Equitable Coloring On Rooted Product Of Graphs

Loura Jency #1, Benedict Michael Raj #2

#Assistant Professor at Department of Mathematics, Loyola College, Chennai, India.

#Head, Department of Mathematics, St. Joseph’s College, Trichy, India.

Abstract - A finite and simple graph $G$ is said to be equitably $k$-colorable if its vertices can be partitioned into $k$ classes $V_1, V_2, ..., V_k$ such that each $V_i$ is an independent set and $|V_i| - |V_j| \leq 1$ holds for every $i, j$. The smallest integer $k$ for which $G$ is equitably chromatic number of $G$ and denoted by $\chi_s(G)$. The equitable chromatic threshold of a graph $G$, denoted by $\chi_e^t(G)$, is the minimum $t$ such that $G$ is equitably $k$-colorable for all $k \geq t$. This paper focuses on the equitable colorability of rooted product of graphs, in particular, exact values or upper bounds of $\chi_s(G)$ and $\chi_e^t(G)$ when $G$ and $H$ are cycles, paths, complete graphs and complete $n$-partite graphs have been found.

Keywords: Equitable coloring, Equitable chromatic number, Equitable chromatic threshold, Rooted product, Cartesian Product.

Introduction

Let $G = (V, E)$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. All the definitions which are not discussed in this paper one may refer [1, 2]. All graph consider in this paper are simple, finite and undirected. Here $P_m, C_m, K_{1,m}, K_{x,m}, K_{r,x_2,..,x_r}$ and $Q_r$ respectively denotes the path, the cycle, the star, the complete bipartite, the complete $r$ partite and the hypercube graph on $k, m, r$ vertices. Graph theory is one of the most interesting branches of mathematics, with wide applications in the domain of physical networks, organic molecules, ecosystems, sociological relationships, databases, or in the flow of control in a computer program. By a same color, it means to assign a color to each vertex of the graph such that no two adjacent vertices have the same color. The minimum number of colors required for coloring of a graph $G$ is called chromatic number and is denoted by $\chi(G)$. The vertices of the same color form a color class. Various types of graph coloring is there, one among them is equitable coloring. This coloring parameter was first introduced by Meyer [3] in 1973. If the set of vertices of a graph $G$ can be partitioned into $k$ classes $V_1, V_2, ..., V_k$ such that each $V_i$ is an independent set and the condition $|V_i| - |V_j| \leq 1$ holds for every pair $(i, j)$, then $G$ is said to be equitably $k$-colorable. The smallest integer $k$ for which $G$ is equitably $k$-colorable is known as the equitable chromatic number of [7-10] $G$ and is denoted by $\chi_e(G)$. It is obvious that $\chi(G) \leq \chi_e(G)$. Note that $\chi(G)$ and $\chi_e(G)$ can vary a lot. For example, $\chi(K_{1,n}) = 2 < 1 + \left[\frac{n}{2}\right] = \chi_e(K_{1,n})$ for $n \geq 3$. The equitable chromatic threshold of a graph $G$, denoted by $\chi_e^t(G)$, is the minimum $t$ such that $G$ is equitably $k$-colorable for all $k \geq t$.

Applications of equitable coloring is found in scheduling and timetabling. Consider, for example, a problem of constructing university timetables. As we know, we can model this problem as coloring the vertices of a graph $G$ whose vertices correspond to classes, edges correspond to time conflicts between classes, and colors to hours. If the set of available rooms is restricted, then we may be forced to partition the vertex set into independent subsets of as near equal size as possible, since then the room usage is the highest. We can find another application of equitable coloring in transportation problems. Here, the vertices represent garbage collection routes and two such vertices are joined by an edge when the corresponding routes should not be run on the same day. The problem of assigning one of the six days of the week to each route becomes the problem of 6-coloring of $G$.

In 1973, Meyer [3] formulated the following conjecture: Equitable Coloring Conjecture [3]. For any connected graph $G$, other than a complete graph or an odd cycle, $\chi_s(G) \leq \Delta(G)$. This conjecture has been verified for all graphs on six or fewer vertices. Lih and Wu [10] proved that the Equitable Coloring Conjecture is true for all bipartite graphs. Wang and Zhang [14] considered a broader class of graphs, namely $r$-partite graphs. They proved that Meyer’s conjecture is true for complete graphs from this class. Also, the conjecture was confirmed for outerplanar graphs [12] and planar graphs with maximum degree at least 13 [13]. We also have a stronger conjecture: Equitable $\Delta -$ Coloring Conjecture[7]. If $G$ is a connected graph of degree $\Delta$, other than a complete graph, an odd cycle or a complete bipartite graph $K_{2n+1,2n+1}$ for any $n \geq 1$, then $G$ is equitably $\Delta -$ Colorable. The Equitable $\Delta -$ Coloring Conjecture holds for some classes of graphs, e.g., bipartite graphs [10], outerplanar graphs with $\Delta \geq 3$ [12] and planar graphs with $\Delta \geq 13$ [13]. The detailed survey of this type of coloring is found in Lih [15].

