Research Article

New Stability Results of Multiplicative Inverse Quartic Functional Equations

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Abstract: The purpose of this investigation is to introduce different forms of multiplicative inverse functional equations, to solve them and to establish the stability results of them in the framework of matrix normed spaces. A suitable counter-example is also provided to prove the instability of the results for a singular case. The applications of the equations dealt in this study are discussed with the fluid resistances of blood vessels and also an important concept in Raman spectroscopy.

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1. Introduction & Preliminaries

Various linear spaces of bounded Hilbert space operators such as mapping spaces, tensor products of operator spaces, quotients spaces in operator theory are abstractly characterized through matrix normed spaces [19]. The characterization of these spaces indicates that they further be considered as spaces of operator. Due to this result, the operator spaces theory has noteworthy application in operator algebra theory [5]. The result obtained in [19] is invoked to the theory of ordered operator spaces [19]. The proof given in [6] is achieved by the technique applied in [13].

Here, we evoke the fundamental ideas of matrix normed spaces. We utilize the ensuing notions:

- $M_r(\mathcal{A})$ is the set of all square matrices of order r in a normed space $\mathcal{A}$;
- $e_n \in M_{1,j}(\mathbb{C})$ denotes n^{th} element is 1, and the other elements are 0;
- $E_{mn} \in M_r(\mathbb{C})$ means $(m,n)^{\text{th}}$-element is 1, and the other elements are 0;
- $E_{mn} \otimes u \in M_r(\mathcal{A})$ indicates $(m,n)^{\text{th}}$-element is u, and the other elements are 0.
- For $u \in M_r(\mathcal{A})$, $v \in M_s(\mathcal{A})$,

$$u \oplus v = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}.$$

Definition 1.1 *Let $\mathcal{A}$ be a normed space with norm $\|\cdot\|$. Then $\mathcal{A}$ is called as a matrix normed space with norm $\|\cdot\|_r$ if and only if $M_r(\mathcal{A})$ is a normed space with norm $\|\cdot\|_r$ for each integer $r > 0$ and $\|\mathcal{X}u\mathcal{Y}\|_s \leq \|\mathcal{X}\|\,\|\mathcal{Y}\|\,\|u\|_r$ is true for $\mathcal{X} \in M_{s,r}(\mathbb{C}), u = (u_{ij}) \in M_r(\mathcal{X})$ and $\mathcal{Y} \in M_{r,s}(\mathbb{C})$.*

Definition 1.2 *$(\mathcal{A}, \|\cdot\|_r)$ is a matrix complete normed space (or matrix Banach space) if and only if $\mathcal{A}$ is a complete normed space (or Banach space) and a matrix normed space with norm $\|\cdot\|_r$.*

Definition 1.3 *Let $\mathcal{A}$ be a matrix normed space with norm $\|\cdot\|_r$. Then $\mathcal{A}$ is said to be an L^∞- matrix normed space if $\|u \oplus v\|_{r+s} = max\{\|u\|_r, \|v\|_s\}$ holds for all $u \in M_r(\mathcal{A})$ and all $v \in M_s(\mathcal{A})$.*

Suppose A_1 and A_2 are vector spaces. Then for a given mapping $q: A_1 \longrightarrow A_2$ and for an integer $r > 0$, define $q_r: M_r(A_1) \longrightarrow M_r(A_2)$ by

$$q_r([u_{mn}]) = [q(u_{mn})]$$

for all $[u_{mn}] \in M_r(A_1)$. More information pertinent to matrix normed spaces are available in [12, 27]

The inspiration for the stability theory of mathematical equations is due to the question raised in [28] regarding homomorphisms in group theory. There are various responses provided in [1, 8, 11, 14, 15] to the question posed in [28]. For the first time, the functional equation

$$p(\theta_1 + \theta_2) = \frac{p(\theta_1)p(\theta_2)}{p(\theta_1)p(\theta_2)} \tag{1.1}$$

where the mapping r is defined in the domain of real numbers excluding zero, is dealt with and its stableness is investigated pertinent to the fundamental stability theory in [17]. The equation (1.1) is called as reciprocal or multiplicative inverse or rational functional equation whose solution is a reciprocal function $p(\theta) = \frac{k}{\theta}$, where $\theta(\neq 0) \in \mathbb{R}$ and k is any real constant.

The non-Archimedean stability of the multiplicative inverse fourth power functional equation

$$q(\theta_1 + \theta_2) = \frac{q(\theta_1)q(\theta_2)}{\left[q(\theta_1)^{\frac{1}{4}} + q(\theta_2)^{\frac{1}{4}}\right]^4} \tag{1.2}$$

is obtained in [10]. It can be easily verified that the multiplicative inverse fourth power mapping $q(\theta) = \frac{1}{\theta^4}$ satisfies equation (1.2).

Later, there are several published papers on solutions, stability results and applications of various forms of multiplicative inverse or rational type or reciprocal functional equations in the literature. For further details, one can refer to [3, 9, 16, 18, 20, 21, 22, 23, 24, 25, 26].

In order to explore applications further, we extend equation (1.2) to new forms as, multiplicative inverse fourth power difference functional equation

$$q\left(\frac{\theta_1+\theta_2}{2}\right) - q(\theta_1 + \theta_2) = \frac{15q(\theta_1)q(\theta_2)}{\left[q(\theta_1)^{\frac{1}{4}} + q(\theta_2)^{\frac{1}{4}}\right]^4} \tag{1.3}$$

and a multiplicative inverse fourth power adjoint functional equation

$$q\left(\frac{\theta_1+\theta_2}{2}\right) + q(\theta_1 + \theta_2) = \frac{17q(\theta_1)q(\theta_2)}{\left[q(\theta_1)^{\frac{1}{4}} + q(\theta_2)^{\frac{1}{4}}\right]^4} \tag{1.4}$$

