Application Of Horadam Polynomial On Sakaguchi Type Bi-Univalent Functions Satisfying Certain Subordination Constraints

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Abstract" In this current investigation, we apply Horadam polynomial to establish sharp upper bound for the second and third coefficient of functions from new subclass of sakaguchi type bi-univalent functions defined in the open unit disk $\mathbb{U}$. Also, we discuss Fekete-Szego inequality for functions belongs to this subclass.

Keywords: Holomorphic function, Univalent functions, Bi-univalent functions, Horadam Polynomial, Starlike functions, Convex functions, Sakaguchi-type functions, Coefficient bounds, Fekete-Szego inequality.

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1 Introduction

Let $\mathbb{U} = \{\xi \ : |\xi| < 1\}$ denote the open unit disk on the complex plane. The class of all holomorphic functions of the form

$$u(\xi) = \xi + a_2\xi^2 + a_3\xi^3 + \cdots \qquad (1)$$

defined in the open unit disk $\mathbb{U}$ with Montel normalization $u(0) = 0 = u'(0) - 1$ is denoted by $\mathcal{A}$ and the class $\mathcal{S} \subset \mathcal{A}$ is the class which consists of univalent functions in $\mathbb{U}$.

The Koebe one quarter theorem [1], states that the image of $\mathbb{U}$ under every univalent function $u \in \mathcal{A}$ contains a disk of radius $\frac{1}{4}$ Thus Koebe one quarter theorem guarantees that for every univalent function $u \in \mathcal{A}$, there exists inverse function $u^{-1} = v$ satisfying

$$u^{-1}\{(u(\xi)\} = \xi, \quad \xi \in \mathbb{U} \quad and \quad u\{u^{-1}(\zeta)\} = \zeta, \quad where \quad |\zeta| < r_u, \quad r_u \geq \frac{1}{4}$$

A function $u \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both u and u^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of all function $u \in \mathcal{A}$ which are bi-univalent functions defined in the unit disk $\mathbb{U}$ and whose Taylor series expansion is given by (1). A simple computation shows that its inverse $v = u^{-1}$ also has the expansion.

$$v(\zeta) = u^{-1}(\zeta) = \zeta - a_2\zeta^2 + (2a_2^2 - a_3)\zeta^3 - (5a_2^3 - 5a_2a_3 + a_4)\zeta^4 + \cdots \qquad (2)$$

Many authors have established and examined subclasses of bi-univalent function and attained sharp bounds for the initial coefficients. (see [2,3,4,5,6])

A holomorphic function u is subordinate to an holomorphic function G in $\mathbb{U}$ denoted as $u \prec G, \ (\xi \in \mathbb{U})$. If $u(\xi) = G(\omega(\xi)), |\xi| < 1$ for some holomorphic schwarz function $\omega(\xi)$ with $\omega(0) = 0$ and $|\omega(\xi)| < 1$. It follows from schwarz lemma that

$$u(\xi) \prec G(\xi) \quad \Leftrightarrow \quad u(0) = G(0) \ and \ u(\mathbb{U}) \subset G(\mathbb{U}), \ \xi \in \mathbb{U}$$

One can refer [1,7] for details of subordination.

The Horadam Polynomial $h_n(\sigma)$ are defined by the following repetition relation (see [9,10]):

$$h_n(\sigma) = x\sigma\, h_{n-1}(\sigma) + y\, h_{n-2}(\sigma), \qquad (\sigma \in \mathbb{R}, \qquad n \in \mathbb{N} - \{1,2\})$$

with

$$h_1(\sigma) = x \quad and \quad h_2(\sigma) = y\sigma \qquad (3)$$

for some real constants a, b, x and y.

