Research Article

Classical and Approximate Theorems of Weighted Space Sampling

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Abstract: We show the same findings in this article, for the authors introduced in 2005 and 2004 that proved the classical sampling theorems, in the space $B^{p}_{\pi w}$ $,1\leq p<\infty$ and Bulzer show that three variants of the indicative analysis sampling theorem are similar in the significance that one can be shown as a corollary of one of the others. in 2014 [1] and [2], but in this work for any function in the space $B^{1,\omega_\alpha}_{\pi,w}$ the space of all functions which are integral by weight function. Consequently, the two sampling theorems in the universal standard are fully identical. The outcome seems to be that our work is successful.

Keyword: Sampling Theorem, Indicative Analysis, Weighted Space.

1. Introduction

With its corresponding aliasing error, the approximate sampling theorem is due to J . L .L. L. Black' (1957)' . The classic Whittaker Kotel, nikov, Shannon theorem contains these theorems, with, $\psi\in\beta^{p}_{\pi w}$, $1\leq p<\infty$, $w>0$, Take into consideration the approximate sampling theorem. For $1\leq p<\infty$, define the space

$$\Psi^{p}=\left\{\psi\in L^{p}\ (\Re)\cap C(\Re)\ ;\ \overset{\wedge}{\psi}\in L'(\Re)\cap L^{p'}\ (\Re)\right\},$$

With $1/p+1/p'=1$, and for $W>0$, the Bernstein feature strap space is limited to $[-\pi w,\pi w]$,

$$\beta^{P}_{\pi,w}=\{\ \psi\in L^{P}\ (\Re)\cap C\ (\Re)\ ;\ \text{supp.}\ \overset{\wedge}{\psi}\subset[-\pi w,\ \pi w]\ \}.$$

Hither $\overset{\wedge}{\psi}(\upsilon)=(1/\sqrt{2\pi})\int_{R}\psi(u)e^{-iu\upsilon}\,du$. The Fourier transformation of ψ should be measured in or under distributional ten, P > 2. The stipulation $\overset{\wedge}{\psi}\in L^{P'}(\Re)$ as specified in the description of Ψ^{p} is Always grateful for the $1\leq P\leq 2$

2- Assertions

Theorem A: Classical Theorem on Sampling

Let $\psi\in B^{P}_{\pi,w}$, $1\leq P<\infty$, $\alpha>0$ then

$$\psi(t)=\sum_{k\in z}\psi\left(\frac{k}{w}\right)\sin(wt-k)\qquad (t\in\Re)\qquad \text{------- (1)}$$

The sequence converges completely and universally on $\Re$. Sampling sequence of samples is the group in (1). ψ measurements. For this theorem's history, see e.g. [3, 4].

If ψ the theorem A is no longer valid. Band not limited. Though at least it can hold on to the cap $w\to\infty$, (see [3, 5, pp. 95, 118 – 122]).

Theorem B: Approximate Sampling Rule or Standardized [6]

Let $\psi\in\Psi^{P}$, $1\leq P<\infty$, and let

$$(R_w\psi)(t) = \frac{1}{\sqrt{2\pi}} \sum_{n\in z}(1-e^{-it2\pi wn}) \int_{(2n-1)\pi w}^{(2n+1)\pi w} \psi(\upsilon)\, e^{i\upsilon t}\, d\upsilon$$

Then

$$\psi(t) = \sum_{k\in z}\psi\left(\frac{k}{w}\right) \sin(wt-k) + (\Re_w\psi)(t) \qquad (t\in\Re) \qquad \text{-------- (2)}$$

Along with the estimate of errors

$$\left|(\Re_w\psi)(t)\right| \le \sqrt{\frac{2}{\pi}} \int_{|\upsilon|\ge\pi w} \left|\hat{\psi}(\upsilon)\right| d\upsilon \qquad (t\in\Re) \qquad \text{-------- (3)}$$

Specifically, one has

$$\psi(t) = \lim_{w\to\infty} \sum_{k\in z}\psi\left(\frac{k}{w}\right) \sin(wt-k) \qquad \text{-------- (4)}$$

Equally for $t\in\Re$

Note that the "aliasing mistake" ($\left|(\Re_w\psi)(t)\right|$).

Treated in the monumental 1908 article by de La Valle Poussin (see [7] pp: 65–156 For publication of his article, and pp. 421-453 Input from a commentary by Butzer –Stens) .