We prove the equivalency of equations (1.3) and (1.4) to achieve their solutions. The stability results of (1.3) and (1.4) are investigated via direct and fixed point techniques in the domain of matrix normed spaces. An apt instance is demonstrated to substantiate the non-stability result. The inferences of equations (1.3) and (1.4) are acquired by employing them in certain occurences in fluid dynamics and Raman spectroscopy. In the entire study, let us assume that $\theta_2 \neq -\theta_1$ to avoid singularity in the main results. Also, unless or otherwise specified, we consider $\mathcal{A}$ to be a matrix normed space containing non-singular square matrices of m with a norm $\|\cdot\|$ so that $\|\theta\| \leq 1$ for all $\theta \in \mathcal{A}$ and $\mathcal{B}$ to be a matrix complete normed space, respectively with norm $\|\cdot\|_r$. Then, applying Taylor's series expansion, we can find $\theta^{\frac{1}{p}}$ after truncating to $(p+1)$ terms [2]. Thus, the rational powers of θ can be computed for all $\theta \in \mathcal{A}$. For a mapping $q: \mathcal{A} \longrightarrow \mathcal{B}$ and for easy computation, let the difference operators $Dq: \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{B}$ and $Dq_r: M_p(\mathcal{A} \times \mathcal{A}) \longrightarrow M_r(\mathcal{B})$ be defined by

$$Dq(\theta_1, \theta_2) = q\left(\frac{\theta_1+\theta_2}{2}\right) - q(\theta_1 + \theta_2) - \frac{15q(\theta_1)q(\theta_2)}{[q(\theta_1)^{1/4} + q(\theta_2)^{1/4}]^4}$$

$$Dq_r\left([\theta_{1_{mn}}], [\theta_{2_{mn}}]\right) = q_r\left(\frac{[\theta_{1_{mn}}] + [\theta_{2_{mn}}]}{2}\right) - q_r\left([\theta_{1_{mn}}] + [\theta_{2_{mn}}]\right) - \frac{15q_r([\theta_{1_{mn}}])q_r([\theta_{2_{mn}}])}{\left[q_r\left([\theta_{1_{mn}}]\right)^{1/4} + q_r[\theta_{2_{mn}}]^{1/4}\right]^4}$$

for all $\theta_1, \theta_2 \in \mathcal{A}$, and all $\theta_1 = [\theta_{1_{mn}}]$, $\theta_2 = [\theta_{2_{mn}}] \in M_r(\mathcal{A})$.

2. Identicalness of equations (1.3) and (1.4)

In the ensuing theorem, we prove that equations (1.3) and (1.4) are equivalent to each other.

Theorem 2.1 *Let $q: \mathbb{R}^{\star} \longrightarrow \mathbb{R}$ be a mapping. Then, the following statements are equivalent.*

- q is solution of *(1.2).*
- q is solution of *(1.3).*
- q is solution of *(1.4).*

Hence, the mapping q is a multiplicative inverse fourth power mapping.

Proof.

1. We assume that q is a solution of (1.2). Then q satisfies (1.2). Now, plugging (θ_1, θ_2) by $\left(\frac{\theta}{2}, \frac{\theta}{2}\right)$ in (1.2) and then multiplying by 16, we obtain

$$q\left(\frac{\theta}{2}\right) = 16q(\theta) \tag{2.1}$$

for all $\theta \in \mathbb{R}^{\star}$. Now, replacing (θ_1, θ_2) by $\left(\frac{\theta_1}{2}, \frac{\theta_2}{2}\right)$ in (1.2) and in lieu of (2.1) in the resulting equation, one finds

$$q\left(\frac{\theta_1+\theta_2}{2}\right) = \frac{16q(\theta_1)q(\theta_2)}{\left[q(\theta_1)^{\frac{1}{4}} + q(\theta_2)^{\frac{1}{4}}\right]^4} \tag{2.2}$$

for all $\theta_1, \theta_2 \in \mathbb{R}^{\star}$. Now, subtracting (1.2) from (2.2), we arrive at (1.3).

2. Suppose q is a solution of (1.4). Then, it satisfies (1.4). Now, if (θ_1, θ_2) is replaced by $\left(\frac{\theta}{2}, \frac{\theta}{2}\right)$ in (1.4) and simplified further, then we have

$$q\left(\frac{\theta}{2}\right) = 16q(\theta) \tag{2.3}$$

for all $\theta \in \mathbb{R}^\star$. Using (2.3) in (1.3) and further simplifying, we obtain

$$q(\theta_1 + \theta_2) = \frac{q(\theta_1)q(\theta_2)}{\left[q(\theta_1)^{\frac{1}{4}} + q(\theta_2)^{\frac{1}{4}}\right]^4} \tag{2.4}$$

for all $\theta_1, \theta_2 \in \mathbb{R}^\star$. Now, reinstating (θ_1, θ_2) by $\left(\frac{\theta_1}{2}, \frac{\theta_2}{2}\right)$ in (2.4) and then employing (1.3), we get

$$q\left(\frac{\theta_1+\theta_2}{2}\right) = \frac{16q(\theta_1)q(\theta_2)}{\left[q(\theta_1)^{\frac{1}{4}} + q(\theta_2)^{\frac{1}{4}}\right]^4} \tag{2.5}$$

for all $\theta_1, \theta_2 \in \mathbb{R}^\star$. Summing (2.5) with (2.4), we obtain (1.4).

3. Suppose q is a solution of (1.4). Then it satisfies (1.4). By the analogous reasoning stated above, when (θ_1, θ_2) is substituted by $\left(\frac{\theta}{2}, \frac{\theta}{2}\right)$ in (1.4) and simplified additionally, we have

$$q\left(\frac{\theta}{2}\right) = 16q(\theta) \tag{2.6}$$

for all $\theta \in \mathbb{R}^\star$. Utilizing the result of (2.6) in (1.4), it leads to (1.2).

Hence q is a multiplicative inverse fourth power mapping.

3. Stabilities of equation (1.3) via direct technique

In this part, we determine the stabilities of equation (1.3) in the domain of matrix normed spaces. The following lemma is a key element to achieve our major results.

Lemma 3.1 ***[4]*** *The following assertions are true:*

- $\|E_{mn} \otimes \theta_1\|_r = \|\theta_1\|$ for $\theta_1 \in \mathcal{A}$.
- $\|\theta_{1_{mn}}\| \leq \left\|[\theta_{1_{mn}}]\right\|_r \leq \sum_{m,n=1}^{r} \|\theta_{1_{mn}}\|$ for $[\theta_{1_{mn}}] \in M_r(\mathcal{A})$.
- $\lim_{r\to\infty} \theta_{1_r} = \theta_1$ if and only if $\lim_{r\to\infty} \theta_{1_{rmn}} = \theta_{1_{mn}}$ for $\theta_{1_r} = [\theta_{1_{rmn}}]$, $\theta_1 = [\theta_{1_{mn}}] \in M_r(\mathcal{A})$.

