G_2$ is equitably k colorable.

Proof. By theorem 2.2 $G_1 \Box G_2$ is equitably $k-$ colorable.$G_1 o G_2$ is a subgraph of $G_1 \Box G_2$ with $V(G_1 o G_2) = V(G_1 \Box G_2)$, such that $E(G_1 o G_2) \subseteq E(G_1 \Box G_2)$. So $G_1 o G_2$ is also equitably $k-$ colorable.

Corollary 2 $\chi_=^*(G_1 o G_2) \leq max\{\chi_=^*(G_1), \chi_=^*(G_2)\}$

Proof**.** By Theorem 2.3 $\chi_=^*(G_1 o G_2) \leq \chi_=^*(G_1 \Box G_2) \leq max\{\chi_=^*(G_1), \chi_=^*(G_2)\}$.

Lemma 3.3 Let $G = G_1 o G_2$, where each G_i is a path,a cycle, a hypercube or a complete graph. Then $\chi(G) = \chi_=(G) = \chi_=^*(G) = max\{\chi(G_i)/i = 1,2\}$.

We generalize the above lemma as follows.

Theorem 3.4 Let $G = G_1 o G_2 o \ldots o G_n$, where each G_i is a path,a cycle, a hypercube or a complete graph. Then $\chi(G) = \chi_=(G) = \chi_=^*(G) = max\{\chi(G_i)/i = 1,2,\ldots,n\}$.

Proof. For all the above graphs $\chi(G_i) = \chi_=(G_i) = \chi_=^*(G_i)$ for each i. $\chi(G) \leq \chi_=(G) \leq \chi_=^*(G) \leq max\{\chi_=^*(G_i)\} = max\{\chi(G_i)\} = \chi(G)$.

Corollary 3. $\chi_=(P_m\ o\ P_n) = 2$.

Proof. We know that $\chi_=(P_m) = \chi_=(P_n) = 2$, $\chi_=^*(P_m) = \chi_=^*(P_n) = 2$ for all $m,n.$ So $2 = \chi_=(P_m) \leq \chi_=(P_m\ o\ P_n) \leq \chi_=^*(P_m\ o\ P_n) = max\{2,2\}$. Hence $\chi_=(P_m\ o\ P_n) = 2$.

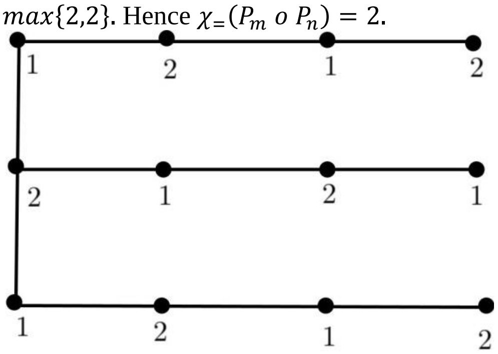

Figure 2: $P_3 o P_4$

Theorem 3.5 Let $G_1(X_1, Y_1)$ and $G_2(X_2, Y_2)$ be two bipartite graphs such that one of them contains atleast one edge and let $|X_2| = |Y_2|$. Then $\chi_=(G_1 o G_2) = 2$.

Proof. The graph $G_1 o G_2$ is given in the following figure 1. The polygons represent independent sets $Z_1, Z_2, Z_3$and Z_4 of $|X_1||X_2|, |X_1||Y_2|, |Y_1||X_2|, |Y_1||Y_2|$ vertices respectively. The lines show the possibilites of existing edges. If $|X_2| = |Y_2|$, we can assign color 1 to the vertices of Z_1 and Z_2 and color 2 to the remaining vertices. The obtained coloring is equitable.

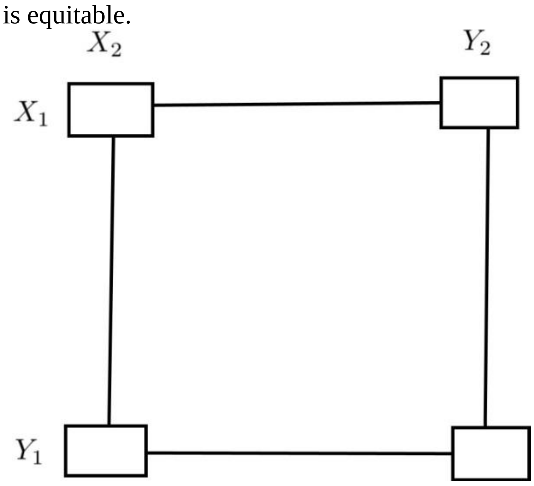

Figure 3: $\chi_=(G_1 o G_2)$

Corollary 3 Let k, m, n and r be positive integers. Then the equitable chromatic numbers of the following graphs are all equal to 2. $C_{2m}oC_{2n}$, $P_m oC_{2n}$, $Q_r oC_{2n}$, $K_{m,k}oC_{2n}$, $K_{1,m}oC_{2n}$, $P_m oC_{2n}$, $P_m oP_{2n}$, $Q_r oP_{2n}$, $K_{m.k}oP_{2n}$, $K_{1,m}oP_{2n}$, $Q_r oQ_r$ where Q_r is a hypercube and $K_{k,m}$ is a complete bipartite graph.

Proof. Immediate from the above theorem.

Theorem 3.6 Let $G_1(X_1, X_2, \ldots, X_r)$ and $G_2(Y_1, Y_2, \ldots, Y_r)$ be any $r-$ partite graphs such that $|Y_1| = |Y_2| = \ldots = |Y_r|$. Then $X_=(G_1 \square G_2) \leq r$.

Proof. Use an $r \times r$ latin square.

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & r-1 & r \\ 2 & 3 & 4 & & r & 1 \\ & \vdots & & \ddots & \vdots & \\ r & 1 & 2 & \cdots & r-2 & r-1 \end{bmatrix}$$

Conclusion

We have found that the exact bound for the equitable chromatic number and equitable chromatic threshold number of rooted product of some fundamental structure graphs.

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