The following work makes provision for the preliminaries on equitable coloring of cartesian, weak tensor, strong tensor, corona products and follows the equitable coloring of rooted products. Using some specific
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Preliminaries

Before we go through the main results, we want several preliminary results.

**Definition 2.1** [11] For graphs G and H, the Cartesian product of G and H is the graph \( G \square H \) with vertex set \( V(G \square H) = \{(x,y): x \in V(G), y \in V(H)\} \), and edge set \( E(G \square H) = \{(x,u)(y,v): x = y \text{ with } uv \in E(H), \text{ or } xy \in E(G) \text{ with } u = v\} \).

**Theorem 2.2** [5] If \( G_1 \) and \( G_2 \) are equitably \( k \)-colorable, then \( G_1 \square G_2 \) is equitably \( k \)-colorable.

**Theorem 2.3** [4] Let G and H be two graphs with \( V(G) = V(H) \) such that \( E(G) \subseteq E(H) \). Then \( \chi_e(G) \leq \chi_e(H) \).

**Theorem 2.4** [4] \( \chi(G_1 \square G_2) = \max\{\chi(G_1), \chi(G_2)\} \).

**Theorem 2.5** [4] \( \chi_e(G_1 \square G_2) = \max\{\chi_e(G_1), \chi_e(G_2)\} \).

**Corollary 1** [4] \( \chi_e(G_1 \square G_2) \leq \max\{\chi_e(G_1), \chi_e(G_2)\} \).

**Definition 2.6** [4] The strong tensor product of graphs G and H is the graph \( G \times H \) with vertex set \( V(G) \times V(H) \) and edge set \( \{(x,y)(x',y'): (x,y), (x',y') \in E(G \square H) \cup E(G \times H)\} \).

**Theorem 2.7** [4] Let \( G_1, G_2 \) be graphs with at least one edge each. Then \( \chi_e(G_1 \boxtimes G_2) \geq \max\{\chi(G_1), \chi(G_2)\} + 2 \).

**Theorem 2.8** [17] The weak tensor product of graphs G and H is the graph \( G \times H \) with vertex set \( V(G) \times V(H) \) and edge set \( \{(x,y)(x',y'): xx' \in E(G) \text{ and } yy' \in E(H)\} \).

**Theorem 2.9** [18] \( \chi(K_m \times K_n) = \min(m,n) \).

**Theorem 2.10** [5] \( \chi_e(K_m \times K_n) = \min(m,n) \).

**Theorem 2.11** [16] The corona of two graphs G and H is the graph \( G \circ H \) formed one copy of G and \( |V(G)| \) copies of H, where the \( i \)-th vertex in the \( i \)-th copy of H.

**Result 1** Let \( G_1 = K_{3,3} \) and \( G_2 = K_{1,1,1,2} \). Then \( \chi_e(G_1 \square G_2) = 4 \).

Graph products are interesting and useful in many situations. For example, Sabidussi [21] showed that any graph has the unique decomposition into prime factors under the Cartesian product. Feigenbaum and Schaffer [19] showed that the strong tensor product admits a polynomial algorithm for decomposing a given connected graph into its factors. An analogous result with respect to weak tensor product is due to Imrich [20]. The complexity of many problems, also equitable coloring, that deal with very large and complicated graphs is reduced greatly if one is able to fully characterize the properties of less complicated prime factors.

Equitable coloring of rooted products

Here we discuss about the equitable coloring on the rooted product of graphs. A graph in which one vertex is fixed as a root vertex to distinguish it from others vertices is called a rooted graph. Let G be a graph with n vertices and H be a sequence of n rooted graphs \( H_1, H_2, \ldots, H_n \). The rooted product graph \( G(H) \) is obtained from the graphs \( G, H_1, H_2, \ldots, H_n \) by identifying the root vertex of \( H_1 \) with the \( i \)-th vertex of G [6].

**Theorem 3.1** Let \( G_1 \) and \( G_2 \) be any two graphs. Then \( \chi_e(G_1 \circ G_2) = \max\{\chi(G_1), \chi(G_2)\} \).

**Proof.** Since \( V(G_1 \circ G_2) = V(G_1 \square G_2) \). \( E(G_1 \circ G_2) \subseteq E(G_1 \square G_2) \). It follows that \( \chi_e(G_1 \circ G_2) \leq \chi_e(G_1 \square G_2) \).

Hence

\[ \chi_e(G_1 \circ G_2) \leq \chi_e(G_1 \square G_2) = \max\{\chi(G_1), \chi(G_2)\}. \] (1)

Also since \( G_1, G_2 \leq G_1 \circ G_2, \chi_e(G_1), \chi_e(G_2) \leq \chi_e(G_1 \circ G_2) \).

