Research Article

Cartesian Product Of $S^{\bullet}-$Valued Graphs

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ABSTRACT: The Notion Of S- Valued Graphs Developed In The Year 2015. Later We Study The Different Products In S-Valued Graphs. In Particular, We Studied The Concept Of S-Valued Graphs By Means Of Cartesian Product. In This Paper, We Precede The Idea Of $S^{\bullet}$ Valued Graphs And Their Cartesian Product Along With Some Regularity Conditions.

Keywords: S− Valued Graphs, Semiring And Cartesian Product.
Ams Classifications: 16y60, 05c25, 05c76

INTRODUCTION

The Theory Of S-Valued Graphs Is Introduced By Dr.M.Chandramouleeswaran In The Year The Year 2015 [4]. Since Then, Many Works Have Been Carried Out Such As Regularity Conditions, Domination Parameters, Connectivity And Colouring Of Graphs By Various Authors [2]. Recently, We Introduced The Idea Of S− Valued Graphs By Means Of Cartesian Product. The Concept Of Defining Cartesian Product Between Two Graphs Is That The Assignment Of Labels To The Vertices And Edges. By Considering The Vertex Valued Function Σ And The Cartesian Product Of Graphs, Edge Labels Have Been Assigned. As Of Now, All The Authors Worked In S−Valued Graphs Are Allotted The Weights To The Vertices From The Members Within The Semiring S And Use The Canonical Preorder On S To Label The Edge Weights.

Here, We Establish A New Kind Of Graphs Namely, $S^{\bullet}-$ Valued Graphs By Labelling The Weights To The Edges By Considering The Binary Operation '•' In The Semiring And It Is Denoted By $G_{S^{\bullet}}$. Thereafter, We Define The Cartesian Product Of $S^{\bullet}-$ Valued Graphs And Study The Regularity Conditions On $S^{\bullet}-$ Valued Graphs.

PRELIMINARIES

Definition 2.1. [3] By A Semiring S, We Mean An Ordered Triplet $(S,+,\cdot)$ Such That
Under "+" And " · " S Is A Monoid With Additive Identity 0.
" · " Is Distributed Over "+" On Both Sides.
$0.X = X.0 = 0\ \forall X \in S.$

1

Definition 2.2. [3] The Relation $\preccurlyeq$ Is Claimed To Be A Canonical Pre-Order In S If For $A, B \in S$, $A \preccurlyeq B$ If And Only If There Exists $c \in S$ such That $A + C = B$.

Definition 2.3. [4] A Semiring Valued Graph Is A Combination Of A Graph $G = (V, E)$ And A Semiring S, Is Defined To Be The Graph $G^S = (V, E, \Sigma, \Psi)$ Where $\Sigma: V \to S$; $\Psi: E \to S$ Is Defined By

$$\psi(x,y) = \begin{cases} min\{\sigma(x), \sigma(y)\} & if\ \sigma(x) \preceq \sigma(y)\ or\ \sigma(y) \preceq \sigma(x) \\ 0 & otherwise \end{cases}$$

Definition 2.4. A Semiring Valued Graph G^S Is

(1) $S^{\bullet}-$vertex regular if $\sigma(v)$ is same $\forall v \in V$.

(2) $S^{\bullet}-$edge regular if $\psi(u,v)$ is same $\forall (u,v) \in E$.

(3) $S^{\bullet}-$regular if both (1) and (2) will be true.

Definition 2.5. [5] A Semiring valued graph G^S is termed to be a degree regular S− valued graph (d_s-regular graph) if $deg_S(v) = (a, n)$, for all $v \in V$ and for some $a \in S$, $n \in N$.