Theorem 3.2 *Let a mapping $q_r: \mathcal{A} \longrightarrow \mathcal{B}$ satisfies*

$$\left\|Dq_r([\theta_{1_{mn}}], [\theta_{2_{mn}}])\right\|_r \leq \sum_{m,n=1}^{r} \psi(\theta_{1_{mn}}, \theta_{2_{mn}}) \tag{3.1}$$

where $\psi: \mathcal{A} \times \mathcal{A} \longrightarrow [0, \infty)$ is a function such that

$$\Upsilon(\theta_1, \theta_2) = 16 \sum_{l=1}^{\infty} \frac{1}{16^l} \psi\left(\frac{\theta_1}{2^l}, \frac{\theta_2}{2^l}\right) < \infty, \tag{3.2}$$

for all $\theta_1, \theta_2 \in \mathcal{A}$, and all $\theta_1 = [\theta_{1_{mn}}]$, $\theta_2 = [\theta_{2_{mn}}] \in M_r(\mathcal{A})$. Then, there a unique solution $Q: \mathcal{A} \longrightarrow \mathcal{B}$ of *(1.3)* exists with the result that

$$\|q_r([\theta_{mn}]) - Q_r([\theta_{mn}])\|_r \leq \sum_{m,n=1}^{r} \Upsilon(\theta_{mn}, \theta_{mn}) \tag{3.3}$$

for all $\theta = [\theta_{mn}] \in M_r(\mathcal{A})$.

Proof. Firstly, let us assume $r = 1$ in (3.1) and proceed to prove the result. Hence, we have

$$\|Dq(\theta_1, \theta_2)\| \leq \psi(\theta_1, \theta_2)$$

for all $\theta_1, \theta_2 \in \mathcal{A}$. Then, a unique multiplicative inverse fourth power mapping $Q: \mathcal{A} \longrightarrow \mathcal{B}$ exists which is unique and

$$\|q(\theta) - Q(\theta)\| \leq \Upsilon(\theta, \theta)$$

for all $\theta \in \mathcal{A}$. Now, let a mapping $Q: \mathcal{A} \longrightarrow \mathcal{B}$ be defined as $Q(\theta) = \lim_{l\to\infty} \frac{1}{16^l} q\left(\frac{\theta}{2^l}\right)$ for all $\theta \in \mathcal{A}$. In view of the outcome of Lemma 3.1, we find

$$\|q_r([\theta_{mn}]) - Q_r([\theta_{mn}])\|_r \leq \sum_{m,n=1}^{r} \|q(\theta_{mn}) - Q(\theta_{mn})\| \leq \sum_{m,n=1}^{r} \psi(\theta_{mn}, \theta_{mn})$$

for all $\theta = [\theta_{mn}] \in M_r(\mathcal{A})$. Therefore, $Q: \mathcal{A} \longrightarrow \mathcal{B}$ is a unique solution of (3.3) and hence it is multiplicative inverse fourth power mapping, as required. This completes the proof.

Corollary 3.3 *Suppose $s > -4$ and $\lambda(\geq 0) \in \mathbb{R}$. Let a mapping $q: \mathcal{A} \longrightarrow \mathcal{B}$ satisfies*

$$\left\|Dq_r([\theta_{1_{mn}}], [\theta_{2_{mn}}])\right\|_r \leq \sum_{m,n=1}^{r} \lambda\left(\|\theta_{1_{mn}}\|^s + \|\theta_{2_{mn}}\|^s\right) \tag{3.4}$$

for all $\theta_1 = [\theta_{1_{mn}}]$, $\theta_2 = [\theta_{2_{mn}}] \in M_r(\mathcal{A})$. Then, a solution $Q: \mathcal{A} \longrightarrow \mathcal{B}$ of *(1.3)* exists which is unique with the result that

$$\|q_r([\theta_{mn}]) - Q_r([\theta_{mn}])\|_r \leq \sum_{m,n=1}^{r} \frac{32\lambda}{2^{s+4}-1} \|\theta_{mn}\|^s$$

for all $\theta = [\theta_{mn}] \in M_r(\mathcal{A})$.

Proof. The required outcome goes along with the proof of Theorem 3.2 by letting $\psi(\theta_{1_{mn}}, \theta_{2_{mn}}) = \lambda\left(\|\theta_{1_{mn}}\|^s + \|\theta_{2_{mn}}\|^s\right)$.

Theorem 3.4 *Let a mapping $q: \mathcal{A} \longrightarrow \mathcal{B}$ satisfies (3.1). Suppose a function $\psi: \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$ satisfies*

$$\Upsilon(\theta_1, \theta_2) = 16 \sum_{l=0}^{\infty} 16^l \psi(2^l\theta_1, 2^l\theta_2) < \infty \tag{3.5}$$

for all $\theta_1, \theta_2 \in \mathcal{A}$. Then, a solution $Q: \mathcal{A} \longrightarrow \mathcal{B}$ of *(1.3)* exists which is unique with the result that

$$\|q_r([\theta_{mn}]) - Q_r([\theta_{mn}])\|_r \leq \sum_{m,n=1}^{r} \Upsilon(\theta_{mn}, \theta_{mn})$$

for all $\theta = [\theta_{mn}] \in M_r(\mathcal{A})$.

Proof. The proof goes through the same way as in Theorem 3.2, and so it is excluded.

Corollary 3.5 *Suppose $s < -4$ and $\lambda(\geq 0) \in \mathbb{R}$. Let a mapping $q: \mathcal{A} \longrightarrow \mathcal{B}$ satisfies (3.4). Then, a solution $Q: \mathcal{A} \longrightarrow \mathcal{B}$ of (1.3) exists which is unique and with the result that*

$$\|q_r([\theta_{mn}]) - Q_r([\theta_{mn}])\|_r \leq \sum_{m,n=1}^{r} \frac{32\lambda}{1-2^{s+4}} \|\theta_{mn}\|^s$$

for all $\theta = [\theta_{mn}] \in M_r(\mathcal{A})$.

Proof. The proof is a direct consequence of Theorem 3.4 by considering $\psi(\theta_{1_{mn}}, \theta_{2_{mn}}) = \lambda\left(\left\|\theta_{1_{mn}}\right\|^s + \left\|\theta_{2_{mn}}\right\|^s\right)$.