The generating function of the Horadam polynomials $h_n(\sigma)$ (see [9,10]) is given by

$$\Pi(\sigma,\xi) = \sum_{n=1}^{\infty} h_n(\sigma)\xi^{n-1} = \frac{a + (b - ax)\sigma\xi}{1 - x\sigma\xi - y\xi^2} \qquad (4)$$

2 Bi-Univalent Function Class $\mathcal{HB}_{\rho,\mu,t}\{\Pi(\sigma,\xi)\}$

In this section, we introduce a new subclass of Sakaguchi type bi-univalent functions with the application of Horadam polynomial by subordination technique and obtain bound for initial Taylor coefficient $|a_2|$ and $|a_3|$ for the function.

Definition 1.
For $0 \leq \rho \leq 1, 0 \leq \mu < 1$ and $|t| \leq 1$, but $t \neq 1$, a function $u \in \Sigma$ of the form (1) is said to be in the class $\mathcal{HB}_{\rho,\mu,t}\{\Pi(\sigma,\xi)\}$, if the following subordination hold:

$$\frac{(1-t)[\rho\mu\,\xi^3\,u'''(\xi)+(2\rho\mu+\rho-\mu)\xi^2\,u''(\xi)+\xi\,u'(\xi)]}{\rho\mu\,\xi^2[u''(\xi)-t^2u''(t\xi)]+(\rho-\mu)\xi[u'(\xi)-tu'(t\xi)]+(1-\rho+\mu)[u(\xi)-u(t\xi)]} \prec \Pi(\sigma,\xi)+1-a \tag{5}$$

and

$$\frac{(1-t)[\rho\mu\zeta^3\,v'''(\zeta)+(2\rho\mu+\rho-\mu)\zeta^2\,v''(\zeta)+\zeta\,v'(\zeta)]}{\rho\mu\zeta^2[v''(\zeta)-t^2v''(t\zeta)]+(\rho-\mu)\zeta[v'(\zeta)-tv'(t\zeta)]+(1-\rho+\mu)[v(\zeta)-v(t\zeta)]} \prec \Pi(\sigma,\zeta)+1-a \tag{6}$$

where v is given by (2).
Specializing the parameter $\rho=0,\ \mu=0,\ t=0$ and $\rho=1,\ \mu=0,\ t=0$, we have the following respectively.
Definition 2.
A function $u \in \Sigma$ of the form (1) is said to be in the class $\mathcal{SHB}_\Sigma\{\Pi(\sigma,\xi)\}$, if the following subordination hold:

$$\frac{\xi\,u'(\xi)}{u(\xi)} \prec \Pi(\sigma,\xi)+1-a$$

and

$$\frac{\zeta\,v'(\zeta)}{v(\zeta)} \prec \Pi(\sigma,\zeta)+1-a$$

where v is given by (2).
Definition 3.
A function $u \in \Sigma$ of the form (1) is said to be in the class $\mathcal{KHB}_\Sigma\{\Pi(\sigma,\xi)\}$, if the following subordination hold:

$$1+\frac{\xi\,u''(\xi)}{u'(\xi)} \prec \Pi(\sigma,\xi)+1-a$$

and

$$1+\frac{\zeta\,v''(\zeta)}{v'(\zeta)} \prec \Pi(\sigma,\zeta)+1-a$$

where v is given by (2).

In the following theorem, we determine the bound for initial Taylor coefficient $|a_2|$ and $|a_3|$ for the function class $\mathcal{HB}_{\rho,\mu,t}\{\Pi(\sigma,\xi)\}$. Later we will reduce these bounds to other classes for special cases.
Theorem 1.
Let u given by (1) be in the class $\mathcal{HB}_{\rho,\mu,t}\{\Pi(\sigma,\xi)\}$. Then

$$|a_2| \leq \frac{|b\sigma|\sqrt{|b\sigma|}}{\sqrt{\left|[2(3\rho\mu+\rho-\mu)+1]\left\{\begin{matrix}(3-T_3)[b^2\sigma^2]\\-(2-T_2)[(2\rho\mu+\rho-\mu)+1]\{b^2\sigma^2T_2+(2-T_2)[xb\sigma^2+ya]\}\end{matrix}\right\}\right|}}$$

