Decomposition of Product Path Graphs Into Graceful Graphs

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Abstract: A decomposition of G is a collection $\psi_g = \{H_1, H_2, ...H_r\}$ such that H_i are edge disjoint and every edge in H_i belongs to G. If each H_i is a graceful graph, then ψ_g is called a graceful decomposition of G. The minimum cardinality of a graceful decomposition of G is called the graceful decomposition number of G and it is denoted by $\pi_g(G)$. In this paper, we define graceful decomposition and graceful decomposition number $\pi_g(G)$ of a graphs. Also, some bounds of $\pi_g(G)$ in product graphs like Cartesian product, composition etc. are investigated.

Keyword: Decomposition, Graceful graphs, Graceful decomposition and Graceful decomposition number.

1. Introduction

A graph is a well-ordered pair $G = (V, E)$, where V is a non-empty finite set, called the set of vertices or nodes of G, and E is a set of unordered pairs (2-element subsets) of V, called the edges of G. If $xy \in E$, x and y are called adjacent and they are incident with the edge xy.

The complete graph on n vertices, denoted by K_n, is a graph on n vertices such that every pair of vertices is connected by an edge. The empty graph on n vertices, denoted by E_n, is a graph on n vertices with no edges. A graph $G' = (V', E')$ is a sub graph of $G = (V, E)$ if and only if $V' \subseteq V$ and $E' \subseteq E$.The order of a graph $G = (V, E)$ is $|V|$, the number of its vertices. The size of G is $|E|$, the quantity of its edges. The degree of a node $x \in V$, represented by $d(x)$, is the quantity of edges incident with it.

A subgraph H of G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a graph $G(V, E)$ and a subset $W \subseteq V$, the subgraph of G induced by W, denoted as $G[W]$, is the graph $H(W, F)$ such that, for all $u, v \in W$, if $uv \in E$, then $uv \in F$. We say H is an induced subgraph of G.

A graph $G(V, E)$ is said to be connected if every pair of vertices is connected by a path. If there is exactly one path connecting each pair of vertices, we say G is a tree. Equivalently, a tree is a connected graph with n − 1 edges. A pathgraph P_n is a connected graph on n vertices such that each vertex has degree at most 2. A cycle graph C_n is a connected graph on n vertices such that every vertex has degree 2.

A complete graph P_n is a graph with n vertices such that every vertex is adjacent to all the others. On the other hand, an independent set is a set of vertices of a graph in which no two vertices are adjacent. We denote In for an independent set with n vertices.

A bipartite graph $G(V, E)$ is a graph such that there exists a partition $P(A, B)$ of V such that every edge of G connects a vertex in A to one in B. Equivalently, G is said to be bipartite if A and B are independent sets. The bipartite graph is also denoted as $G(A, B, E)$.

A graceful labelling of a graph G is a vertex labelling $f : V \to [0,1]$ such that f is injective and the edge labelling $f^* : E \to [1, m]$ defined by $f^*(uv) = |f(u) - f(v)|$ is also injective. If a graph G admits a graceful labelling, we say G is a graceful graph.

In this paper we define graceful decomposition and graceful decomposition number $\pi_g(G)$ of a graph G. Also investigate some bounds of $\pi_g(G)$ in product graphs like Cartesian product, composition etc.

2. Graceful Decomposition

In this section we define graceful decomposition of a graph $G(V,E)$ some and investigate some bounds of graceful decomposition number in $G(V,E)$.

Definition 2.1: Let $\psi_g = \{H_1, H_2, ...H_r\}$ be a decomposition of a graph G. If each H_i is a graceful graph, then ψ_g is called a graceful decomposition of G. The minimum cardinality of a graceful decomposition of G is called the graceful decomposition number of G and it is denoted by $\pi_g(G)$.

Definition 2.2: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs. The join $G_1 + G_2$ of G_1 *and* G_2 with disjoint vertex set $V_1 \& V_2$ and the edge set E of $G_1 + G_2$ is defined by the two vertices (u_i, v_j) if one of the following conditions are satisfied

i) $u_i v_j \in E_1$.