The ensuing lemma plays a key role in proving our major results.

Lemma 3.6 ***[27]**. Let F be a L^∞-matrix normed space. Then $\|[\theta_{mn}]\|_r \leq \|[\|\theta_{mn}\|]\|_r$ for all $[\theta_{mn}] \in M_r(F)$.*

Theorem 3.7 *Assume that $\mathcal{B}$ to be a L^∞-normed Banach space. Let a mapping $q_r: \mathcal{A} \longrightarrow \mathcal{B}$ satisfies*

$$\left\|Dq_r([\theta_{1_{mn}}], [\theta_{2_{mn}}])\right\|_r \leq \left\|[\psi(\theta_{1_{mn}}, \theta_{2_{mn}})]\right\|_r \tag{3.6}$$

where $\varphi: \mathcal{A} \times \mathcal{A} \longrightarrow [0, \infty)$ is a function which satisfies *(3.1)* for all $\theta_1 = [\theta_{1_{mn}}]$, $\theta_2 = [\theta_{2_{mn}}] \in M_r(\mathcal{A})$. Then, a solution $Q: \mathcal{A} \longrightarrow \mathcal{B}$ of *3* exists which is unique with the result that

$$\|[q(\theta_{mn}) - Q(\theta_{mn})]\|_r \leq \|[\Upsilon(\theta_{mn}, \theta_{mn})]\|_r \tag{3.7}$$

for all $\theta = [\theta_{mn}] \in M_r(\mathcal{A})$. Here Υ is assumed as in Theorem *3.2*.

Proof. Using the identical arguments applied to prove Theorem 3.2, we find that a multiplicative inverse fourth power mapping $Q: \mathcal{A} \longrightarrow \mathcal{B}$ exists and unique so that $\|q(\theta) - Q(\theta)\| \leq \Upsilon(\theta, \theta)$ for all $\theta \in \mathcal{A}$. The mapping $Q: \mathcal{A} \longrightarrow \mathcal{B}$ is given by $Q(\theta) = \lim_{l \to \infty} \frac{1}{16^l} q\left(\frac{\theta}{2^l}\right)$ for all $\theta \in \mathcal{A}$. It is not hard to demonstrate that if $0 \leq u_{mn} \leq v_{mn}$ for all m, n, then

$$\|[u_{mn}]\|_r \leq \|[v_{mn}]\|_r. \tag{3.8}$$

By Lemma 3.6 and inequality (3.8), we have

$$\|[q(\theta_{mn}) - Q(\theta_{mn})]\|_r \leq \|[\|q(\theta_{mn}) - Q(\theta_{mn})\|]\|_r \leq \left\|\Upsilon(\theta_{mn}, \theta_{ij})\right\|_r$$

for all $\theta = [\theta_{mn}] \in M_r(\mathcal{A})$. Hence, we have the inequality (3.7), which concludes the proof.

Corollary 3.8 *Let $\mathcal{B}$ be a L^∞-complete normed space. Let $s > -4$ and $\lambda(\geq 0) \in \mathbb{R}$. Let a mapping $q: \mathcal{A} \longrightarrow \mathcal{B}$ satisfies*

$$\left\|Dq_p([\theta_{1_{mn}}], [\theta_{2_{mn}}])\right\|_r \leq \left\|[\lambda\left(\left\|\theta_{1_{mn}}\right\|^s + \left\|\theta_{2_{mn}}\right\|^s\right)]\right\|_r \tag{3.9}$$

for all $\theta_1 = [\theta_{1_{mn}}]$, $\theta_2 = [\theta_{2_{mn}}] \in M_r(\mathcal{A})$. Then, a solution $Q: \mathcal{A} \longrightarrow \mathcal{B}$ of *(1.3)* exists which is unique with the result that

$$\|q_r([\theta_{mn}]) - Q_r([\theta_{mn}])\|_r \leq \left\|\left[\frac{32\lambda}{2^{s+4}-1} \|\theta_{mn}\|^s\right]\right\|_r$$

for all $\theta = [\theta_{mn}] \in M_r(\mathcal{A})$.

Proof. The aspired result is achieved through Theorem 3.7 by taking $\psi(\theta_1, \theta_2) = \lambda(\|\theta_1\|^s + \|\theta_2\|^s)$.

Theorem 3.9 *Assume that $\mathcal{B}$ to be a L^∞-complete normed space. Let a mapping $q: \mathcal{A} \longrightarrow \mathcal{B}$ satisfies (3.6) and $\psi: \mathcal{A} \times \mathcal{A} \longrightarrow [0, \infty)$ is a function satisfying (3.5). Then, a solution $Q: \mathcal{A} \longrightarrow \mathcal{B}$ of (1.3) exists whihc is unique with the result that*

$$\|[q(\theta_{mn}) - Q(\theta_{mn})]\|_r \leq \|[\Upsilon(\theta_{mn}, \theta_{mn})]\|_r$$

for all $\theta = [\theta_{mn}] \in M_r(\mathcal{A})$. Here Υ is defined as in Theorem *3.4*.

Proof. The required result follows via the proof of Theorem 3.7, and so the details are neglected.

Corollary 3.10 *Let $\mathcal{B}$ be a L^∞-complete normed space. Let $s < -4$ and $\lambda(\geq 0) \in \mathbb{R}$. Let a mapping $q: \mathcal{A} \longrightarrow \mathcal{B}$ satisfies (3.9). Then, a solution $Q: \mathcal{A} \longrightarrow \mathcal{B}$ of (1.3) exists whihc is unique with the result that*

$$\|q_r([\theta_{mn}] - Q_r([\theta_{mn}]))\|_p \leq \left\|\left[\frac{32\lambda}{1-2^{s+4}} \|\theta_{mn}\|^s\right]\right\|_r$$

for all $\theta = [\theta_{mn}] \in M_r(\mathcal{A})$.

Proof. The similar arguments as in the proof of Theorem 3.9 will lead to the proof of this corollary by taking $\psi(\theta_{1_{mn}}, \theta_{2_{mn}}) = \lambda\left(\left\|\theta_{1_{mn}}\right\|^s + \left\|\theta_{2_{mn}}\right\|^s\right)$.