and

$$|a_3| \leq \frac{|b\sigma|}{|3-T_3|[2(3\rho\mu+\rho-\mu)+1]}+\frac{|b^2\sigma^2|}{(2-T_2)^2[(2\rho\mu+\rho-\mu)+1]^2}$$

where

$$T_n = \frac{1-t^n}{1-t} = 1+t+t^2+\cdots+t^{n-1} \tag{7}$$

Proof.
Let $u \in \mathcal{HB}_{\rho,\mu,t}\{\Pi(\sigma,\xi)\}$. Then there are two holomorphic schwarz functions $f, g : \mathbb{U} \to \mathbb{U}$ given by

$$f(\xi)=\alpha_1\xi+\alpha_2\xi^2+\alpha_3\xi^3+\cdots \qquad (\xi \in \mathbb{U}) \tag{8}$$

$$g(\zeta)=\beta_1\zeta+\beta_2\zeta^2+\beta_3\zeta^3+\cdots \qquad (\zeta \in \mathbb{U}) \tag{9}$$

with $f(0)=g(0)=0$ and $|f(\xi)|<1,\ \ |g(\zeta)|<1 \quad (\xi,\zeta \in \mathbb{U})$
Hence, we have

$$|\alpha_i|<1 \ \ and \quad |\beta_i|<1, \quad \forall \quad i \in \mathbb{N} \tag{10}$$

Now using (8) and (9) in (5) and (6), we have

$$\frac{(1-t)[\rho\mu\,\xi^3\,u'''(\xi)+(2\rho\mu+\rho-\mu)\xi^2\,u''(\xi)+\xi\,u'(\xi)]}{\rho\mu\,\xi^2[u''(\xi)-t^2u''(t\xi)]+(\rho-\mu)\xi[u'(\xi)-tu'(t\xi)]+(1-\rho+\mu)[u(\xi)-u(t\xi)]} = \Pi(\sigma,f(\xi))+1-a \tag{11}$$

and

$$\frac{(1-t)[\rho\mu\,\zeta^3\,v'''(\zeta)+(2\rho\mu+\rho-\mu)\zeta^2\,v''(\zeta)+\zeta\,v'(\zeta)]}{\rho\mu\,\zeta^2[v''(\zeta)-t^2v''(t\zeta)]+(\rho-\mu)\zeta[v'(\zeta)-tv'(t\zeta)]+(1-\rho+\mu)[v(\zeta)-v(t\zeta)]} = \Pi(\sigma,g(\zeta))+1-a \tag{12}$$

where $\xi,\zeta\in\mathbb{U}$ and v is given by (2).

Now (11) $\Longrightarrow$

$$1+(2-T_2)[(2\rho\mu+\rho-\mu)+1]a_2\,\xi-\begin{Bmatrix}(2-T_2)[(2\rho\mu+\rho-\mu)+1]^2a_2^2\,T_2\\-(3-T_3)[2(3\rho\mu+\rho-\mu)+1]a_3\end{Bmatrix}\xi^2+\cdots=\Pi(\sigma,f(\xi))+1-a \tag{13}$$

where

$$\begin{aligned}\Pi\big(\sigma,f(\xi)\big)+1-a &= 1-a+h_1(\sigma)+h_2(\sigma)f(\xi)+h_3(\sigma)f^2(\xi)+\cdots\\ &= 1+h_2(\sigma)\alpha_1\,\xi+[h_2(\sigma)\alpha_2+h_3(\sigma)\alpha_1^2]\xi^2+\cdots\end{aligned} \tag{14}$$

Equating coefficients of ξ and ξ^2 from (13) and (14), we get

$$(2-T_2)[(2\rho\mu+\rho-\mu)+1]a_2=h_2(\sigma)\alpha_1 \tag{15}$$

$$\begin{Bmatrix}(3-T_3)[2(3\rho\mu+\rho-\mu)+1]a_3\\-(2-T_2)[(2\rho\mu+\rho-\mu)+1]^2a_2^2\,T_2\end{Bmatrix}=h_2(\sigma)\alpha_2+h_3(\sigma)\alpha_1^2 \tag{16}$$