3. Main Results

For $1\le p<\infty$, $\alpha>0$, let ψ be any function & define

$L^{P,\alpha} = \{\psi : \|\psi\|_{p,\alpha} = \left[\int |\psi(\upsilon)\ .\omega_\alpha|^P\ d\upsilon\right]^{1/p} < \infty$ And define $\omega_\alpha = \{\psi : \|\psi\|_{p,\alpha}\ exists\}$

Desire of space

$$\Psi^{p,\alpha} = \left\{\psi\in L^{p,\alpha}(\Re)\cap C(\Re)\ ,\ \hat{\psi}\in L^1(\Re)\ \cap L^{p',\alpha'}(\Re)\right\}$$

Where $\Re$ The set of all actual numbers and then all real numbers is C ($\Re$) space defined for the set of all complex continuous, valued functions $\Re$.

The "Bernstein space" of function band Limited to $[-\pi w\ ,\pi w]$,is St.

$$B_{\pi,w}^{p,\omega_\alpha} = \left\{\psi\in L^{p,\alpha}(\Re)\cap C(\Re)\ ,\ \sup\ \hat{\psi}\subset[-\pi w\ ,\pi w]\right\}$$

Here $\hat{\psi}(\upsilon) = (1/\sqrt{2\pi})\int_R \psi(u).\, e^{-i\upsilon u}\, .\, \omega_\alpha .du$, denotes the transformation of Fourier of ψ.

In the description of ($\hat{\psi}\in L^{p',\alpha'(\Re)}$), the condition ($\Psi^{p,\alpha}$) is always fulfilled for $1\le p\le 2$.

Now we have to show that:

Theorem 1

Let $\psi\in B_{\pi,w}^{p,\omega_\alpha}$, $1\le p<\infty$, $\alpha>0$, then

$$\psi\,\omega_\alpha(t) = \sum_{k\in z}\psi\left(\frac{k}{w}\right)\omega_\alpha Sin(wt-k). \qquad (t\in\Re) \qquad \text{--------- (5)}$$

The series is

$$\psi(t) = \sum_{k=z}\omega_\alpha\psi\left(\frac{k}{w}\right)\sin(wt-k)\,\omega_{-\alpha}(t)$$

Proof

Let $G=\psi(t).\,\omega_\alpha$ then G is bounded and by theorem a (1) we get

$$G\ (t) = \sum_{k \in z} G\left(\frac{k}{w}\right)\ Sin(wt-k) \quad \text{Thus}$$

$$G(t) = \psi\,\omega_\alpha\left(\frac{k}{w}\right) = \sum_{k\in z}\ \psi\,\omega_\alpha\left(\frac{k}{w}\right)\sin(wt-k)$$

end this series is absolute conv.
thus

$$\psi(t) = \sum_{k\in z} \psi\,\omega_\alpha\left(\frac{k}{w}\right)\sin\ (wt-k).\omega_{-\alpha}(t)$$

□

Theorem 2

Let $\psi \in \Psi^{p,\alpha}$, $1 \le P < \infty$, $\alpha > 0$ and let

$$(\Re_w\psi)(t) = \frac{1}{\sqrt{2\pi}} \sum_{n\in z}(1-e^{-it\,2\pi wn}) \int_{(2n-1)\pi w}^{(2n+1)\pi w} \hat{\psi}(\upsilon).\,e^{it\upsilon}\ .\ \omega_\alpha(\upsilon).d\upsilon$$

Then

$$\psi(t) = \sum_{k\ \in z}\psi\,\omega_\alpha\left(\frac{k}{w}\right)\sin(wt-k)\ \omega_{-\alpha}(t) + (\Re_w\psi)(t) \quad (t\in\Re) \qquad \text{--------- (6)}$$

In comparison to the error estimation

$$\left|(\Re_w\psi)(t)\right| \le \sqrt{\frac{2}{\pi}} \int_{|\upsilon|\ge\pi w} \left|\hat{\psi}(\upsilon)\right| d\upsilon \qquad (t\in\Re)$$

Proof

Let $G = \psi(t).\,\omega_\alpha(t)$ then G is bounded

$$(R_w G)(t) = \frac{1}{\sqrt{2\pi}} \sum_{n\ \in z}(1-e^{it\,2\pi\,wn}) \int_{(2n-1)\pi w}^{(2n+1)\pi w} \hat{G}\ .\ e^{it\upsilon}.d\upsilon$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{n\in z}(1-e^{it2\pi wn}) \int_{(2n-1)\pi w}^{(2n+1)\pi \hat{w}} \omega_\alpha \hat{\psi}(v)d(v)$$

Then

$$G(t) = \sum_{k\in z} G\left(\frac{k}{w}\right)\ \sin(wt-k) + (R_w G)(t) \qquad (t\in R)$$

and by theorem B

$$(\Re_w\psi)(t) = \frac{1}{\sqrt{2\pi}} \sum_{n\,\varepsilon\, z}(1-e^{-it2\pi wn}) \int_{(2n-1)\pi w}^{(2n+1)\pi w} \hat{\psi}(\upsilon).e^{it\upsilon}\ .\ \omega_\alpha(\upsilon)\,d\upsilon$$

$$\psi(t) = \sum_{k\in\ z}\psi\left(\frac{k}{w}\right)\ Sin\ (wt-k).\omega_\alpha(t) + (\Re_w\psi)(t) \qquad (t\in\Re)$$