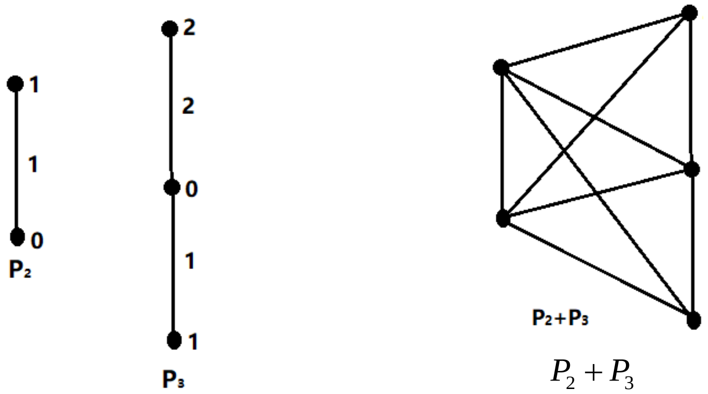
ii) $u_i v_j \in E_2$.

iii) $u_i \in V_1 \; \& v_j \in V_2$, $u_i v_j \in E$

Theorem 2.1: A graph $P_n + P_m$ is a join of two path graceful graphs with (m>n) can be decomposed in to at least 'm' number of P_m ,graceful graphs. Then the graceful decomposition number $\pi_g(P_n + P_m) \geq 3.$

Proof: Let P_n and P_m be two path graceful graphs of order m and n (m>n)respectively and $P_n + P_m$ is a join of P_n and P_m with edge set E. Therefore $E = E_1 \cap E_2 \cap S(K_{m,n})$, here $S(K_{m,n})$ is a size of a bipartite complete graph $K_{m,n}$. Note that P_n and P_m be two graceful graphs and complete bipartite graphs $K_{m,n}$ also graceful graph. The complete bipartite graphs $K_{m,n}$ can be decomposed in to m number of P_m. This implies $\psi_g \supseteq \left\{ \bigcup_{i=1}^{m} P_{mi} \right\}$ and $\left|\psi_g\right| \geq \left| \left\{ \bigcup_{i=1}^{m} P_{mi} \right\} \right|$. Therefore we get $\pi_g(P_n + P_m) \geq m.$ Note that P_n and P_m are graceful graph also decomposed in to P_n and P_m paths, hence we get $\pi_g(P_n + P_m) \geq m.$

Illustration 2.1: The Join of two graceful graphs $P_2 \& P_3$ is given in figure.2.1



$P_2 + P_3$

The graph $P_2 + P_3$ is decomposed in to isomorphic graphs of P_2 , P_3 and $K_{3,,2}$. Therefore the set $\psi_g = \{P_1, P_2, K_{3,2}\}$

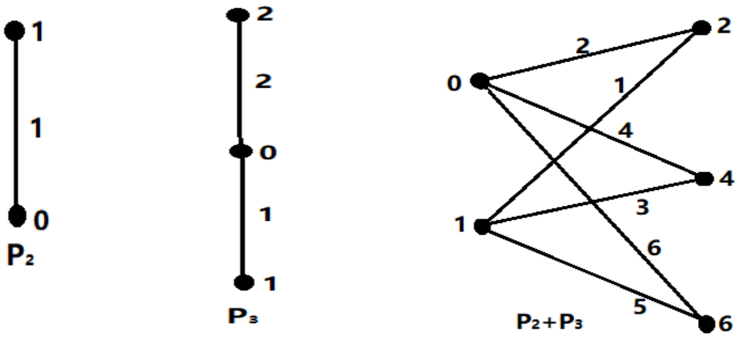


Figure.2.1: Graceful decomposition of $P_2 + P_3$

Definition 2.3: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs. The Cartesian product $G_1 \times G_2$ of G_1 *and* G_2, is a graph with vertex set $V = V_1 \times V_2$ and the edge set of $G_1 \times G_2$ is defined by the two vertices (u_i, v_j) & (u_k, v_l) if one of the following conditions are satisfied

i) $u_1 = v_1$ and u_2, v_2 are adjacent vertices in $G_2 = (V_2, E_2)$.

ii) $u_2 = v_2$ and u_1, v_1 are adjacent vertices in $G_1 = (V_1, E_1)$.

