Research Article

Coefficient Inequalities using Lucas Polynomials

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Abstract: The aim of this paper is to apply Lucas polynomial, in order to obtain the initial coefficients on $|a_2|$ and $|a_3|$ belonging to the new subclass $L_\Sigma(\lambda,\mu)$ of analytic, univalent and Sakaguchi functions as defined in the open unit disc Δ. Furthermore, the Fekete – Szego inequality is also investigated for this subclass.
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1 Introduction

In ([4], [5]), for any variable quantity x, Lucas polynomials $L_n(x)$ are explained recursively as follows:

$$L_n(x) := \begin{cases} 2, n=0 \\ x, n=1 \\ xL_{n-1}(x)+L_{n-2}(x), n\ge 2 \end{cases}$$

from which the first few Lucas polynomials can be identified as

$$L_0(x)=2\ ,\ L_1(x)=x\,,\ L_2(x)=x^2+2$$
$$L_3(x)=x^3+3x\,,\ L_4(x)=x^4+4x^2+2\,,\ldots.. \quad (1)$$

By letting $x=1$ in the Lucas polynomials the Lucas numbers are deduced. The ordinary generating function of the Lucas polynomials is

$$\Psi_{\{L_n(x)\}}(z)=\sum_{n=0}^{\infty}L_n(x)z^n=\frac{2-xz}{1-z(x+z)}.$$

Various authors have analyzed the properties of the Lucas polynomials and obtained many fruitful results. It is widely accepted that many number and polynomial sequences can be generated by recurrence relations of second order. Of these the important sequences remain celebrated sequences of Lucas. These sequences of polynomials and numbers are of great importance in different branches such as physics, engineering, architecture, nature, art, number theory, combinatory and numerical analysis. These sequences have been carefully considered in several papers from a theoretical point of view (see, [7, 8, 10, 16, 17]).

Let *A* be the family of functions *f* that are analytic in the open unit disk $\Delta=\{z\in C:|z|<1\}$ of the form:

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k \quad (2)$$

For $h(z)\in A$, given by

$$h(z)=z+\sum_{k=2}^{\infty}h_k z^k$$

Let *S* mean the subclass of *A* consisting of univalent functions in Δ. It is well known (refer[1]) that every function of $f \in S$ virtually possesses an inverse of *f*, defined by $f^{-1}[f(z)] = z, (z \in \Delta)$ and $f[f^{-1}(w)] = w, (|w| < r_0(f); r_0(f) \geq \frac{1}{4})$, where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + ... \qquad (3)$$

When the function $f \in A$ is bi univalent, both f and f^{-1} are univalent in Δ. Let $\sum$ be the class of bi univalent functions in Δ given by (2). In fact, Srivastava et al.[15] have revived the study of analytic and bi univalent functions in recent years. Many researchers investigated and propounded various subclasses of bi univalent functions and fixed the initial coefficients $|a_2|$ and $|a_3|$ (refer [3, 6, 9, 12, 13, and 14]).

For analytic functions *f* and *g* , *f* is said to be subordinate to *g*, denoted $f(z) \prec g(z)$, if there is an analytic function *w* such that $w(0) = 0, |w(z)| < 1$ and $f(z) = g(w(z))$.

A function $f \in S$ is said to be Bazilevic function if it satisfies (see[12]):
:

$$\Re\left(\frac{z^{1-\lambda} f'(z)}{(f(z))^{1-\lambda}}\right) > 0, \quad (z \in \Delta, \lambda \geq 0)$$

This class of the function was denoted by B_λ . Consequently when $\lambda = 0$, the class of starlike function is obtained.

Frasin [2] investigated the inequalities of coefficient for certain classes of Sakaguchi type functions that satisfy geometrical condition as

$$\text{Re}\left\{\frac{(s-t)z(f'(z))}{f(sz) - f(tz)}\right\} > \alpha \qquad (4)$$

for complex numbers s, t but s ≠ t and α (0 ≤ α < 1) .