Definition 2.6. [6] The Cartesian Product Of G and H Is A Graph, Denoted By G□H Whose Vertex Set Is $V(G) \times V(H)$.. Two Vertices (G, H) And (G', H') Are Adjacent If $G = G'$ And $HH' \in E(H)$ Or $GG' \in E(H)$ And $H = H'$. Thus

$$V(G \square H) = \{(G, H)/G \in V(G) And h \in V(H)\}.$$

$E(G \square H) = \{(G,H)(G',H')/G = G', HH' \in E(G) \text{ Or } GG' \in E(H), H = H'\}.$

$S^{\bullet}-$Valued Graphs

Definition 3.1. Let $G = V, E \neq \Phi$ Be A Given Graph And Let S Be Any Semiring, We Define A $S^{\bullet}-$Valued Graph $G^{S^{\bullet}} = (V, E, \Sigma, \Psi)$, Where $\Sigma: V \to S$ And $\Psi: E \to S$ Is Such That $\Psi(X,Y) = \big(\Sigma(X) \bullet \Sigma(Y)\big)$ For $(X,Y) \in E \subseteq V \times V$.

Example 3.2. Let $S = B(5,3) = (\{0,1,2,3,4\}, +, \cdot)$ With The Binary Operations $\oplus$ And $\odot$ Defined By

$$A \oplus B = \begin{cases} A + B \text{ If } A + B \leq 4 \\ \quad C \quad \text{If } C \equiv A + B (\text{Mod } 2), 3 \leq C \leq 4 \end{cases}$$

And $$A \odot B = \begin{cases} A \cdot B \text{ If } \quad A \cdot B \leq 4 \\ \quad C \qquad \text{If } C \equiv A \cdot B (\text{Mod } 2),\ 3 \leq C \leq 4 \end{cases}$$

Then Its Cayley's Tables Is Given Below

$\oplus$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	3
2	2	3	4	3	4
3	3	4	3	4	3
4	4	3	4	3	4

$\odot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	4	4
3	0	3	4	3	4
4	0	4	4	4	4

Consider The Graph $G = (V, E)$, Where The Vertex Set And Edge Set Are Given By $V = \{v_1, v_2, v_3, v_4, v_5\}$ And $E = \{(v_1,v_2), (v_1,v_4), (v_1,v_5), (v_2,v_3), (v_3,v_4), (v_4,v_5)\}$.

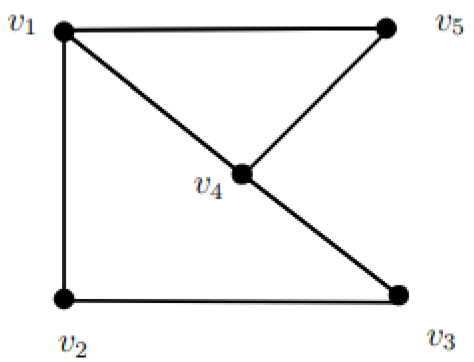
The $S^{\bullet}$ $-$Valued Graph $G^{S^{\bullet}}$ With Respect To The Graph G Is Given As Follows:

Define $\sigma: V \to S$ By $\sigma(v_1) = 3; \sigma(v_2) = 2; \sigma(v_3) = 4; \sigma(v_4) = 1; \sigma(v_5) = 3.$

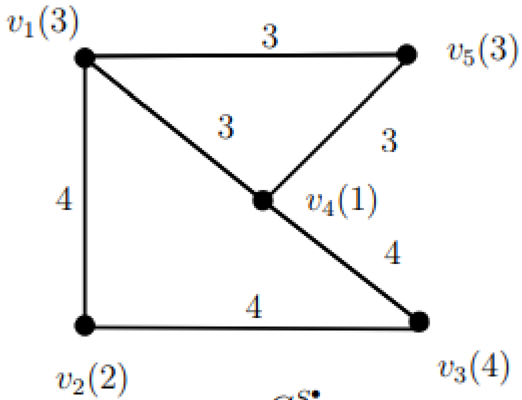
The $S^{\bullet}$ $-$Vertex Set Of $G^{S^{\bullet}}$ *is* $V = \{1,2,3,4\}$.

Define $\psi: E \to S$, Then The $S^{\bullet}$ $-$Edge Set Of $G^{S^{\bullet}}$ *is* $E = \{3,4\}$.