Theorem 2.2: A graph $P_m \times P_n$ is a Cartesian product of two graceful graphs P_m & P_n with order m and n can be decomposed in to at least $(m+n)$ graceful graphs (i.e. $\pi_g(G_1 \times G_2) \geq (m+n)$).

Proof: Let P_m and P_n be two path graceful graphs of order m and n (m > n) respectively and $P_n \times P_m$ and is a Cartesian product of P_n & P_m with edge set E the one of the following conditions are satisfied

i) $u_1 = v_1$ and u_2, v_2 are adjacent vertices in $G_2 = (V_2, E_2)$.

ii) $u_2 = v_2$ and u_1, v_1 are adjacent vertices in $G_1 = (V_1, E_1)$.

Case (i): If $u_1 = v_1$ and u_2, v_2 are adjacent vertices in $G_2 = (V_2, E_2)$

If $u_1 = v_1$ and u_2, v_2 are adjacent vertices in $G_2 = (V_2, E_2)$. Let the sub graph H_i is isomorphic to the graph $G_2 = (V_2, E_2)$. The graph $G_2 = (V_2, E_2)$ be a graceful graph this implies H_i is also a graceful graph. This implies $H_i \subset \psi$

Case (ii): If $u_2 = v_2$ u_1, v_1 are adjacent vertices in $G_1 = (V_1, E_1)$

If $u_2 = v_2$ u_1, v_1 are adjacent vertices in $G_1 = (V_1, E_1)$. Let the sub graph H_j is isomorphic to the graph $G_1 = (V_1, E_1)$. The graph $G_1 = (V_1, E_1)$ is a graceful graph this implies H_j is also a graceful graph. This implies $H_j \subset \psi$.

From case (i) and (ii), we get $\psi = \left\{ \left(\bigcup_{i=1}^{m} H_i \right) \cup \left(\bigcup_{j=1}^{n} H_j \right) \right\}$ this implies $|\psi| = \sum_{i=1}^{m} H_i + \sum_{j=1}^{n} H_j = m + n$. Hence we get $\pi_g(G_1 \times G_2) = (m+n)$.

Illustration 2.2: The Cartesian product of two graceful graphs P_2 & P_3 is given in Figure.2.2

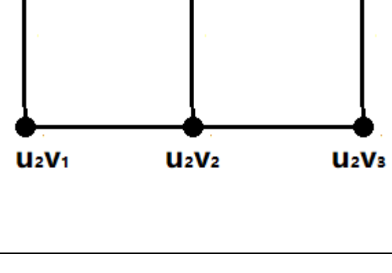
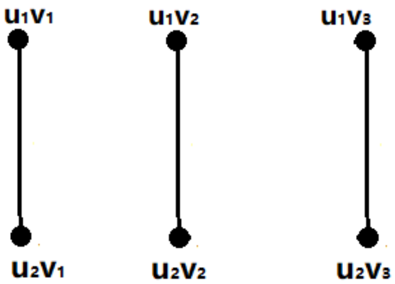
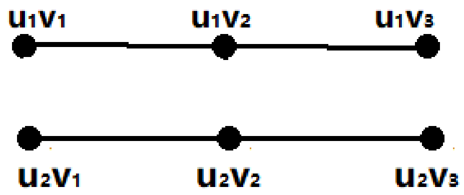


Figure.2.2: $P_2 \times P_3$

The graph $P_2 \times P_3$ is decomposed in to isomorphic graphs of P_2 and P_3, the set ψ contains n times P_2 and m times P_3 as follows.

Isomorphic graphs of P_2 *Isomorphic graphs of* P_3





The graph $P_2 \times P_3$ is decomposed in to $O(G_2)$ number of G_1 graphs, $O(G_1)$ number of G_2 Graphs.