□

Here we inquire the other way around: Does hypothesis 1 mean theorem 2, the outcome of theorem 3, but we have to prove the following outcome:

Lemma 1

Let $\psi_2(t) = \frac{1}{\sqrt{2\pi}} \int_{|v|>\pi w} \hat{\psi}(\upsilon).e^{i\upsilon t}.\,\omega_\alpha(\upsilon)\,d\upsilon$

The sampling sequence, next,

$$(S_w\psi_2)(t) = \sum_{k=-\infty}^{\infty} \psi_2\left(\frac{k}{w}\right) Sin\ (wt-k).\omega_\alpha(t)$$

$$= \lim_{N\to\infty} \sum_{k=-N}^{N} \psi_2\left(\frac{k}{w}\right) Sin(wt-k).\omega_\alpha(t) \qquad \text{-------- (7)}$$

For everything converging ($t \in R$) and rewritable as:

$$(S_w\psi_2)(t) = \frac{1}{\sqrt{2\pi}} \sum_{n\in z} e^{-it2\pi wn} \int_{(2n-1)\pi w}^{(2n+1)\pi w} \hat{f}(\upsilon).e^{it\upsilon}.\omega_\alpha(\upsilon)\,d\upsilon \qquad (t\in R)$$

Proof

The ($t \in R$) periodic extension of the function ($2\pi w$) from the interval ($\upsilon \to e^{i\upsilon t}\ .\ \omega_\alpha(\upsilon)$) to the entire real axis R is for fixed letter $-\pi w, \pi w$ gt, i.e.

$$g_t(\upsilon) = e^{it(\upsilon-2\pi wn)} \qquad (\upsilon\in((2j-1)\pi w\ ,\ (2j+1)\pi w]\ j\in z) \qquad \text{------- (8)}$$

Gt's Fourier coefficients are sin ($wt-k$) $k \in Z$ and, since g_t is of bounded variety, At each continuity point in its sequence of Fourier converges to (g_t) with partial amounts bound evenly (see [5 p, p .28]

Consequently, we have

$$g_t(\upsilon) = \sum_{k=-\infty}^{\infty} Sin\ (wt-k)\ .\omega_\alpha(t).e^{ik\upsilon/w} \qquad (\ \upsilon \neq\ (2j+1)\ \pi\ w,\ j\in Z) \qquad \text{------- (9)}$$

$$\sum_{k=-\infty}^{\infty} Sin(wt-k)\,\omega_\alpha(t).e^{ik\upsilon/w} \le C \qquad (\upsilon\in R,\ n\in N\ ,C>0 \qquad \text{------- (10)}$$

Now for both the series in issue, it have (9)

$$\lim_{N\to\infty}\sum_{k=-N}^{N}\psi_2\left(\frac{k}{w}\right) Sin(wt-k).\ \omega_\alpha(t)$$

$$= \lim_{N\to\infty}\sum_{k=-N}^{N}\left\{\frac{1}{\sqrt{2\pi}}\int_{|\upsilon|>\pi w}\hat{\psi}(\upsilon).\ e^{ikv/w}\ .\ \omega_\alpha(\upsilon)\ d\upsilon\right\} Sin\ (wt-k)$$

$$= \lim_{N\to\infty}\left\{\frac{1}{\sqrt{2\pi}}\int_{|\upsilon|>\pi w}\hat{\psi}(\upsilon)\sum_{k=-N}^{N}e^{ik\upsilon/w}.\ Sin\ (wt-k).\ \omega_\alpha(\upsilon)\ d\upsilon\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{|\upsilon|>\pi w}\hat{\psi}(\upsilon)\lim_{N\to\infty}\sum_{k=-N}^{N}e^{ik\upsilon/w}.\ Sin\ (wt-k).\ \omega_\alpha(\upsilon)\ d\upsilon$$

$$= \frac{1}{\sqrt{2\pi}}\int_{|\upsilon|>\pi w}\hat{\psi}(\upsilon)\ g_t(\upsilon)\ d\upsilon \qquad \text{-------- (11)}$$

Lebesgue's dominant principle of approximation justifies the trade of limit and integral (10).

$$\left|\hat{f}(\upsilon) - \sum_{k=-N}^{N} Sin(wt-k).\omega_\alpha(\upsilon).e^{i\upsilon k/w}\right| \le c\left|\hat{\psi}(\upsilon)\right| \in \overset{1}{L}(R)\ \ (n\in N)$$