4. Stabilities of (1.3) via fixed point technique

Employing fixed point method, we obtain the stabilities of equation (1.3) in the framework of matrix normed spaces.

Theorem 4.1 *Let a mapping $q: \mathcal{A} \longrightarrow \mathcal{B}$ satisfies*

$$\left\|Dq_r([\theta_{1_{mn}}], [\theta_{2_{mn}}])\right\|_r \leq \sum_{m,n=1}^{r} \psi(\theta_{1_{mn}}, \theta_{2_{mn}}) \tag{4.1}$$

where $\psi: \mathcal{A} \times \mathcal{A} \longrightarrow [0, \infty)$ is a function such that there exists an $P < 1$ with

$$\psi(\theta_1, \theta_2) \leq \frac{L}{16} \psi\left(\frac{\theta_1}{2}, \frac{\theta_2}{2}\right), \tag{4.2}$$

for all $\theta_1, \theta_2 \in \mathcal{A}$, for all $\theta_1 = [\theta_{1_{mn}}], \theta_2 = [\theta_{2_{mn}}] \in M_r(\mathcal{A})$. Then, there a solution $Q: \mathcal{A} \longrightarrow \mathcal{B}$ of *(1.3)* which is unique with the result that

$$\|q_r([\theta_{mn}]) - Q_r([\theta_{mn}])\|_r \leq \sum_{m,n=1}^{r} \frac{16}{1-P} \psi(\theta_{mn}, \theta_{mn}) \quad (4.3)$$

for all $\theta = [\theta_{mn}] \in M_r(\mathcal{A})$.

Proof. First, let us consider $r = 1$ in (4.1) and proceed to prove the result. Then, we have

$$\|Dq(\theta_1, \theta_2)\| \leq \psi(\theta_1, \theta_2)$$

for all $\theta_1, \theta_2 \in \mathcal{A}$. Substituting $\theta_1 = \theta_2 = \theta$ in above inequality and then multiplying by 16 on its both sides, we get

$$\|q(\theta) - 16q(2\theta)\| \leq 16\psi(\theta, \theta) \quad (4.4)$$

for all $\theta \in \mathcal{A}$. Suppose the following is the generalized metric defined on a set $X = \{h: \mathcal{A} \longrightarrow \mathcal{B}\}$:

$$d(u, v) = \inf\{\lambda \in \mathbb{R}_+ : \|u(\theta) - v(\theta)\| \leq \lambda\psi(\theta, \theta), \forall \theta \in \mathcal{A}\},$$

with the convention that $\inf \psi = +\infty$. It is not hard to show the completeness of X. Suppose a mapping J from X to X with the property that $Ju(\theta) = 16u(\theta)$, for all $\theta \in \mathcal{A}$. Assume $d(u, v) = \lambda$ for given $u, v \in X$. Then, we have

$$\|u(\theta) - v(\theta)\| \leq \lambda\psi(\theta, \theta)$$

for all $\theta \in \mathcal{A}$. Therefore, we find

$$\|Ju(\theta) - Jv(\theta)\| = \|16u(2\theta) - 16v(2\theta)\| \leq \lambda\psi(\theta, \theta)$$

for all $\theta \in \mathcal{A}$. Consequently, $d(u, v) = \lambda$ produces $d(Ju, Jv) \leq \theta\lambda$. This indicates that

$$d(Ju, Jv) \leq \lambda d(u, v)$$

1. Q is a fixed point of the mapping J, that is,

$$Q(2\theta) = \frac{1}{16} Q(\theta) \quad (4.5)$$

for all $\theta \in \mathcal{A}$. We notice that the mapping Q is unique and it is fixed point of J in the set

$$M = \{g \in S: d(Q, g) < \infty\}.$$

This indicates the uniquess of Q and hence it satisifes (5) with a $\lambda \in (0, \infty)$ satisfying $\|q(\theta) - Q(\theta)\| \leq \lambda\psi(\theta, \theta)$, for all $\theta \in \mathcal{A}$;

2. $d(J^l q, Q)$ approaces 0 as l tends to ∞. This produces the existence of the limit $\lim_{l \to \infty} 16^l q(2^l \theta)$ approaching to $Q(\theta)$ for all $\theta \in \mathcal{A}$;
3. $d(q, Q) \leq \frac{1}{1-P} d(q, Jq)$, which gives the inequality $d(q, Q) \leq \frac{16}{1-P}$.

   Hence, we have

$$\|q(\theta) - Q(\theta)\| \leq \frac{16}{1-P} \psi(\theta, \theta) \quad (4.6)$$

for all $\theta \in \mathcal{A}$. In lieu of (4.2) and (4.1), we obtain that

$$\begin{aligned} \|DQ(\theta_1, \theta_2)\| &= \lim_{l \to \infty} 16^l \|Dq(2^l \theta_1, 2^l \theta_2)\| \\ &\leq \lim_{l \to \infty} 16^l \psi(2^l \theta_1, 2^l \theta_2) \\ &\leq \lim_{l \to \infty} \frac{16^l \psi(\theta_1, \theta_2)}{2^l P^l} = 0 \end{aligned}$$

for all $\theta_1, \theta_2 \in \mathcal{A}$. Thus, we have

$$Q\left(\frac{\theta_1 + \theta_2}{2}\right) - Q(\theta_1 + \theta_2) = \frac{15Q(\theta_1)Q(\theta_2)}{[Q(\theta_1)^{\frac{1}{4}} + Q(\theta_2)^{\frac{1}{4}}]^4}$$

for all $\theta_1, \theta_2 \in \mathcal{A}$, which indicates that $Q: \mathcal{A} \longrightarrow \mathcal{B}$ is a multiplicative inverse fourth power mapping. By Lemma 3.1 and (4.6), we have

$$\begin{aligned} \|q_r([\theta_{mn}]) - Q_r([\theta_{mn}])\|_r &\leq \sum_{m,n=1}^{p} \|q(\theta_{mn}) - Q(\theta_{mn})\| \\ &\leq \sum_{m,n=1}^{n} \frac{16}{1-P} \psi(\theta_{mn}, \theta_{mn}) \end{aligned}$$

for all $\theta = [\theta_{mn}] \in M_r(\mathcal{A})$, which shows that $Q: \mathcal{A} \longrightarrow \mathcal{B}$ is unique satisfying (4.3).