Definition 2.4: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs. The Composition $G_1 \circ G_2$ of G_1 *and* G_2, is a graph with vertex set $V = V_1 \times V_2$ and the edges in $G_1 \circ G_2$ is defined by the two vertices (u_1, u_2) & (v_1, v_2) if one of the following conditions are satisfied

i) $u_1 = v_1$ and u_2, v_2 are adjacent vertices in $G_2 = (V_2, E_2)$.
ii) $u_2 = v_2$ and u_1, v_1 are adjacent vertices in $G_1 = (V_1, E_1)$.
iii) u_1, v_1 are adjacent vertices in $G_1 = (V_1, E_1)$.

Theorem 2.3: A graph $G_1 \circ G_2$ is a Composition of two graceful graphs G_1 & G_2 with order m and n, can be decomposed in to at least $(mn + m + n)$ graceful graphs (i.e. $\pi_g(G_1 \circ G_2) \geq (mn + m + n)$).

Proof: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graceful graphs of order m and n respectively and $G_1 \circ G_2$ is a Composition of G_1 *and* G_2 with edge set E the one of the following conditions are satisfied

i) $u_1 = v_1$ and u_2, v_2 are adjacent vertices in $G_2 = (V_2, E_2)$.
ii) $u_2 = v_2$ and u_1, v_1 are adjacent vertices in $G_1 = (V_1, E_1)$.
iii) u_1, v_1 are adjacent vertices in $G_1 = (V_1, E_1)$.

Case (i): If $u_1 = v_1$ and u_2, v_2 are adjacent vertices in $G_2 = (V_2, E_2)$

If $u_1 = v_1$ and u_2, v_2 are adjacent vertices in $G_2 = (V_2, E_2)$. Let the sub graph H_i is isomorphic to the graph $G_2 = (V_2, E_2)$. The graph $G_2 = (V_2, E_2)$ is a graceful graph this implies H_i is also a graceful graph. This implies $H_i \subset \psi$

Case (ii): If $u_2 = v_2$ u_1, v_1 are adjacent vertices in $G_1 = (V_1, E_1)$

If $u_2 = v_2$ u_1, v_1 are adjacent vertices in $G_1 = (V_1, E_1)$. Let the sub graph H_j is isomorphic to the graph $G_1 = (V_1, E_1)$. The graph $G_1 = (V_1, E_1)$ be a graceful graph this implies H_j is also a graceful graph. This implies $H_j \subset \psi$.

Case (iii): If u_1, v_1 are adjacent vertices in $G_1 = (V_1, E_1)$.

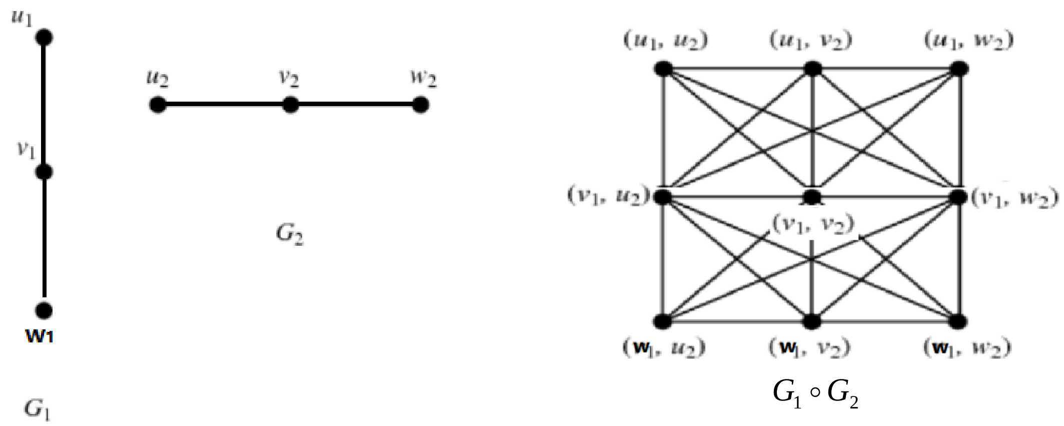
If u_1, v_1 are adjacent vertices in $G_1 = (V_1, E_1)$. The graph $G_1 = (V_1, E_1)$ be a graceful graph therefore we get mn number graceful graph isomorphic to $G_1 = (V_1, E_1)$. Hence we get mn times of $G_1 = (V_1, E_1)$.