The objective of this paper is to introduce convolution in Sakaguchi type of new subclass $L_\Sigma(\lambda, \mu)$ of $\sum$ and find estimates on the coefficient $|a_2|$ and $|a_3|$ for functions in the new subclass. The Fekete – Szego functional $|a_3 - \eta a_2^2|$ for this subclass is also obtained.

2 The class $L_\Sigma(\lambda, \mu)$

This section defines the class $L_\Sigma(\lambda, \mu)$ and attempts to find the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in this class.

Definition 2.1 : For $\lambda \geq 0, |\mu| \leq 1$ but $\mu \neq 1$, the function *f* is said to be in the class $L_\Sigma(\lambda, \mu)$, if the following conditions are satisfied:

$$\frac{((1-\mu)z)^{1-\lambda}(f'(z))}{(f(z) - f(\mu z))^{1-\lambda}} \prec \Psi_{\{L_n(x)\}}(z) - 1$$

and

$$\frac{((1-\mu)w)^{1-\lambda}(h'(w))}{(h(w) - h(\mu w))^{1-\lambda}} \prec \Psi_{\{L_n(x)\}}(w) - 1 \qquad (5)$$

where

$$h(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + ...$$

Remark 2.1: For $\mu = 0$, the function $f \in L_\Sigma(\lambda, \mu)$, if the following subordination conditions are satisfied :

$$\frac{z^{1-\lambda}\left(f'(z)\right)}{\left(f(z)\right)^{1-\lambda}} \prec \Psi_{\{L_n(x)\}}(z)-1$$

and

$$\frac{w^{1-\lambda}\left(h'(w)\right)}{\left(h(w)\right)^{1-\lambda}} \prec \Psi_{\{L_n(x)\}}(w)-1$$

which were studied by Sahsene Altinkaya et al [11] when the parameters k = 0 and μ = 1.

Remark 2.2: For $\lambda = 0$, the function $f \in L_\Sigma(\lambda,\mu)$, if the following subordination conditions are satisfied :

$$\frac{z\left(f'(z)\right)}{f(z)} \prec \Psi_{\{L_n(x)\}}(z)-1$$

and

$$\frac{w\left(h'(w)\right)}{h(w)} \prec \Psi_{\{L_n(x)\}}(w)-1$$

that are studied by Sahsene Altinkaya et al [11].

Theorem 2.1: For $\lambda \geq 0$, $|\mu| \leq 1$ but $\mu \neq 1$, let *f* given by (2) be in the class $L_\Sigma(\lambda,\mu)$. Then

$$|a_2| \leq \frac{|x|\sqrt{2|x|}}{\sqrt{\left|\begin{array}{l}\left(\begin{array}{l}2\left(3-(1-\lambda)(1+\mu+\mu^2)\right)-(1-\lambda)(1+\mu)\left(2(1-\mu)+\lambda(1+\mu)\right)\\ -2\left(2-(1-\lambda)(1+\mu)\right)^2\end{array}\right)x^2\\ -4\left(2-(1-\lambda)(1+\mu)\right)^2\end{array}\right|}}$$

and

$$|a_3| \leq \frac{|x|}{\left(3-(1-\lambda)(1+\mu+\mu^2)\right)} + \frac{x^2}{\left(2-(1-\lambda)(1+\mu)\right)^2}$$

Proof. Let $f \in L_\Sigma(\lambda,\mu)$. Then there are two analytic functions ϕ, ζ such as $\phi(0)=\zeta(0)=0$ and $|\phi(z)<1|$, $|\zeta(w)|<1$ for all $z, w \in \Delta$, which can be written as

$$\frac{\left((1-\mu)z\right)^{1-\lambda}\left(f'(z)\right)}{\left(f(z)-f(\mu z)\right)^{1-\lambda}} = -1+L_0(x)+L_1(x)\phi(z)+L_2(x)\phi^2(x)+\ldots. \tag{6}$$

and

$$\frac{\left((1-\mu)w\right)^{1-\lambda}\left(h'(w)\right)}{\left(h(w)-h(\mu w)\right)^{1-\lambda}} = -1+L_0(x)+L_1(x)\zeta(w)+L_2(x)\zeta^2(w)+\ldots.. \tag{7}$$