Corollary 4.2 *Let s, λ be positive real numbers with $s < 4$. Let $q: \mathcal{A} \longrightarrow \mathcal{B}$ be a mapping such that*

$$\left\|Dq_r([\theta_{1_{mn}}], [\theta_{2_{mn}}])\right\|_r \leq \sum_{m,n=1}^{r} [\lambda(\left\|\theta_{1_{mn}}\right\|^s + \left\|\theta_{2_{mn}}\right\|^s)] \quad (4.7)$$

for all $\theta_1 = [\theta_{1_{mn}}]$, $\theta_2 = [\theta_{2_{mn}}] \in M_r(\mathcal{A})$. Then, there exists a unique mapping $Q: \mathcal{A} \longrightarrow \mathcal{B}$ satisfying *(1.3)* such that

$$\|q_r([\theta_{mn}]) - Q_r([\theta_{mn}])\|_r \leq \sum_{m,n=1}^{p} \frac{32\lambda}{1-2^{s+4}} \|\theta_{mn}\|^s$$

for all $\theta = [\theta_{mn}] \in M_r(\mathcal{A})$.

Proof. The proof of this corollary is obtained via Theorem 4.1 by letting $\psi(\theta_{1_{mn}}, \theta_{2_{mn}}) = \lambda(\left\|\theta_{1_{mn}}\right\|^s + \left\|\theta_{2_{mn}}\right\|^s)$ for all $\theta_1, \theta_2 \in \mathcal{A}$. Then, we can choose $P = 2^{s+4}$ to obtain the desired result.

Theorem 4.3 *Suppose a mapping $q: \mathcal{A} \longrightarrow \mathcal{B}$ satisfies (4.1). Assume that there exists a function $\psi: \mathcal{A} \times \mathcal{A} \longrightarrow [0, \infty)$ with a $P < 1$ such that*

$$\psi(\theta_1, \theta_2) \leq 16L\psi(2\theta_1, 2\theta_2) \quad (4.8)$$

for all $\theta_1, \theta_2 \in \mathcal{A}$. Then, a unique mapping $Q: \mathcal{A} \longrightarrow \mathcal{B}$ exists and satisfies *(1.3)* such that

$$\|q_r([\theta_{mn}]) - Q_r([\theta_{mn}])\|_r \leq \sum_{m,n=1}^{p} \frac{16}{1-P}\psi(\theta_{mn}, \theta_{mn})$$

for all $\theta = [\theta_{mn}] \in M_r(\mathcal{A})$.

Proof. Suppose (X, d) is the generalized metric space defined as in Theorem 4.1. Now, we consider a mapping $J: X \longrightarrow X$ such that

$$Jg(\theta) = \frac{1}{16} g\left(\frac{\theta}{2}\right)$$

for all $\theta \in \mathcal{A}$. By virtue of (4.4), we find that

$$\left\|q(\theta) - \frac{1}{16} q\left(\frac{\theta}{2}\right)\right\| \leq \psi\left(\frac{\theta}{2}, \frac{\theta}{2}\right) \leq \frac{P}{16}\psi(\theta, \theta)$$

for all $\theta \in \mathcal{A}$. Then $d(q, Q) \leq \frac{1}{16}$. Therefore, we have

$$d(q, JQ) \leq \frac{16P}{1-P}.$$

The enduring part of proof is obtained through similar arguments as in Theorem 4.1.

Corollary 4.4 *Let s, λ be positive real numbers with $s > -4$. Let $q: \mathcal{A} \longrightarrow \mathcal{B}$ be a mapping satisfying (4.7). Then, there exists a unique mapping $Q: \mathcal{A} \longrightarrow \mathcal{B}$ satisfying (1.3) with the result that*

$$\|q([\theta_{mn}]) - Q([\theta_{mn}])\|_r \leq \sum_{m,n=1}^{p} \frac{32\lambda}{2^{s+4}-1}\|\theta_{mn}\|^s$$

for all $\theta = [\theta_{mn}] \in M_r(\mathcal{A})$.

Proof. It is easy to prove this corollary through Theorem 4.3 by taking $\psi(\theta_{1_{mn}}, \theta_{2_{mn}}) = \lambda\left(\left\|\theta_{1_{mn}}\right\|^s + \left\|\theta_{2_{mn}}\right\|^s\right)$ and then choosing $P = 2^{-(s+4)}$.

In the ensuing outcomes, let us assume that $\mathcal{B}$ is an L^∞-complete normed space and $q: \mathcal{A} \longrightarrow \mathcal{B}$ is a mapping.

Theorem 4.5 *Let a function $\psi: \mathcal{A} \times \mathcal{A} \longrightarrow [0, \infty)$ satisfies (4.2) and*

$$\left\|Dq_r([\theta_{1_{mn}}], [\theta_{2_{mn}}])\right\|_r \leq \left\|\psi(\theta_{1_{mn}}, \theta_{2_{mn}})\right\|_r \tag{4.9}$$

for all $\theta_1 = [\theta_{1_{mn}}], \theta_2 = [\theta_{2_{mn}}] \in M_r(\mathcal{A})$. Then, a solution $Q: \mathcal{A} \longrightarrow \mathcal{B}$ of *(1.3)* exists which is unique and satisfying the following approximation

$$\left\|q([\theta_{1_{mn}}]) - Q([\theta_{1_{mn}}])\right\|_p \leq \left\|\frac{16}{1-P}\psi(\theta_{1_{mn}}, \theta_{1_{mn}})\right\|_p \tag{4.10}$$

for all $\theta_1 = [\theta_{1_{mn}}] \in M_r(\mathcal{A})$.

Proof. Using the similar arguments akin to the proof of Theorem 4.1, a solution $Q: \mathcal{A} \longrightarrow \mathcal{B}$ of *(1.3)* exists such that

$$\|q([\theta_{mn}]) - Q([\theta_{mn}])\| \leq \left\|\frac{16}{1-P}\psi(\theta_{mn}, \theta_{mn})\right\|$$

for all $\theta = [\theta_{mn}] \in M_r(\mathcal{A})$. It is easy to show that if $0 \leq u_{mn} \leq v_{mn}$ for all m, n, then

$$\|[u_{mn}]\|_r \leq \|[v_{mn}]\|_r. \tag{4.11}$$

By Lemma 3.1 and (4.11), we have

$$\|[q(\theta_{mn}) - Q(\theta_{mn})]\|_r \leq \|[\|q(\theta_{mn}) - Q(\theta_{mn})\|]\|_r \leq \left\|\frac{16}{1-P}\psi(\theta_{mn}, \theta_{mn})\right\|_r$$

for all $\theta = [\theta_{mn}] \in M_r(\mathcal{A})$. Hence, we acquire the inequality (4.10).