The Graph G And Its Corresponding $S^{\bullet}$ $-$Valued Graph $G_1^{S^{\bullet}}$ Is Given Below:



G



$G^{S^{\bullet}}$

4. CARTESIAN PRODUCT OF $S^{\bullet}$ $-$VALUED GRAPHS

Definition 4.1. Let $G_1^{S^{\bullet}} = (V_1, E_1, \sigma_1, \psi_1)$ and $G_2^{S^{\bullet}} = (V_2, E_2, \sigma_2, \psi_2)$ be two $S^{\bullet}$ $-$valued graphs. Then the Cartesian product of $S^{\bullet}$ $-$valued graph is denoted by

$G_{\square}^{S^{\bullet}} = G_1^{S^{\bullet}} \square G_2^{S^{\bullet}} = (V = V_1 \times V_2, E = E_1 \times E_2, \sigma = \sigma_1 \times \sigma_2, \psi = \psi_1 \times \psi_2)$.

Where $V = \{w_{ij} = (v_i, u_j) | v_i \in V_1 \text{ and } u_j \in V_2\}$ And Two Vertices w_{ij} And w_{kl} Are Adjacent If $i = k$ And $u_j u_l \in E_2$ Or $j = l$ And $v_i v_k \in E_1$.

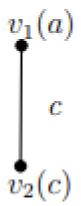
That Is $E = \{E_{ij}^{kl} | \text{ if either } i = k \text{ and } u_j u_l \in E_2 \text{ or } j = l \text{ and } v_i v_k \in E_1\}$,

Define $\sigma = V_1 \times V_2 \to S \times S$ By $\sigma(v_i, u_j) = \sigma(w_{ij}) = \Big(\sigma_1(v_i) \bullet \sigma_2(u_j)\Big)$

$$\psi: E \to S \text{ By } \psi(e_{ij}^{kl}) = \psi\Big((v_i,u_j)(v_k,u_l)\Big) = \begin{cases} \sigma_1(v_i) \bullet \psi_2(u_j, u_l) \text{ if } i = k \text{ and } u_j u_l \in E_2 \\ \psi_1(v_i, v_k) \bullet \sigma_2(u_j) \text{ if } j = l \text{ and } v_i v_k \in E_1 \end{cases}$$

Example 4.2 . Let $S = (\{0, a, b, c\}, +, \cdot)$ Be The Semiring With Its Cayley Tables

+	0	A	B	C



0	0	A	B	C
A	A	B	C	C
B	B	C	C	C
C	C	C	C	C

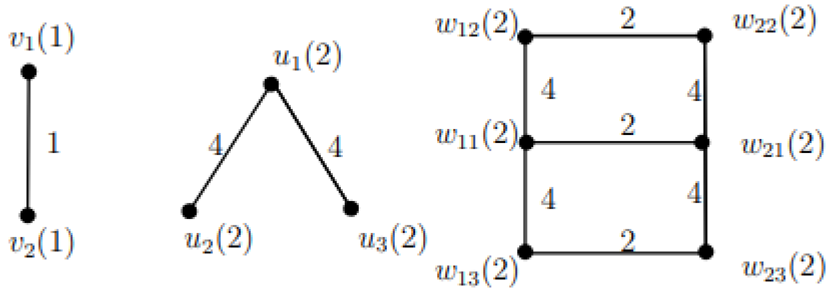
·	0	A	B	C
0	0	0	0	0
A	0	A	B	C
B	0	B	C	C
C	0	C	C	C

Theorem 4.3. The cartesian product of any two $S^{\bullet}$-Vertex regular graph is $S^{\bullet}$- vertex regular.

Proof: Let $G_1^{S^\bullet} = (V_1, E_1, \sigma_1, \psi_1)$ And $G_2^{S^\bullet} = (V_2, E_2, \sigma_2, \psi_2)$ Be Two Given $S^\bullet$-Regular Graphs.

