Research Article

Solving Coupled Fractional Differential Equations Using Differential Transform Method.

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Abstract : This paper presents the solution of coupled equations which are of fractional order using differential transform method. In this paper we extend the scope of differential transform method to system of fractional differential equations so that we get the analytical solutions. The coupled fractional differential equations of a physical system, namely, coupled fractional oscillator with some applications is given via differential transform method. Here we introduce the solution of coupled oscillation of equal fractional order which can be enhanced to unequal fractional order.

Key words: differential transform method, coupled fractional differential equations, coupled oscillation.

2010 subject classification: 34C15, 35R11

1. Introduction

In this paper we have considered systemFDE's

$$D_*^{\alpha 1} x_1\,(t) = f_1(t_1, x_1, x_2 \dots x_n)$$

$D_*^{\alpha 2} x_2\,(t) = f_2(t_1, x_1, x_2 \dots x_n)$...

$D_*^{\alpha n} x_1\,(t) = f_n(t_1, x_1, x_2 \dots x_n)$(1)

Where $D_*^{\alpha i}$is derivative of $x_i$in the sense of Caputo $0 < \alpha_i < 1$ subject to initial conditions.

The differential equationsof fractional order have been studied from many years due to their tremendous number applications not only in fluid mechanics and physics but in biology and engineering as well. Application of FDE of nonlinear in nature has been concluded in [1-6].

The DTM first came into existence in engineering domain which has been fully evaluated in [7]. There is a comparison between DTM and ADM for solving FDE as discussed in [8].

The Riemann- Liouvillefractional integration of order β is given by

$J_{x_0}^{x} f(x) = \frac{1}{\Gamma(\beta)} \int_{x_0}^{x} (x-t)^{\beta-1} f(t)dt, \beta > 0, x > 0$ (2) Given blow are the next two equations of Riemann- Liouville and Caputo fractional derivative of order β respectively:

$D_{x_0}^{\beta} f(x) = \frac{d^p}{dx^p} [J^{p-\beta} f(x)]$(3)

$D_{x_0}^{\beta} f(x) = J^{p-\beta} [\frac{d^p}{dx^p} f(x)]$ (4)

Where $p - 1 < \beta \le p$and $p \epsilon N$

In the formation of problem on initial and boundary conditions,Caputo fractional derivative has been chosen, and the two operators coincide for homogeneous initial conditions.

2. Fractional differential transform method.

Theinitial development of FDTMin [8] is as follows:

Thefractional differentiation of Riemann- Liouville sense is defined by

$D_{x_0}^{\gamma} f(x) = \frac{1}{\Gamma(p-r)} \frac{d^p}{dx^p} [\int_{x_0}^{x} \frac{f(t)}{(x-t)^{1+r-p}} dt],$ (5)

For $p-1 \leq \gamma \leq p$and $p \epsilon z^{+}, x > x_0$. We can expressit in power series as follows:

$f(x) = \sum_{k=0}^{\infty} F(k)(x-x_0)^{\frac{k}{\beta}}$(6)

Where β is order of the fraction and F(k) is fractional differential transform off(x).

Avoiding initial and boundary condition of fractional order, we define derivate in Caputo sense. Relation between operators of Riemann- Liouville and Caputo is given as

$D_{*x_0}^{\gamma} f(x) = D_{x_0}^{\gamma} [f(x) - \sum_{k-0}^{p-1} \frac{1}{k!} (x-x_0)^k f^k(x_0)]$.(7)

Setting$f(x) = f(x) - \sum_{k-0}^{p-1} \frac{1}{k!} (x-x_0)^k f^k(x_0)$ in Eq. 5 and using Eq. 7 we obtain Caputo sense as

$$D_{*x_0}^{\gamma} f(x) = \frac{1}{\Gamma(p-r)} \frac{d^p}{dx^p} [\int_{x_0}^{x} \frac{f(t) - \sum_{k-0}^{p-1} \frac{1}{k!} (t-x_0)^k f^k(x_o)}{(x-t)^{1+r-p}} dt]$$

Applying initial conditions to interger order derivatives,

$$F(k) = \begin{cases} \text{if} \frac{k}{\beta} \epsilon Z^{+}, \frac{1}{\left(\frac{k}{\beta}\right)!} \left[\frac{d^{\frac{k}{\beta}} f(x)}{dx^{\frac{k}{\beta}}} \right]_{x=x0} \\ \text{if} \frac{k}{\beta} \notin Z^{+} \quad 0 \end{cases} \quad (8)$$

We can obtain below mentioned theorems from 5 and 6, for proof and details see [8].