From case (i) and (ii), we get $\psi = \left\{ \left(\bigcup_{i=1}^{m} H_i \right) \cup \left(\bigcup_{j=1}^{n} H_j \right) \cup \left(\bigcup_{j=1}^{n} \left(H_{1j}, H_{2j}, \ldots H_{mj} \right) \right) \right\}$ this

implies $|\psi| = \sum_{i=1}^{m} H_i + \sum_{j=1}^{n} H_j + \sum_{j=1}^{n} \sum_{i=1}^{m} H_{ij} = m + n + mn$. Hence we get

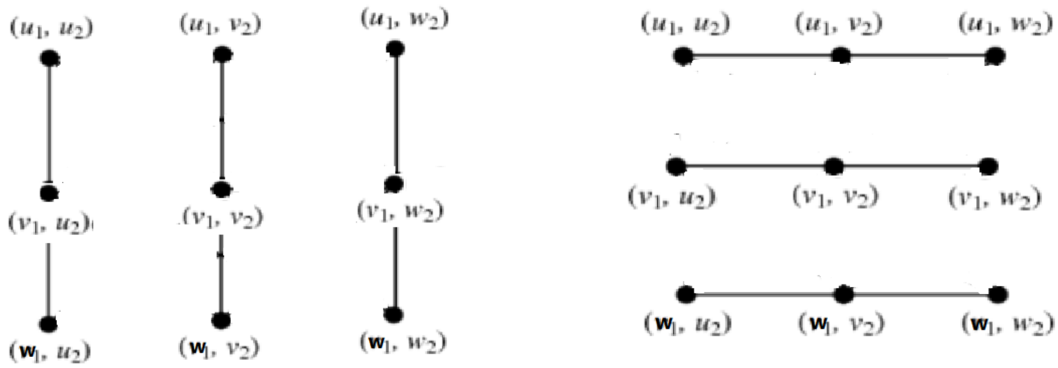
$\pi_g(G_1 \circ G_2) \geq (m + n + mn)$.

Illustration 2.3: The Cartesian product of two graceful graphs P_2 & P_3 is given in Figure.2.3