Corollary 4.6 *Let $s, \lambda > 0$ be real numbers with $s < -4$. Let the mapping q satisfies*

$$\left\|Dq_r([\theta_{1_{mn}}], [\theta_{2_{mn}}])\right\|_r \leq \left\|\left[\lambda(\left\|\theta_{1_{mn}}\right\|^s + \left\|\theta_{2_{mn}}\right\|^s)\right]\right\|_r \tag{4.12}$$

for all $\theta_1 = [\theta_{1_{mn}}], \theta_2 = [\theta_{2_{mn}}] \in M_r(\mathcal{A})$. Then, a solution $Q: \mathcal{A} \longrightarrow \mathcal{B}$ of *(1.3)* which is unique and satisfying the ensuing approximation

$$\|q_r([\theta_{mn}]) - Q_r([\theta_{mn}])\|_r \leq \left\|\left[\frac{32\lambda}{1-2^{s+4}}\|\theta_{mn}\|^s\right]\right\|_r$$

for all $\theta = [\theta_{mn}] \in M_r(\mathcal{A})$.

Theorem 4.7 *Let the mapping q and a function $\psi: \mathcal{A} \times \mathcal{A} \longrightarrow [0, \infty)$ satisfy (4.8) and (4.9), respectively. Then, a solution $Q: \mathcal{A} \longrightarrow \mathcal{B}$ of (1.3) exists which is unique and satisfies the following inequality*

$$\|q([\theta_{mn}]) - Q([\theta_{mn}])\|_r \leq \left\|\frac{16}{1-P}\psi(\theta_{mn}, \theta_{mn})\right\|_r \tag{4.13}$$

for all $\theta = [\theta_{mn}] \in M_r(\mathcal{A})$.

Proof. The desire result is obtained similar to Theorem 4.5.

Corollary 4.8 *Let $s > -4, \lambda > 0$ be real numbers. Let the mapping q satisfies (4.12). Then, a unique mapping $Q: \mathcal{A} \longrightarrow \mathcal{B}$ exists with the result that*

$$\|q_r([\theta_{mn}]) - Q_r([\theta_{mn}])\|_r \leq \left\|\left[\frac{32\lambda}{1-2^{s+4}}\|\theta_{mn}\|^s\right]\right\|_r$$

for all $\theta = [\theta_{mn}] \in M_r(\mathcal{A})$.

Proof. The proof follows by taking $\psi(\theta_{1_{mn}}, \theta_{2_{mn}}) = \lambda(\|\theta_{1_{mn}}\|^s + \|\theta_{2_{mn}}\|^s)$ and then choosing $P = 2^{-(s+4)}$ in Theorem 4.7.

Remark 4.9 *The stability results associated with equation (1.4) are similar to the results of equation (1.3). Hence, we omit the results of equation (1.4).*

5. An instance for the failure of stability of equation (1.3)

Persuaded through the excellent illustration proved in [7], in this section, we demonstrate an apt example to prove the failure of stability of equation (1.3) for the critical case when $s = -4$ in Corollaries 3.3, 3.5, 3.8 and 3.10.

Theorem 5.1 *Let a mapping $\phi: \mathbb{R}^\star \longrightarrow \mathbb{R}$ be defined as follows:*

$$\phi(\theta) = \begin{cases} \frac{\mu}{\theta^4}, & if \ 1 < \theta < \infty \\ \mu, & otherwise \end{cases}$$

where μ is a positive constant and a mapping $q: \mathbb{R}^\star \longrightarrow \mathbb{R}$ defined by

$$q(\theta) = \sum_{k=0}^{\infty} 16^{-k} \phi(2^{-k}\theta) \qquad for\ all \quad \theta \in \mathbb{R}^\star.$$

Then the mapping q satisfies the ensuing approximation

$$|Dq(\theta_1, \theta_2)| \leq \frac{752}{15}\mu\left(\left|\frac{1}{\theta_1}\right|^4 + \left|\frac{1}{\theta_2}\right|^4\right) \tag{5.1}$$

for every $\theta_1, \theta_2 \in \mathbb{R}^\star$. Then, a multiplicative inverse fourth power mapping $Q: \mathbb{R}^\star \longrightarrow \mathbb{R}$ and a positive constant ρ do not exist so that

$$|q(\theta) - Q(\theta)| \leq \rho\left|\frac{1}{\theta}\right|^4 \qquad for\ every \quad \theta \in \mathbb{R}^\star. \tag{5.2}$$

Proof. Firstly, let us show that q is bounded. In lieu of the mapping q's definition, we obtain $|q(\theta)| \leq \sum_{k=0}^{\infty} 16^{-k}\phi(2^{-k}\theta) = \sum_{k=0}^{\infty} \frac{\mu}{16^k} = \frac{16\mu}{15}$ which indicates that q is bounded. Let us show that q satisfies the inequality (1). Suppose $\left|\frac{1}{\theta_1}\right|^4 + \left|\frac{1}{\theta_2}\right|^4 \geq 1$. Then the left hand side of (1) is not greater than or equal to $\frac{752\mu}{15}$. On the contrary, suppose that $0 < \left|\frac{1}{\theta_1}\right|^4 + \left|\frac{1}{\theta_2}\right|^4 < 1$. Then there exists a positive integer r such that

$$\frac{1}{16^{r+1}} \leq \left|\frac{1}{\theta_1}\right|^4 + \left|\frac{1}{\theta_2}\right|^4 < \frac{1}{16^r}, \tag{5.3}$$

which in turn gives rise to $16^r\left(\frac{1}{\theta_1}\right)^4 < 1, 16^r\left(\frac{1}{\theta_1}\right)^4 < 1$ and hencet, we have

$$\frac{1}{16^{r-1}}(\theta_1) > 1, \frac{1}{16^{r-1}}(\theta_2) > 1, \frac{1}{16^{r-1}}(\theta_1 + \theta_2) > 1, \frac{1}{16^{r-1}}\left(\frac{\theta_1+\theta_2}{2}\right) > 1.$$