From the equalities (6) and (7).it is obtained that

$$\frac{\left((1-\mu)z\right)^{1-\lambda}\left(f'(z)\right)}{\left(f(z)-f(\mu z)\right)^{1-\lambda}} = 1+L_1(x)r_1z+\left[L_1(x)r_2+L_2(x)r_1^2\right]z^2+\ldots \tag{8}$$

and

$$\frac{\left((1-\mu)w\right)^{1-\lambda}\left(h'(w)\right)}{\left(h(w)-h(\mu w)\right)^{1-\lambda}} = 1+L_1(x)s_1w+\left[L_1(x)s_2+L_2(x)s_1^2\right]w^2+\ldots \tag{9}$$

It is well known that if

$$|\phi(z)| = |r_1z+r_2z^2+r_3z^3+\ldots| < 1, (z \in \Delta)$$

and

$$\left|\zeta(w)\right| = \left|s_1 w + s_2 w^2 + s_3 w^3 + \ldots\right| < 1, (w \in \Delta)$$

then

$$|r_k < 1| \text{ and } |s_k| < 1 \quad \text{for } k \in \mathrm{N} \tag{10}$$

Comparing the coefficients in (8) and (9),

$$(2 - (1-\lambda)(1+\mu))a_2 = L_1(x) r_1 \tag{11}$$

$$\left(3 - (1-\lambda)(1+\mu+\mu^2)\right)a_3 - \frac{(1-\lambda)(1+\mu)}{2}(2(1-\mu) + \lambda(1+\mu))a_2^2 = L_1(x) r_2 + L_2(x) r_1^2 \tag{12}$$

$$-(2 - (1-\lambda)(1+\mu))a_2 = L_1(x) s_1 \tag{13}$$

$$\left(2\left(3 - (1-\lambda)(1+\mu+\mu^2)\right) - \frac{(1-\lambda)(1+\mu)}{2}(2(1-\mu) + \lambda(1+\mu))\right)a_2^2 - \left(3 - (1-\lambda)(1+\mu+\mu^2)\right)a_3 = L_1(x) s_2 + L_2(x) s_1^2 \tag{14}$$

It follows from (11) and (13) that

$$r_1 = -s_1 \tag{15}$$

and

$$2(2 - (1-\lambda)(1+\mu))^2 a_2^2 = L_1^2(x)(r_1^2 + s_1^2) \tag{16}$$

By summing (12) and (14), it is obtained that

$$\left(2\left(3 - (1-\lambda)(1+\mu+\mu^2)\right) - (1-\lambda)(1+\mu)(2(1-\mu) + \lambda(1+\mu))\right)a_2^2 = L_1(x)(r_2 + s_2) + L_2(x)(r_1^2 + s_1^2), \tag{17}$$

By substituting the values of $(r_1^2 + s_1^2)$ from (16) in the right side of (17), it is deduced that

$$\begin{bmatrix} 2\left(3 - (1-\lambda)(1+\mu+\mu^2)\right) - (1-\lambda)(1+\mu)(2(1-\mu) + \lambda(1+\mu))L_1^2(x) \\ -2(2 - (1-\lambda)(1+\mu))^2 L_2(x) \end{bmatrix} a_2^2 = L_1^3(x)(r_2 + s_2) \tag{18}$$

Moreover by computations using (1), (10) and (18), it is found that

$$|a_2| \le \frac{|x|\sqrt{2|x|}}{\sqrt{\left| \begin{pmatrix} 2\left(3 - (1-\lambda)(1+\mu+\mu^2)\right) - (1-\lambda)(1+\mu)(2(1-\mu) + \lambda(1+\mu)) \\ -2(2 - (1-\lambda)(1+\mu))^2 \end{pmatrix} x^2 - 4(2 - (1-\lambda)(1+\mu))^2 \right|}}.$$