Now (12) $\Longrightarrow$

$$\begin{aligned}1+(2-T_2)[(2\rho\mu+\rho-\mu)+1]a_2\,\zeta-\begin{Bmatrix}(2-T_2)[(2\rho\mu+\rho-\mu)+1]^2a_2^2\,T_2\\-(3-T_3)[2(3\rho\mu+\rho-\mu)+1](2a_2^2-a_3)\end{Bmatrix}\zeta^2+\cdots\\ =\Pi(\sigma,g(\zeta))+1-a\end{aligned} \tag{17}$$

where $\Pi\big(\sigma,g(\zeta)\big)+1-a = 1-a+h_1(\sigma)+h_2(\sigma)g(\zeta)+h_3(\sigma)g^2(\zeta)+\cdots$

$$= 1+h_2(\sigma)\beta_1\,\zeta+[h_2(\sigma)\beta_2+h_3(\sigma)\beta_1^2]\zeta^2+\cdots \tag{18}$$

Equating coefficients of ζ and ζ^2 from (17) and (18), we get

$$-(2-T_2)[(2\rho\mu+\rho-\mu)+1]a_2=h_2(\sigma)\beta_1 \tag{19}$$

$$\begin{Bmatrix}(3-T_3)[2(3\rho\mu+\rho-\mu)+1](2a_2^2-a_3)\\-(2-T_2)[(2\rho\mu+\rho-\mu)+1]^2a_2^2\,T_2\end{Bmatrix}=h_2(\sigma)\beta_2+h_3(\sigma)\beta_1^2 \tag{20}$$

From (15) and (19), we have

$$\alpha_1=-\beta_1 \tag{21}$$

Now $(15)^2+(19)^2 \Longrightarrow$

$$2a_2^2=\frac{(\alpha_1^2+\beta_1^2)\;h_2^2(\sigma)}{(2-T_2)^2[(2\rho\mu+\rho-\mu)+1]^2}$$

using (21) in the above, we get

$$2a_2^2=\frac{(2\alpha_1^2)\;h_2^2(\sigma)}{(2-T_2)^2[(2\rho\mu+\rho-\mu)+1]^2} \tag{22}$$

$$\Longrightarrow\quad \alpha_1^2=\frac{(2-T_2)^2[(2\rho\mu+\rho-\mu)+1]^2\,a_2^2}{h_2^2(\sigma)} \tag{23}$$

Now by summing (16) and (20)

$$2\begin{Bmatrix}(3-T_3)[2(3\rho\mu+\rho-\mu)+1]\\-(2-T_2)[(2\rho\mu+\rho-\mu)+1]^2\,T_2\end{Bmatrix}a_2^2 = h_2(\sigma)[\alpha_2+\beta_2]+h_3(\sigma)[\alpha_1^2+\beta_1^2]$$

Since by (21), we have

$$2\begin{Bmatrix}(3-T_3)[2(3\rho\mu+\rho-\mu)+1]\\-(2-T_2)[(2\rho\mu+\rho-\mu)+1]^2\,T_2\end{Bmatrix}a_2^2 = h_2(\sigma)[\alpha_2+\beta_2]+h_3(\sigma)[2\alpha_1^2] \tag{24}$$

By substituting (23) in (24), we have

$$2\begin{Bmatrix}(3-T_3)h_2^2(\sigma)[2(3\rho\mu+\rho-\mu)+1]\\-(2-T_2)[(2\rho\mu+\rho-\mu)+1]^2\{h_2^2(\sigma)T_2+(2-T_2)h_3(\sigma)\}\end{Bmatrix}a_2^2=h_2^3(\sigma)[\alpha_2+\beta_2] \tag{25}$$

Therefore, by using (10), we obtain

$$|a_2|\le\frac{|b\sigma|\sqrt{|b\sigma|}}{\sqrt{\left|[2(3\rho\mu+\rho-\mu)+1]\begin{Bmatrix}(3-T_3)[b^2\sigma^2]\\-(2-T_2)[(2\rho\mu+\rho-\mu)+1]\{b^2\sigma^2T_2+(2-T_2)[xb\sigma^2+ya]\}\end{Bmatrix}\right|}}$$