Theorem 1. If $h(x) = u(x) \pm s(x)$implies$H(k) = U(k) \pm S(k)$

Theorem 2. If $h(x) = u(x)s(x)$, implies$H(k) = \sum_{l=0}^{k} U(l)S(k-l)$

Theorem 3. If $h(x) = u_1(x)u_2(x) \dots \dots \dots . u_n(x)$ then

$$H(k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \sum_{k_{n-3}=0}^{k_{n-2}} \sum_{k_{n-2}=0}^{k_{n-1}} . \dots \dots \dots \dots . \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} U_1(k_1)U_2(k_2-k_1) \dots U_n(k-k_{n-1})$$

Theorem 4.If $h(x) = (x-x_0)^m$then $H(k) = \varphi(k-\beta m)$

$$\varphi(\text{k}) = \begin{cases} 1 & if\ k = 0 \\ 0 & if\ k \neq 0 \end{cases}$$

Theorem 5.If $h(x) = D_{x_0}^{\gamma}[r(x)]$ implies$(k) = \frac{\Gamma\left(r+1+\frac{k}{\beta}\right)}{\Gamma\left(1+\frac{k}{\beta}\right)} U(k+\beta r)$.

Theorem 6.In most general form, fractional derivatives can be produced from

$h(x) = \frac{d^{r1}}{dx^{r1}}[u_1(x)]\frac{d^{r2}}{dx^{r2}}[u_2(x)] \ldots \frac{d^{r_{n-1}}}{dx^{r_{n-1}}}[u_{n-1}(x)]\frac{d^{r_n}}{dx^{r_n}}[u_n(x)]$, implies

$$H(k) = \sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\sum_{k_{n-3}=0}^{k_{n-2}}\sum_{k_{n-2}=0}^{k_{n-1}} \cdot \ldots \sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}\frac{\Gamma\left(r_1+1+\frac{k_1}{\beta}\right)}{\Gamma\left(1+\frac{k_1}{\beta}\right)}\frac{\Gamma\left(r_2+1+\frac{k_2-k_1}{\beta}\right)}{\Gamma\left(1+\frac{k_2-k_1}{\beta}\right)}\cdots\frac{\Gamma\left(r_2+1+\frac{k-k_n}{\beta}\right)}{\Gamma\left(1+\frac{k-k_n}{\beta}\right)}$$

$$U_1(k_1+\beta r_1)U_2(k_2-k_1+\beta r_2)\ldots U_n(k-k_{n-1}+\beta r_n);\beta r_j \epsilon z^+ for\ I = 1,2\ldots n$$

3. Coupled fractional differential equation

We introduce the DTM to solve linear inhomogeneous coupled fractional differential equations. Our discussion is restricted to only linear non-homogenous ODE's of fractional order. As given in [9] Coupled oscillators are solved using various techniques. Here we solve the same oscillator equation with DTM to make it more convenient.

To proceed with, linear-coupled oscillator system is given by

$$D^{\gamma_1}x_1(t) = -\omega_2 x_1(t) + \overline{\omega}_2\{x_2(t) - x_1(t)\},$$

$$D^{\gamma_1}x_2(t) = -\omega_2 x_2(t) + \overline{\omega}_2\{x_1(t) - x_2(t)\}$$

Here we use the substitution as

$$-\omega_2 + \overline{\omega}_2 = \mu \,\&\overline{\omega}_2 = \in$$

The above equations become

$$D^{\gamma_1}x_1(t) = -\mu x_1(t) + \in x_2(t)$$

$D^{\gamma_2}x_2(t) = -\mu x_2(t) + \in x_1(t)$(9)

Applying parameters;$\omega = 1$, $\overline{\omega} = 0.5$, $x_1(0) = 0$, $x_2(0) = 1$

$$D^{\gamma_1}x_1(0) = 0$$

$$D^{\gamma_2}x_2(0) = 0.1$$

System (1.9) can be changed by using Theorems 1 to 5

$$X_1(k+\gamma_1\beta_1) = \frac{\boldsymbol{\Gamma}(1+\frac{k}{\beta_1})}{\boldsymbol{\Gamma}(\gamma_1+1+\frac{k}{\beta_1})}[-\mu X_1(k) + \in X_2(k)]$$

$X_2(k+\gamma_2\beta_2) = \frac{\boldsymbol{\Gamma}(1+\frac{k}{\beta_2})}{\boldsymbol{\Gamma}(\gamma_1+1+\frac{k}{\beta_2})}[\in X_1(k) - \mu X_2(k)]$(10)

$$X_1(k) = 0 \qquad for\ k = 1,2,3\ldots\gamma_1\beta_1 - 1$$

$$X_2(k) = 0 \qquad for\ k = 1,2,3\ldots\gamma_2\beta_2 - 1$$

$X_2(0) = 1$ (11)