Decomposition of $G_1 \circ G_2$

Isomorphic to G_1 *Isomorphic to* G_2



Isomorphic to 'mn' times of G_1

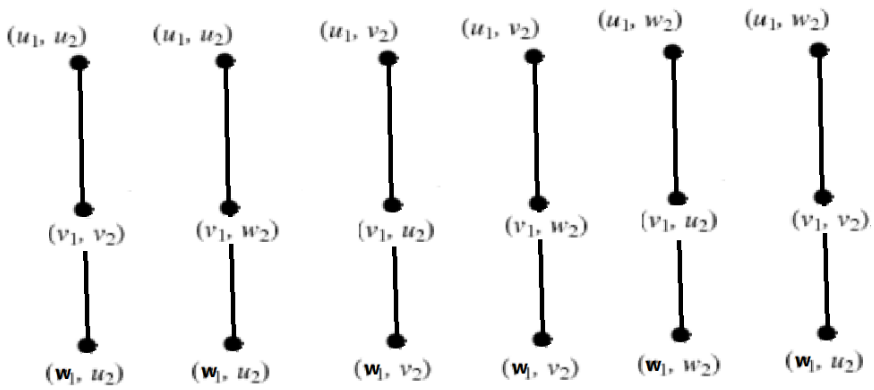


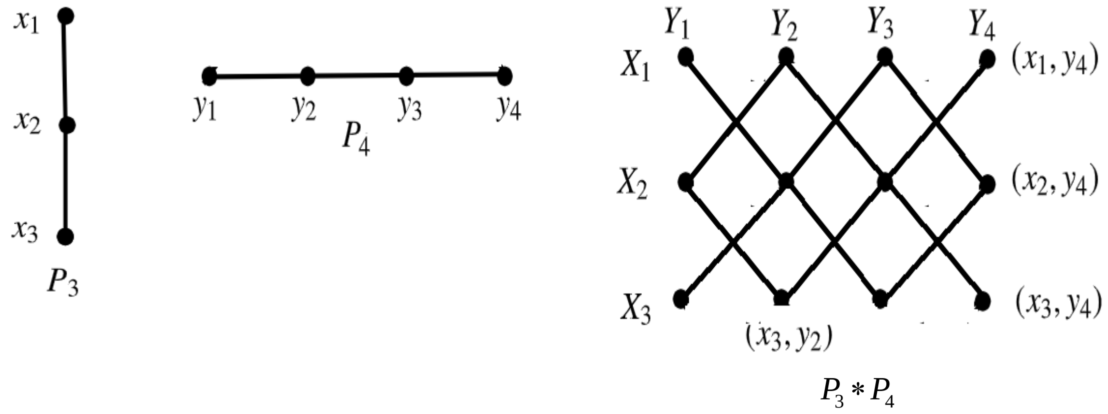
Figure.2.3

Definition 2.5: For two simple graphs G and H their tensor product is denoted by $G * H$, has vertex set $V = V_1 \times V_2$ in which (g_1, h_1) and (g_2, h_2) are adjacent whenever $g_1 g_2$ is an edge in G and $h_1 h_2$ is an edge in H

Theorem 2.4: A graph P_m is a tensor product of two graceful graphs with order $(m > n)$, can be decomposed in to (m) number of P_m graceful graphs (i.e. $\pi_g(P_m * P_n) = (m)$).

Proof: A graph $P_m * P_n$ is a tensor product of two graceful graphs with $(m > n)$. Let the vertex (u_1, v_1) and (u_2, v_2) are adjacent whenever u_1u_2 is an edge in P_m and v_1v_2 is an edge in P_n. By the definition we identify 'm' number of P_m in tensor product P_m .Hence we get $\pi_g(P_m * P_n) = (m)$.

Illustration 2.4: The tensor product of two graceful graphs P_2 & P_3 is given in Figure.2.4



P_4 Decomposition of $P_3 * P_4$

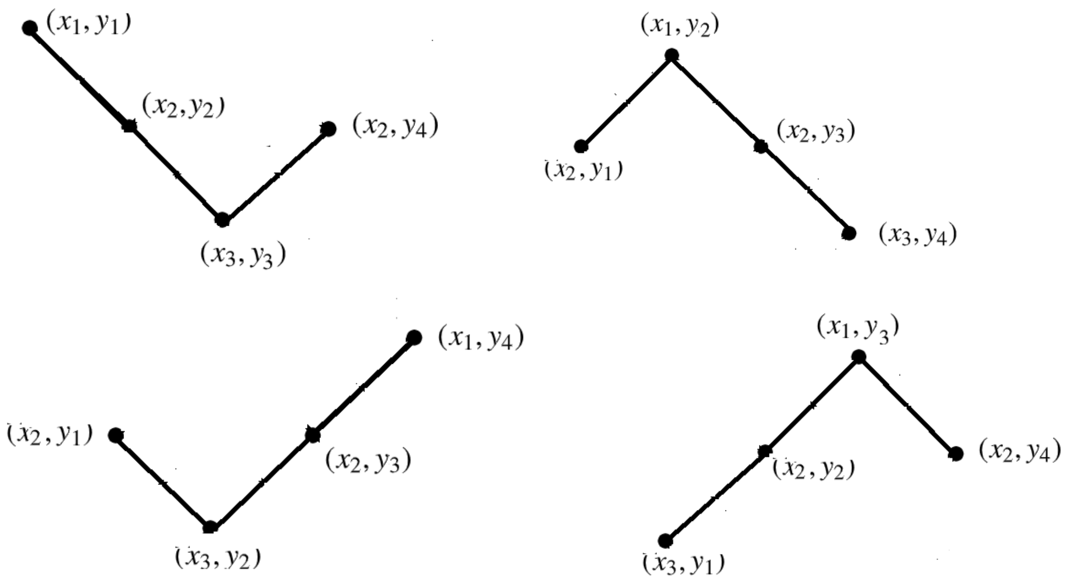


Figure.2.4

Definition 2.6: The Strong product $G \otimes H$ of graphs G and H has the vertex set $V(G \otimes H) = V(G) \times V(H)$ and $(a, x)(b, y)$ is an edge of $G \otimes H$ ere satisfied one of the following condition.

i) $a = b$ and $xy \in E(H)$.

ii) $ab \in E(G)$ and $x = y$.

iii) $ab \in E(G)$ and $xy \in E(H)$.