By subtracting (12) and (14) , it is obtained that

$$2\left(3 - (1-\lambda)(1+\mu+\mu^2)\right)(a_3 - a_2^2) = L_1(x)(r_2 - s_2) + L_2(x)(r_1^2 - s_1^2) \tag{19}$$

Thus applying (1), it is concluded that

$$|a_3| \le \frac{|x|}{\left(3 - (1-\lambda)(1+\mu+\mu^2)\right)} + \frac{x^2}{(2 - (1-\lambda)(1+\mu))^2}$$

□

By setting $\mu = 0$ in Theorem 1, it is claimed that

Corollary 2.1: If f belongs to $L_\Sigma(\lambda)$,then

$$|a_2| \le \frac{|x|\sqrt{2|x|}}{\sqrt{\left|\left(\lambda^2 + 3\lambda + 2 - 2(1+\lambda)^2\right)x^2 - 4(1+\lambda)^2\right|}},$$

and

$$|a_3| \le \frac{|x|}{2+\lambda} + \frac{x^2}{(1+\lambda)^2}.$$

which was studied by Sahsene Altinkaya et al [11] when the parameters *k = 0* and *μ = 1*.
By putting *λ = 0*, in Theorem 1, it is obtained that,
Remark 2.1 : If *f* is belongs to L_Σ,, then

$$|a_2| \le |x|\sqrt{\frac{|x|}{2}},$$

and

$$|a_3| \le \frac{|x|}{2} + x^2.$$

which was studied by Sahsene Altinkaya et al [11].

3 Fekete- Szego Inequalities for the function class $L_\Sigma(\lambda,\mu)$.

In this section, it is attempted to provide Fekete – Szego inequalities for functions in the class $L_\Sigma(\lambda,\mu)$. These inequalities are given in the following theorem.

Theorem 3.1: For $\eta \in \Re$, the function $f \in L_\Sigma(\lambda,\mu,\eta)$. Then

$$|a_3-\eta a_2^2| \le \begin{cases} \dfrac{|x|}{3-(1-\lambda)(1+\mu+\mu^2)}, \\ \quad |\eta-1| \le \left| \begin{array}{l} \dfrac{2(3-(1-\lambda)(1+\mu+\mu^2))-(1-\lambda)(1+\mu)(2(1-\mu)+\lambda(1+\mu))}{4(3-(1-\lambda)(1+\mu+\mu^2))} \\ -\dfrac{(2-(1-\lambda)(1+\mu))^2}{2(3-(1-\lambda)(1+\mu+\mu^2))} - \dfrac{(2-(1-\lambda)(1+\mu))^2}{x^2(3-(1-\lambda)(1+\mu+\mu^2))} \end{array} \right| \\ \dfrac{4x^2(1-\eta)}{\left[\begin{array}{l} \left\{\begin{array}{l} 2(3-(1-\lambda)(1+\mu+\mu^2))-(1-\lambda)(1+\mu)(2(1-\mu)+\lambda(1+\mu)) \\ -2(2-(1-\lambda)(1+\mu))^2 \end{array}\right\} x^2 \\ -4(2-(1-\lambda)(1+\mu))^2 \end{array}\right]}, \\ \quad |\eta-1| \ge \left| \begin{array}{l} \dfrac{2(3-(1-\lambda)(1+\mu+\mu^2))-(1-\lambda)(1+\mu)(2(1-\mu)+\lambda(1+\mu))}{4(3-(1-\lambda)(1+\mu+\mu^2))} \\ -\dfrac{(2-(1-\lambda)(1+\mu))^2}{2(3-(1-\lambda)(1+\mu+\mu^2))} - \dfrac{(2-(1-\lambda)(1+\mu))^2}{x^2(3-(1-\lambda)(1+\mu+\mu^2))} \end{array} \right| \end{cases}$$