Now we have to find bound for $|a_3|$, Lets subtract (19) from (15), then we get

$$2(3-T_3)[2(3\rho\mu+\rho-\mu)+1]\{a_3-a_2^2\}=h_2(\sigma)[\alpha_2-\beta_2] \tag{26}$$

Hence, we get

$$(3-T_3)[2(3\rho\mu+\rho-\mu)+1]|a_3| \leq \frac{b\sigma[\alpha_2-\beta_2]}{2}+(3-T_3)[2(3\rho\mu+\rho-\mu)+1]|a_2|^2 \quad (27)$$

Now use (22) in (27), we obtain

$$|a_3| \leq \frac{|b\sigma|}{|3-T_3|[2(3\rho\mu+\rho-\mu)+1]}+\frac{|b^2\sigma^2|}{(2-T_2)^2[(2\rho\mu+\rho-\mu)+1]^2}$$

where T_2, T_3 are given by (7).

If we take the parameters $\rho=0, \mu=0, t=0$ and $\rho=1, \mu=0, t=0$, in the above theorem, we have the following bounds of initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function classes $\mathcal{SHB}_\Sigma\{\Pi(\sigma,\xi)\}$ and $\mathcal{KHB}_\Sigma\{\Pi(\sigma,\xi)\}$ respectively

Corollary 1.

Let u given by (1) be in the class $\mathcal{SHB}_\Sigma\{\Pi(\sigma,\xi)\}$, Then

$$|a_2| \leq \frac{|b\sigma|\sqrt{|b\sigma|}}{\sqrt{b^2\sigma^2-(xb\sigma^2+ya)}}$$

and

$$|a_3| \leq \frac{|b\sigma|}{2}+b^2\sigma^2$$

Corollary 2.

Let u given by (1) be in the class $\mathcal{KHB}_\Sigma\{\Pi(\sigma,\xi)\}$, Then

$$|a_2| \leq \frac{|b\sigma|\sqrt{|b\sigma|}}{\sqrt{2b^2\sigma^2-4(xb\sigma^2+ya)}}$$

and

$$|a_3| \leq \frac{|b\sigma|}{6}+\frac{b^2\sigma^2}{4}$$

3 Fekete-Szego Inequalities for the Function Class $\mathcal{HB}_{\rho,\mu,t}\{\Pi(\sigma,\xi)\}$

Fekete and Szego [12] introduced the generalized functional $|a_3-\lambda\, a_2^2|$, where λ is some real number. Due to Zaprawa [13], in the following theorem we determine the Fekete-Szego functional for $u \in \mathcal{HB}_{\rho,\mu,t}\{\Pi(\sigma,\xi)\}$.

Theorem 2.

Let u given by (1) be in the class $\mathcal{HB}_{\rho,\mu,t}\{\Pi(\sigma,\xi)\}$ and $\lambda \in \mathbb{R}$. Then we have

$$|a_3-\lambda\, a_2^2| \leq \begin{cases} \frac{|b\sigma|}{|3-T_3|[2(3\rho\mu+\rho-\mu)+1]}, & \text{if } |\phi(\lambda,\sigma)| \leq \frac{1}{2(3-T_3)[2(3\rho\mu+\rho-\mu)+1]} \\ 2|b\sigma||\phi(\lambda,\sigma)|, & \text{if } |\phi(\lambda,\sigma)| \geq \frac{1}{2(3-T_3)[2(3\rho\mu+\rho-\mu)+1]} \end{cases}$$

where

$$\phi(\lambda,\sigma)=\frac{(1-\lambda)h_2^2(\sigma)}{2\left\{(3-T_3)h_2^2(\sigma)[2(3\rho\mu+\rho-\mu)+1]-(2-T_2)[(2\rho\mu+\rho-\mu)+1]^2\left\{h_2^2(\sigma)T_2+(2-T_2)h_3(\sigma)\right\}\right\}} \quad (28)$$

and T_2, T_3 are given by (7)

Proof.