Claim: $G_1^{S^\bullet} \square G_2^{S^\bullet}$ Is $S^\bullet$-Vertex Regular
Since $G_1^{S^\bullet} and\ G_2^{S^\bullet}$ Are $S^\bullet$-Regular Graphs, $\sigma_1(v_i) = s_1$ And $\sigma_2(u_j) = s_2$, For Some $s_1, s_2 \in S$.
Now By Definition, $\sigma(w_{ij}) = \sigma_1(v_i) \bullet \sigma_2(u_j) = s_1 \bullet s_2 = s$, For Some $s \in S$.
This Implies That $\sigma(w_{ij})$ Is Equal For All $w_{ij} \in V$, For All i, j.
Hence $G_1^{S^\bullet} \square G_2^{S^\bullet}$ Is Vertex Regular.

Remark 4.4. Under Cartesian Product, The Cartesian Product Two S− Vertex Regular Graphs Is S− Edge Regular.
But In $S^\bullet$ Valued Graphs, The Cartesian Product Two $S^\bullet$- Vertex Regular Graphs Is Need Not Be $S^\bullet$-Edge Regular.
For Example, Consider The Semiring Discussed In The Example 3.2



Theorem 4.5. The Cartesian Product Of Any Two $S^\bullet$− Vertex Regular Graphs Is $S^\bullet$− Edge Regular Only If The Semiring Considered Must Be Multiplicatively Idempotent.
Proof: Let $G_1^{S^\bullet} = (V_1, E_1, \sigma_1, \psi_1)$ And $G_2^{S^\bullet} = (V_2, E_2, \sigma_2, \psi_2)$ Be Any Two $S^\bullet$- Vertex Regular Graphs, $\sigma_1(v_i) = s_1$ And $\sigma_2(u_j) = s_2$, For Some $s_1, s_2 \in S$.
Claim: $G_1^{S^\bullet} \square G_2^{S^\bullet}$ Is $S^\bullet$- Regular
Now By Definition, $\sigma(w_{ij}) = \sigma_1(v_i) \bullet \sigma_2(u_j) = s_1 \bullet s_2 = s$, For Some $s \in S$.
This Implies That $\sigma(w_{ij})$ Is Equal For All $w_{ij} \in V$, For All i, j.
Hence $G_1^{S^\bullet} \square G_2^{S^\bullet}$ Is Vertex Regular.

By Definition

$$\psi(e_{ij}^{kl}) = \psi((v_i, u_j)(v_k, u_l)) = \begin{cases} \sigma_1(v_i) \bullet \psi_2(u_j, u_l)\ if\ i = k\ and\ u_j u_l \in E_2 \\ \psi_1(v_i, v_k) \bullet \sigma_2(u_j)\ if\ j = l\ and\ v_i v_k \in E_1 \end{cases}$$

$$= \begin{cases} \sigma_1(v_i) \bullet (\sigma_2(u_j) \bullet \sigma_2(u_l)) \ if \ i = k \ and \ u_j u_l \in E_2 \\ (\sigma_1(v_i) \bullet \sigma_1(v_k)) \bullet \ \sigma_2(u_j) \ if \ j = l \ and \ v_i v_k \in E_1 \end{cases}$$

$$= \begin{cases} \sigma_1(v_i) \bullet \sigma_2(u_j) \ if \ i = k \ and \ u_j u_l \in E_2 \\ \sigma_1(v_i) \bullet \sigma_2(u_j) \ if \ j = l \ \text{And} \ v_i v_k \in E_1 \end{cases}$$

This Is True For All $e_{ij}^{kl} \in E$, Therefore $G_1^{S^\bullet} \square G_2^{S^\bullet}$ Is $S^\bullet$- Edge Regular And Hence $S^\bullet$- Regular.

Theorem4.7. The Cartesian Product Of Any Two S•–Edge Regular Graphs Is S•–Edge Regular Only If The Semiring Considered Must Be Additively Idempotent.

Proof: Let $G_1^{S^\bullet} = (V_1, E_1, \sigma_1, \psi_1)$ And $G_2^{S^\bullet} = (V_2, E_2, \sigma_2, \psi_2)$ Be Any Two $S^\bullet$- Edge Regular Graphs.