Proof : From (14) and (15), it is observed that

$$\begin{aligned} a_3-\eta a_2^2 &= \frac{(1-\eta)L_1^3(x)(r_2+s_2)}{\left[\begin{array}{l} \{2(3-(1-\lambda)(1+\mu+\mu^2))-(1-\lambda)(1+\mu)(2(1-\mu)+\lambda(1+\mu))\}L_1^2(x) \\ -2(2-(1-\lambda)(1+\mu))^2 L_3(x) \end{array}\right]} \\ &\quad + \frac{L_1(x)(r_2-s_2)}{2(3-(1-\lambda)(1+\mu+\mu^2))}. \\ &= L_1(x)\left[\left(\xi(\eta)+\frac{1}{2(3-(1-\lambda)(1+\mu+\mu^2))}\right)r_2 + \left(\xi(\eta)-\frac{1}{2(3-(1-\lambda)(1+\mu+\mu^2))}\right)s_2\right] \end{aligned}$$

Where

$$\xi(\eta)=\frac{L_1^2(x)(1-\eta)}{\left[\begin{array}{l}\{2(3-(1-\lambda)(1+\mu+\mu^2))-(1-\lambda)(1+\mu)(2(1-\mu)+\lambda(1+\mu))\}L_1^2(x)\\-2(2-(1-\lambda)(1+\mu))^2L_3(x)\end{array}\right]}$$

Then, in view of (1) , it is concluded that

$$\left|a_3-\eta a_2^2\right|\le\begin{cases}\dfrac{|x|}{3-(1-\lambda)(1+\mu+\mu^2)},0\le|\xi(\eta)|\le\dfrac{1}{2(3-(1-\lambda)(1+\mu+\mu^2))}\\ 2|x||\xi(\eta)|,|\xi(\eta)|\ge\dfrac{1}{2(3-(1-\lambda)(1+\mu+\mu^2))}\end{cases}.$$

□

Setting $\mu = 0$ in Theorem 3.1, we have

Corollary 3.1: For $\eta\in\Re$, let the functions $f\in L_\Sigma(\lambda,\mu)$. Then

$$\left|a_3-\eta a_2^2\right|\le\begin{cases}\dfrac{|x|}{2+\lambda},|\eta-1|\le\left|\dfrac{1+\lambda}{4}-\dfrac{(1+\lambda)^2}{2(2+\lambda)}-\dfrac{(1+\lambda)^2}{x^2(2+\lambda)}\right|\\ \dfrac{4x^2(1-\eta)}{\left|(\lambda^2+3\lambda+2-2(1+\lambda)^2)x^2-4(1+\lambda)^2\right|},|\eta-1|\ge\left|\dfrac{1+\lambda}{4}-\dfrac{(1+\lambda)^2}{2(2+\lambda)}-\dfrac{(1+\lambda)^2}{x^2(2+\lambda)}\right|\end{cases}$$

which was investigated by Sahsene Altinkaya et al [11].

Taking the parameter $\lambda = 0$ in the above Corollary 3.1

Remark 3.1: For $\eta\in\Re$, let the function $f\in L_\Sigma(\eta)$, then

$$\left|a_3-\eta a_2^2\right|\le\begin{cases}\dfrac{|x|}{2},|\eta-1|\le\dfrac{1}{|2x^2|}\\ x^2(1-\eta),|\eta-1|\ge\left|\dfrac{1}{2x^2}\right|\end{cases}$$

That was obtained by Sahsene Altinkaya et al [11].

Putting $\eta = 1$ in Theorem 3.1, we have

Corollary 3.2: If the function $f\in L_\Sigma(\lambda,\mu,1)$, then

$$\left|a_3-a_2^2\right|\le\frac{|x|}{3-(1-\lambda)(1+\mu+\mu^2)}$$

which was investigated by Sahsene Altinkaya et al [11].

By setting the parameters $\lambda = 0$ and $\mu = 0$.

Remark 3.2: Let the function $f\in L_\Sigma$, then

$$\left|a_3-a_2^2\right|\le\frac{|x|}{2}.$$

which was studied by Sahsene Altinkaya et al [11].

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