Theorem 2.5: A graph $P_m \otimes P_n$ is a Strong productof two graceful graphs with m > n, can be decomposed in to at least $(2m + n)$ graceful graphs (i.e. $\pi_g(P_m \otimes P_n) \geq (2m + n)$).

Proof: Let $P_m = (V_1, E_1)$ and $P_m = (V_2, E_2)$ be two graceful graphs of order m and n respectively and $P_m \otimes P_n$ is a Strong productof $P_m \, and \, P_n$ with edges $(a, x)(b, y) \in E$ and the set is satisfied the one of the following conditions.

i) $a = b$ and $xy \in P_m$.

ii) $ab \in P_n$ and $x = y$.

iii) $ab \in P_n$ and $xy \in P_m$.

Case (i): If $a = b$ and $xy \in P_m$ are adjacent vertices in P_m.

If $a = b$ and $xy \in P_m$ are adjacent vertices in P_m. Let the sub graph formed by these set of edges is H_i isomorphic to the graph P_m. The graph P_m is a graceful graph this implies H_i is also a graceful graph. This implies $H_i \subset \psi$

Case (ii): If $ab \in P_n$ are adjacent vertices in P_n and $x = y$.

If $ab \in P_n$ are adjacent vertices in P_n and $x = y$ .Let the sub graph formed by these set of edges is H_j isomorphic to the graph P_n. The graph P_n is a graceful graph this implies H_j is also a graceful graph. This implies $H_j \subset \psi$.

Case (iii): If $ab \in P_n$ are adjacent vertices in P_n .and $xy \in P_m$ are adjacent vertices in P_m.

If $ab \in P_n$ are adjacent vertices in P_n .and $xy \in P_m$ are adjacent vertices in P_m. The graph P_m is a graceful graph therefore we get m number graceful graph isomorphic to P_m Hence we get m times of P_m.

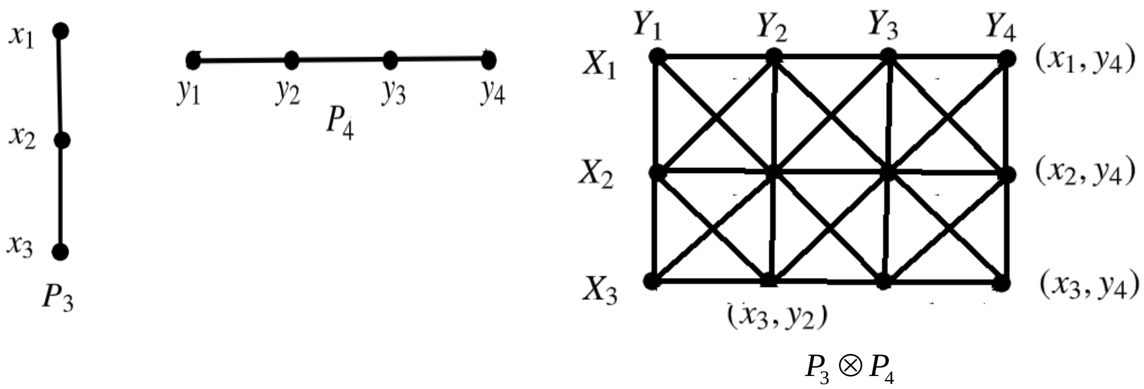
From case (i) and (ii), we get $\psi = \left\{ \left(\bigcup_{i=1}^{m} P_{ni} \right) \cup \left(\bigcup_{j=1}^{n} P_{mj} \right) \cup \left(\bigcup_{i=1}^{m} P_{mi} \right) \right\}$ this implies

$$|\psi| = \sum_{i=1}^{m} P_{ni} + \sum_{j=1}^{n} P_{mj} + \sum_{i=1}^{m} P_{mi} .$$

$$|\psi| \quad = m + n + m = 2m + n$$

Paths $P_m \,\&\, P_n$ are also decomposed in to graceful graphs .Hence we get $\pi_g (P_m \otimes P_n) \geq (2m + n)$.