From (25) and (26), we obtain

$$a_3-a_2^2 = \frac{h_2(\sigma)[\alpha_2-\beta_2]}{2(3-T_3)[2(3\rho\mu+\rho-\mu)+1]}$$

$$a_3-\lambda\, a_2^2=\frac{h_2(\sigma)[\alpha_2-\beta_2]}{2(3-T_3)[2(3\rho\mu+\rho-\mu)+1]}+(1-\lambda)a_2^2$$

$$= h_2(\sigma)\left[\frac{\alpha_2-\beta_2}{2(3-T_3)[2(3\rho\mu+\rho-\mu)+1]}+(\alpha_2+\beta_2)\phi(\lambda,\sigma)\right]$$

$$= h_2(\sigma)\left[\left(\frac{1}{2(3-T_3)[2(3\rho\mu+\rho-\mu)+1]}+\phi(\lambda,\sigma)\right)\alpha_2 + \left(\phi(\lambda,\sigma)-\frac{1}{2(3-T_3)[2(3\rho\mu+\rho-\mu)+1]}\right)\beta_2\right]$$

Then, by taking modulus, we conclude that

$$|a_3 - \lambda a_2^2| \leq \begin{cases} \dfrac{|b\sigma|}{|3-T_3|[2(3\rho\mu+\rho-\mu)+1]}, & \text{if } |\phi(\lambda,\sigma)| \leq \dfrac{1}{2(3-T_3)[2(3\rho\mu+\rho-\mu)+1]} \\ 2|b\sigma||\phi(\lambda,\sigma)|, & \text{if } |\phi(\lambda,\sigma)| \geq \dfrac{1}{2(3-T_3)[2(3\rho\mu+\rho-\mu)+1]} \end{cases}$$

where $\phi(\lambda,\sigma)$ is given by (28).

If we take the parameters $\rho = 0,\ \mu = 0,\ t = 0$ and $\rho = 1,\ \mu = 0,\ t = 0$, in the above theorem, we have the following Fekete-Szego inequalities for the function classes $\mathcal{SHB}_\Sigma\ \{\Pi(\sigma,\xi)\}$ and $\mathcal{KHB}_\Sigma\ \{\Pi(\sigma,\xi)\}$, respectively.

Corollary 3.

Let u given by (1) be in the class $\mathcal{SHB}_\Sigma\ \{\Pi(\sigma,\xi)\}$ and $\lambda \in \mathbb{R}$, Then we have

$$|a_3 - \lambda a_2^2| \leq \begin{cases} \dfrac{|b\sigma|}{2}, & if \quad |1-\lambda| \leq \dfrac{|b^2\sigma^2 - (xb\sigma^2 + ya)|}{2|b^2\sigma^2|} \\ \dfrac{|1-\lambda||b^3\sigma^3|}{|b^2\sigma^2 - (xb\sigma^2 + ya)|}, & if \ |1-\lambda| \geq \dfrac{|b^2\sigma^2 - (xb\sigma^2 + ya)|}{2|b^2\sigma^2|} \end{cases}$$

Corollary 4.

Let u given by (1) be in the class $\mathcal{KHB}_\Sigma\ \{\Pi(\sigma,\xi)\}$ and $\lambda \in \mathbb{R}$, Then we have

$$|a_3 - \lambda a_2^2| \leq \begin{cases} \dfrac{|b\sigma|}{6}, & if \quad |1-\lambda| \leq \dfrac{|b^2\sigma^2 - 2(xb\sigma^2 + ya)|}{3|b^2\sigma^2|} \\ \dfrac{|1-\lambda||b^3\sigma^3|}{2|b^2\sigma^2 - 2(xb\sigma^2 + ya)|}, & if \ |1-\lambda| \geq \dfrac{|b^2\sigma^2 - 2(xb\sigma^2 + ya)|}{3|b^2\sigma^2|} \end{cases}$$

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