Claim: $G_1^{S^\bullet} \square G_2^{S^\bullet}$ Is $S^\bullet$- Edge Regular.

By Definition

$$\psi(e_{ij}^{kl}) = \psi((v_i, u_j)(v_k, u_l)) = \begin{cases} \sigma_1(v_i) \bullet \psi_2(u_j, u_l) \ if \ i = k \ and \ \mathrm{U_J U_L} \in \mathrm{E_2} \\ \Psi_1(\mathrm{V_I}, \mathrm{V_K}) \bullet \Sigma_2(\mathrm{U_J}) \ \text{If J = L And} \ \mathrm{V_I V_K} \in \mathrm{E_1} \end{cases}$$

$$= \begin{cases} \Sigma_1(\mathrm{V_I}) \bullet (\Sigma_2(\mathrm{U_J}) \bullet \Sigma_2(\mathrm{U_L})) \ \text{If I = K And} \ \mathrm{U_J U_L} \in \mathrm{E_2} \\ (\Sigma_1(\mathrm{V_I}) \bullet \Sigma_1(\mathrm{V_K})) \bullet \ \Sigma_2(\mathrm{U_J}) \ \text{If J = L And} \ \mathrm{V_I V_K} \in \mathrm{E_1} \end{cases}$$

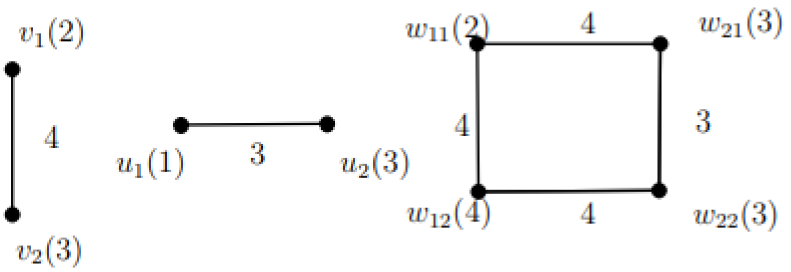
$$= \begin{cases} \Sigma_1(\mathrm{V_I}) \bullet \Sigma_2(\mathrm{U_J}) \ \text{If I = K And} \ \mathrm{U_J U_L} \in \mathrm{E_2} \\ \Sigma_1(\mathrm{V_I}) \bullet \Sigma_2(\mathrm{U_J}) \ \text{If J = L And} \ \mathrm{V_I V_K} \in \mathrm{E_1} \end{cases}$$

This Is True For All $\mathrm{E_{Ij}^{Kl}} \in \mathrm{E}$, Therefore $\mathrm{G_1^{S^\bullet}} \square \mathrm{G_2^{S^\bullet}}$ Is S•- Edge Regular.

Corollary 4.8. If $\mathrm{G_1^{S^\bullet}}, \mathrm{G_2^{S^\bullet}}$ Are Two S•–Vertex Regular Graphs And S Is An Additively Idempotent Semiring Then Their Cartesian Product Is $\mathrm{G_\square^{S^\bullet}}$ Is S•–Regular.

Remark 4.9. If $\mathrm{G_1^{S^\bullet}}, \mathrm{G_2^{S^\bullet}}$ Are Two S•-Regular Graphs And S Is An Additively Idempotent Semiring, Then S•-Regular And S-Regular Coincides.

Example 4.10. The Cartesian product of any two d_S- regular $S^\bullet$-valued graphs is not be d_S- regular. Consider the following two d_S- regular $S^\bullet$-valued graphs $G_1^{S^\bullet}$ and $G_2^{S^\bullet}$.



Here $\mathrm{D_s(W_{11})}$=(4,2); $\mathrm{D_s(W_{21})}$=(3,2); $\mathrm{D_s(W_{22})}$=(3,2); $\mathrm{D_s(W_{12})}$=(4,2).

Here The Degrees Of The Vertices Are Different; We Observe That $\mathrm{G_\square^{S^\bullet}}$ Is Not $\mathrm{D_s}$-Regular.

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