For $\gamma_1 = 1$ and $\gamma_2 = 1$;$\beta_1 = 1$ and $\beta_2 = 1$

$$x_1(0) = 0 \quad ; \quad x_2(0) = 1$$

$$x_1(1) = \in \quad ; \quad x_2(1) = -\mu$$

$$x_1\left(\frac{1}{2}\right) = \frac{1}{2}[-\in \mu - \mu \in] = -\mu \in$$

$$x_2(1) = \frac{1}{2}[\mu^2 + \in^2]$$

Then following series can be obtained for$x_1(t)$and $x_2(t)$, respectively.

$$x_1(t) = \in t - \in \mu t^2 + \cdots$$

$$x_2(t) = -1 - \mu t + \frac{1}{2}[\mu^2 + \in^2]t^2 + \cdots$$

For γ_1 = 1.4 and γ_2 = 1.7 we have

$$X_1(t+14) = \frac{\boldsymbol{\Gamma}(1+k/10)}{\boldsymbol{\Gamma}(14+1+k/10)}[-\mu X_1(k) + \in X_2(k)]$$

$$X_2(t+17) = \frac{\boldsymbol{\Gamma}(1+k/10)}{\boldsymbol{\Gamma}(1.7+1+k/10)}[X_1(k) - \mu X_2(k)]$$

For k up to 50 and then using in above equation we get the below mentioned series for $x_1(t)$ and $x_2(t)$, respectively.

$$x_1(t) = \frac{\in}{\boldsymbol{\Gamma}(2.4)} t^{\frac{14}{10}} - \frac{\in \mu}{\boldsymbol{\Gamma}(3.8)} t^{\frac{28}{10}} - \frac{\in \mu}{\boldsymbol{\Gamma}(2.7)} t^{\frac{31}{10}} + \frac{\boldsymbol{\Gamma}(4.1)}{\boldsymbol{\Gamma}(5.5)}\left[\frac{\in \mu^2}{\boldsymbol{\Gamma}(2.7)} + \frac{\in^3}{\boldsymbol{\Gamma}(4.1)}\right] t^{\frac{45}{10}} + \frac{\in \mu^2}{\boldsymbol{\Gamma}(4.4)} t^{\frac{48}{10}} + \cdots$$

$$x_2(t) = 1 - \frac{\mu}{\boldsymbol{\Gamma}(2.7)} t^{\frac{17}{10}} + \frac{\in^2}{\boldsymbol{\Gamma}(4.1)} t^{\frac{31}{10}} + \frac{\mu^2}{\boldsymbol{\Gamma}(4.4)} t^{\frac{34}{10}} - \frac{\mu \in^2}{\boldsymbol{\Gamma}(5.5)} t^{\frac{45}{10}} + \frac{\boldsymbol{\Gamma}(4.1)}{\boldsymbol{\Gamma}(5.8)}\left[\frac{\mu \in^2}{\boldsymbol{\Gamma}(2.7)} + \frac{\mu \in^2}{\boldsymbol{\Gamma}(4.1)}\right] t^{\frac{48}{10}} + \ldots$$

4. Conclusion

The present work assures the applicability of solving coupled fractional differential equation by DTM. Earlier it was done with various numerical methods like Laplace transformation method, adjoint method etc. The much greater advantage of this method of solving coupled fractional differential equation over adomain decomposition method is that we do not get adomain polynomials. This method is a reliable one and full of techniques and much easier in solving fractional differential equations of coupled form both linear and nonlinear.

References

1. X. Gao, J.Yu, Synchronization Of Two Coupled Fractional-Order Chaotic Oscillators, Chaos Sol. Fract. 26 (1) (2005) 141–145.
2. J.H. He, Approximate Analytical Solution For Seepage Flow With Fractional Derivatives In Porous Media, Comput. Methods Appl. Mech. Engng.167 (1998) 57–68.
3. H. Jafari, V. Daftardar-Gejji, Solving A System Of Nonlinear Fractional Differential Equations Using Adomian Decomposition, J. Comput. Appl. Math. 196 (2) (2006) 644–651.
4. J.G. Lu, Chaotic Dynamics And Synchronization Of Fractional-Order Arneodo's Systems, Chaos Sol. Fract. 26 (4) (2005) 1125–1133.
5. J.G. Lu, G. Chen, A Note On The Fractional-Order Chen System, Chaos Sol. Fract. 27 (3) (2006) 685–688.
6. I. Podlubny, Fractional Differential Equations, Academic Press, Newyork, 1999.
7. J.K. Zhou, Differentialtransformation And Itsapplications For Electrical Circuits, Huazhong University Press,Wuhan, China, 1986 (In Chinese).
8. A. Arikoglu, I. Ozkol, Solution Of Fractional Differential Equations By Using Differential Transform Method, Chaos Sol.Fract.(2006), Doi:10.1016/J.Chaos. 2006.09.004
9. Solving Linear Coupled Fractional Differential Equations By Direct Operational Method And Some Applications S. C. Lim,1 Chai Hok Eab,2 K. H. Mak,1 Ming Li,3 And S. Y. Chen4