Therefore, for every $k = 0,1, \dots, r-1$, we notice that

$$\frac{1}{16^r}(\theta_1) > 1, \frac{1}{16^r}(\theta_2) > 1, \frac{1}{16^r}(\theta_1 + \theta_2) > 1, \frac{1}{16^r}\left(\frac{\theta_1+\theta_2}{2}\right) > 1$$

and $Dq(16^{-k}\theta_1, 16^{-k}\theta_2) = 0$. for $k = 0,1,2, \dots, r-1$. In view of the definition of q and (5.3), we obtain

$$\begin{aligned} |Dq(\theta_1, \theta_2)| &\leq \sum_{k=r}^{\infty} \frac{\mu}{16^k} + \sum_{k=r}^{\infty} \frac{\mu}{16^k} + \frac{15}{16}\sum_{k=r}^{\infty} \frac{\mu}{16^k} \\ &\leq \frac{47\mu}{15}\frac{1}{16^k} \leq \frac{752\mu}{15}\frac{1}{16^{k+1}} \leq \frac{752\mu}{15}(|\frac{1}{\theta_1}|^4 + |\frac{1}{\theta_2}|^4) \end{aligned}$$

for every $\theta_1, \theta_2 \in \mathbb{R}^\star$, which indicates that q satisfies (5.1) for all $\theta_1, \theta_2 \in \mathbb{R}^\star$. Next is to prove that (1.3) fails to be stable for $s = -4$ in Corollaries 3.3, 3.5, 3.8 and 3.10. Suppose a reciprocal fourth power mapping $Q: \mathbb{R}^\star \longrightarrow \mathbb{R}$ and a constant $\rho > 0$ satisfy (5.2). Since q is bounded above and using the result of Corollaries 3.3, 3.5, 3.8 and 3.10, the mapping $Q(\theta)$ must be a reciprocal fourth power mapping and $Q(\theta) = \frac{k}{\theta^4}$ for any $\theta \in \mathbb{R}^\star$. Therefore, we arrive at

$$|q(\theta)| \leq (\rho + |k|)\left|\frac{1}{\theta}\right|^4. \tag{5.4}$$

At the same time, we can choose an integer $m > 0$ with $m\mu > \rho + |r|$. Suppose $0 < theta < 2^{m-1}$, then $1 < 2^{-k}\theta < \infty$ for every $k = 0,1, \dots, m-1$. Hence, for this θ, we obtain

$$q(\theta) = \sum_{k=0}^{\infty} \frac{\phi(2^{-k}\theta)}{16^k} \geq \sum_{k=0}^{m-1} \frac{(2^{-k}\theta)^4}{16^k} = m\mu\frac{1}{\theta^4} > (\rho + |k|)\frac{1}{\theta^4}$$

which is a contradiction to (5.4). Hence, (5.3) fails to be stable for $s = -4$ in Corollaries 3.3, 3.5, 3.8 and 3.10.

6. Pertinence of equations (1.3) and (1.4) in fluid dynamics and Raman Spectroscopy

In this section, we present the pertinence of equations (1.3) and (1.4) in various field such as fluid dynamics and Raman spectroscopy.

6.1. Fluid Dynamics

The fundamental factors that are necessary to find the blood flow resistance (R) within a single vessel are the radius of the vessel (r), the length of the vessel (L) and the viscoscity of the blood (η). Among the above three factors, the most signifcant factor is the radius of the vessel in terms of quantity and physiology. Since the vascular smooth muscle in the wall of the blood vessel contracts and expands, the radius of the blood vessel is a primary factor. Also, a very small change in the radius of vessel leads to large change in resistance, where as length of vessel does not change significantly and viscosity of blood normally stays within a small range (except when hematocrit changes). Then the fluid resistance is directionary proportional to η and L and inversely proportional to r^4, which is given by

$$R = \frac{c\eta L}{r^4}$$

where c is constant of proportionality. Suppose η and L are kept constant, then the fluid resistance is given by

$$R = \frac{k}{r^4}.$$

where k is a constant. If r_1 and r_2 are the radii of two blood vessels, then the fluid resistances in those blood vessels can be considered as $q(r_1)$ and $q(r_2)$, respectively, where q is a reciprocal fourth power mapping. Also, $q\left(\frac{r_1+r_2}{2}\right)$ and $q(r_1+r_2)$ represent the fluid resistances of vessel radius $\frac{r_1+r_2}{2}$ and r_1+r_2, respectively. The exposition of equation (1.3) is the difference between the fluid resistances $q\left(\frac{r_1+r_2}{2}\right)$ and $q(r_1+r_2)$ is given by the ratio of $15q(r_1)q(r_2)$ and $\left[q(r_1)^{\frac{1}{4}}+q(r_2)^{\frac{1}{4}}\right]^4$. Similarly, the explication of equation (4) is the sum of the fluid resistances $q\left(\frac{r_1+r_2}{2}\right)$ and $q(r_1+r_2)$ is equal to the ratio of $17q(r_1)q(r_2)$ and $\left[q(r_1)^{\frac{1}{4}}+q(r_2)^{\frac{1}{4}}\right]^4$.

6.1. Raman Spectroscopy

In Raman spectroscopy, the solution of equations (1.3) and (1.4) can be applied in studying the nongraphite samples with distinct crystallite sizes and laser energies. The disorder-induced Raman bands D and G are denoted by I_D and I_G. Then the intensity ratio of these disorder-induced Raman bands I_D/I_G is proportional to the multiplicative inverse fourth power of the laser energies.

7. Discussion and Conclusions

As of our knowledge, our findings in this study are novel in the field of stability theory. This is our antecedent endevavour to deal with new type of reciprocal fourth power functional equations. It is shown that the equations (1.3) and (1.4) are equivalent to each other to conclude that their solution is reciprocal fourth power mapping. The stability results of different forms of reciprocal functional equations are obtained by many mathematicians in various spaces. But, in this work, we have considered matrix normed spaces to analyze the results of equations (1.3) and (1.4) and they are found to be stable except for a singular case. For the failure of stability result when critical case arises, we have illustrated an apt example. At the end of this study, we have presented pertinency of equations (1.3) and (1.4) in fluid dynamics and Raman spectroscopy.

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