Hence

\[ \Box \]
From (1) and (2) \(\chi(G_1 \circ G_2) = M\max\{\chi(G_1), \chi(G_2)\}\).

**Note:** In view of the Theorem 2.1, \(\chi(G_1 \circ G_2) \leq \chi(G_1 \sqcup G_2)\).

The equality need not hold for every pair of graphs. Here is an example where equality fails. Let \(G_1 = K_{3,3}\) and \(G_2 = K_{1,1,2}\). We have \(\chi(G_1) = 2\) and \(\chi(G_2) = 3\) but \(\chi(G_1 \sqcup G_2) = 4\), \(\chi(G_1 \circ G_2) = 3\). Hence \(\chi(G_1 \circ G_2) < \chi(G_1 \sqcup G_2)\).

**Theorem 3.2** If \(G_1\) and \(G_2\) are equitably \(k\) - colorable then \(G_1 \circ G_2\) is equitably \(k\) colorable.

**Proof.** By theorem 2.2 \(G_1 \sqcup G_2\) is equitably \(k\) - colorable. \(G_1 \circ G_2\) is a subgraph of \(G_1 \sqcup G_2\) with \(V(G_1 \circ G_2) = V(G_1 \sqcup G_2)\), such that \(E(G_1 \circ G_2) \subseteq E(G_1 \sqcup G_2)\). So \(G_1 \circ G_2\) is also equitably \(k\) - colorable.

**Corollary 2** \(\chi^*_z(G_1 \circ G_2) \leq \max\{\chi^*_z(G_1), \chi^*_z(G_2)\}\).

**Lemma 3.3** Let \(G = G_i \circ G_j\), where each \(G_i\) is a path, a cycle, a hypercube or a complete graph. Then \(\chi(G) = \chi(G_i) = \chi^*_z(G_i) = \max\{\chi(G_i)/i = 1, 2\}\).

We generalize the above lemma as follows.

**Theorem 3.4** Let \(G = G_1 \circ G_2 \circ \ldots \circ G_n\), where each \(G_i\) is a path, a cycle, a hypercube or a complete graph. Then \(\chi(G) = \chi(G_i) = \chi^*_z(G_i) = \max\{\chi(G_i)/i = 1, 2, \ldots, n\}\).

**Proof.** For all the above graphs \(\chi(G_i) = \chi(G_i) = \chi^*_z(G_i)\) for each \(i\). \(\chi(G) \leq \chi(G_i) \leq \chi^*_z(G) \leq \max\{\chi^*_z(G_i)\} = \max\{\chi(G_i)\} = \chi(G)\).

**Corollary 3** \(\chi(P_m \circ P_n) = 2\).

**Theorem 3.5** Let \(G_1(X_1, Y_1)\) and \(G_2(X_2, Y_2)\) be two bipartite graphs such that one of them contains at last one edge and let \(|X_2| = |Y_2|\). Then \(\chi(G_1 \circ G_2) = 2\).

**Proof.** The graph \(G_1 \circ G_2\) is given in the following figure 1. The polygons represent independent sets \(Z_1, Z_2, Z_3, \) and \(Z_4\) of \(|X_1||X_2|, |X_1||Y_2|, |Y_1||X_2|, |Y_1||Y_2|\) vertices respectively. The lines show the possibilities of existing edges. If \(|X_2| = |Y_2|\), we can assign color 1 to the vertices of \(Z_1\) and \(Z_2\) and color 2 to the remaining vertices. The obtained coloring is equitable.

**Corollary 3** Let \(k, m, n\) and \(r\) be positive integers. Then the equitable chromatic numbers of the following graphs are all equal to 2. \(C_{2m} \circ C_{2n}, P_m \circ C_{2n}, Q_r \circ C_{2n}, K_{m,k} \circ C_{2n}, K_{1, m} \circ C_{2m}, P_m \circ C_{2n}, P_m \circ P_{2n}, Q_r \circ P_{2n}, K_{m,k} \circ P_{2n}, K_{1, m} \circ P_{2m}, Q_r \circ Q_t\) where \(Q_r\) is a hypercube and \(K_{k,m}\) is a complete bipartite graph.
Proof. Immediate from the above theorem.

**Theorem 3.6** Let $G_1(X_1, X_2, \ldots, X_r)$ and $G_2(Y_1, Y_2, \ldots, Y_r)$ be any $r$-partite graphs such that $|Y_1| = |Y_2| = \ldots = |Y_r|$. Then $X_r(G_1 \square G_2) \leq r$.

**Proof.** Use an $r \times r$ Latin square.

$$
\begin{array}{cccc}
1 & 2 & 3 & \cdots & r-1 & r \\
2 & 3 & 4 & \cdots & r & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
r & 1 & 2 & \cdots & r-2 & r-1
\end{array}
$$

**Conclusion**
We have found that the exact bound for the equitable chromatic number and equitable chromatic threshold number of rooted product of some fundamental structure graphs.

**References**