10. F. Mainardi, Fractional Calculus Andwaves In Linear Viscoelasticity, Imperial College Press, London, Uk, 2010.
11. J. Klafter, S. C. Lim, And R. Metzler, Eds., Fractional Dynamics: Recent Advances, World Scientific, Singapore, 2011.
12. W. M. Ahmad And J. C. Sprott, “Chaos In Fractional-Order Autonomous Nonlinear Systems,” Chaos, Solitons And Fractals, Vol. 16, No. 2, Pp. 339–351, 2003.
13. T. T. Hartley, C. F. Lorenzo, And Qammer, “Chaos In A Fractional Order Chua’s System,” Ieee Transactions On Circuits And Systems I, Vol. 42, No. 8, Pp. 485–490, 1995.
14. I. Grigorenko And E. Grigorenko, “Chaotic Dynamics Of The Fractional Lorenz System,” Physical Review Letters, Vol. 91, No. 3, Pp. 034101/1–034101/4, 2003.
15. C. Li And G. Chen, “Chaos And Hyperchaos In The Fractional-Order R¨ Ossler Equations,” Physica A, Vol. 341, No. 1–4, Pp. 55–61, 2004
16. M. Caputo, Linear Models Of Dissipation Whose Q Is Almost Frequency Independent. Part Ii, J. Roy. Austral. Soc. 13 (1967) 529–539.
17. S. Momani, Analytic And Approximate Solutions Of The Space- And Time Fractional Telegraph Equations, Appl. Math. Comput. 170 (2) (2005) 1126–1134.
18. R.Hilfer. Applications Of Fractional Calculus In Physics. Academic Press, Orlando, 1999.
19. S.Abbasbandy. An Approximation Solution Of A Nonlinear Equation With Riemann- Liouville’s Fractional Derivatives By He’s Variational Iteration Method. J. Compu. And App. Math., 207(1):53–58, 2007.
20. [20] S.J.Liao. The Proposed Homotopy Analysis Technique For The Solution Of Nonlinear Problems.Phd Thesis, Shanghai Jiao Tong University, 1992.
21. [21] S.Momani And Z.Odibat. Analytical Solution Of A Time-Fractional Navier-Stokes Equation By Adomian Decomposition Method. Appl. Math. Comput., 177(2):488–494, 2006.
22. Aboodh, K. S., Et Al. "Solving Delay Differential Equations By Aboodh Transformation Method." *International Journal Of Applied Mathematics & Statistical Sciences* (Ijamss) 7.2 (2018): 55-64.
23. Mohamad, H. A., And Sattar Naser Ketab. "Oscillation And Nonoscillation Properties Of Solutions Of Third Order Linear Neutral Differential Equations.” International Journal Of Applied Mathematics & Statistical Sciences (Ijamss) 5.3 (2016): 31-38
24. Shaalini, J. Vinci, And A. Emimal Kanaga Pushpam. "Application Of Rkhem Method For Solving Delay Differential Equations With Constant Lags." International Journal Of Applied Mathematics & Statistical Sciences (Ijamss) 5.3 (2016) 39-46
25. Adio, Ak. "Application Of Reduced Differential Transformation Method For Solving System Of Nonlinear Partial Differential Equations (Pdes)." International Journal Of Applied Mathematics & Statistical Sciences (Ijamss) 5.4 (2016) 63-70
26. Napoleao, Diovana Aparecida Dos Santos, Carolina De Souza Ramunno, And Helcio Jose. "Slurry Degradation Analysis Via Fenton And Photo-Fenton By Stochastic Differential Equations." Impact: International Journal Of Research In Engineering & Technology (Impact: Ijret) 5.8 (2017) 9-16
27. Jawahar, G. Gomathi. "Qualitative Analysis On Second Order Neutral Delay Difference Equations." International Journal Of Mechanical And Production Engineering Research And Development (Ijmperd) 9.2 (2019) 659-664