This illustrates the Lemma's first section. The second part is now easy to follow since, in view of (10)

$$(S_w\psi_2)(t) = \frac{1}{\sqrt{2\pi}}\int_{|\upsilon|>\pi w}\hat{\psi}(\upsilon).g_t(\upsilon)\ d\upsilon$$

$$= \sum_{n\in z}\frac{1}{\sqrt{2\pi}}\int_{(2n-1)\pi w}^{(2n+1)\pi w}\hat{\psi}(\upsilon)\,g_t(\upsilon)\,d\upsilon$$

$$= \sum_{n \in z} \frac{1}{\sqrt{2\pi}} \int_{(2n-1)\pi w}^{(2n-1)\pi w} \hat{\psi}(\upsilon).\omega_\alpha(\upsilon).\, e^{it(\upsilon-2n\pi w)} d\upsilon)$$

The final equation is correct (8)

Theorem 3

In the presumption ($\psi \in F^{p,\alpha} \quad 1 \leq p < \infty$), (1) implies the approximate sampling theorem theorem (2)

Proof

First assume $p \geq 2$, since $\psi \in F^{p,\alpha}$, $\alpha > 0$

The reversal form for Fourier

$$\psi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\psi}(\upsilon).e^{i\upsilon t}.\omega_\alpha(\upsilon)\, d\upsilon$$

$$= \frac{1}{\sqrt{2\pi}} \int_{|v| \leq \pi w} \hat{\psi}(\upsilon).e^{i\upsilon t}.\omega_\alpha(\upsilon)\, d\upsilon + \frac{1}{\sqrt{2\pi}} \int_{|v| > \pi w} \hat{\psi}(\upsilon).e^{i\upsilon t}.\omega_\alpha\, d\upsilon$$

$$= \psi_1(t) + \psi_2(t) \qquad \text{-------- (12)}$$

Say now $\hat{\psi} \in L^{p'}(R)$ implies $\psi_1 \in B^{p,\alpha}_{\pi w}$, and therefore it can be extended with the classical sampling theorem (Theorem 1).

$$\psi_1(t) = \sum_{k=-\infty}^{\infty} \psi_1\left(\frac{k}{w}\right) Sin(wt-k).\, \omega_\alpha(\upsilon) = (S_w \psi_1)(t)$$

With respect to the Lemma sample sequence f2 (1)

$$(S_w \psi_2)(t) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \varepsilon z} e^{-it2\pi wn} \int_{(2n-1)\pi w}^{(2n+1)\pi w} \hat{\psi}(\upsilon).e^{it\upsilon}.\omega_\alpha(\upsilon)\, d\upsilon \qquad (t \in R)$$

$$\psi(t) = \sum_{k=-\infty}^{\infty} \psi_1\left(\frac{k}{w}\right) Sin(wt-k).\omega_\alpha(t) + \psi_2(t)$$

$$= \sum_{k=-\infty}^{\infty} \left[\psi\left(\frac{k}{w}\right) Sin(wt-k) - \left\{ (S_w \psi_2)(t) - \psi_2(t) \right\} \right] \qquad \text{-------- (13)}$$

As regards the word in curly brackets, we get the definition of f2 by our Lemma

$$\left\{ (S_w \psi_2)(t) - \psi_2(t) \right\} = \frac{1}{\sqrt{2\pi}} \sum_{n\, \varepsilon\, z} e^{-it2\pi wn} \int_{(2n-1)\pi w}^{(2n+1)} \hat{\psi}(\upsilon).e^{it\upsilon}.\omega_\alpha(\upsilon)\, d\upsilon$$

$$= -\frac{1}{\sqrt{2\pi}} \int_{|\upsilon| > \pi w} \hat{\psi}(\upsilon).e^{i\upsilon t}.\, \omega_\alpha(\upsilon)\, d\upsilon$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{n\, \varepsilon\, z} (e^{-it2\pi wn} - 1) \int_{(2n-1)\pi w}^{(2n+1)\pi w} \hat{\psi}(\upsilon).e^{it\upsilon}.\omega_\alpha(v)\, d\upsilon$$

$= -(R_w\, \psi)(t)$

If (2) and a simple approximate rest (row)(t) of the error is inserted into (13), then (($R_w \psi$) (t)) if ($1 \leq p < 2$), ($p' > 2$) and the inference, ($\hat{\psi} \in Lp'^{p,\alpha}\ (R)$) doesn't normally indicate (ψ_1) as a consequence of which iv (1) is not actually ($\psi_1 \in L^{p,\alpha}(R)$) and the reasons mentioned above are not applicable;

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