Illustration 2.5: The strong product of two graceful graphs $P_2 \,\&\, P_3$ and its possible decomposition are given in Figure.2.5



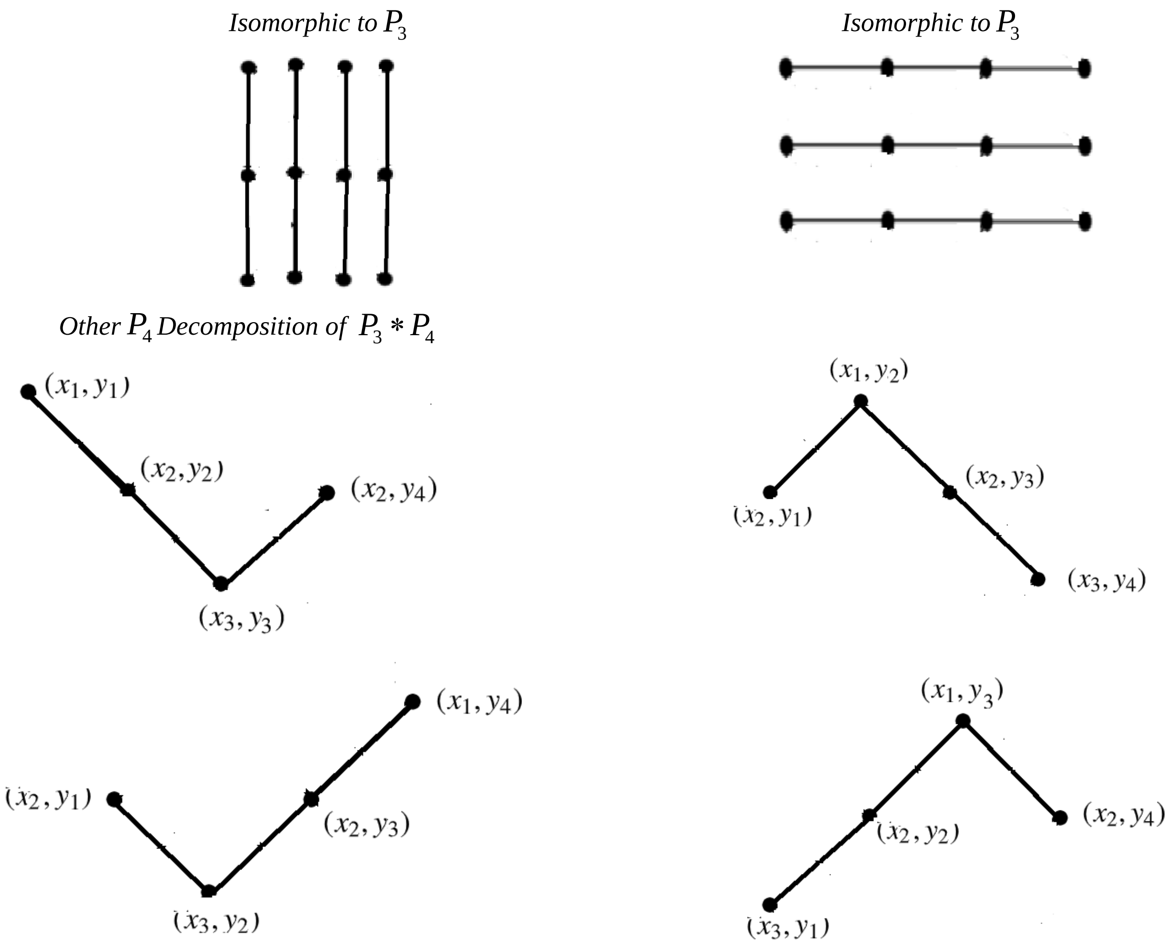


Figure.2.5

Conclusion:

In this paper, we define graceful decomposition and graceful decomposition number $\pi_g(G)$ of a graph G . Also, some bounds of $\pi_g(G)$ in product graphs like Cartesian product, composition etc. are discussed. In future, we will define different types of decomposition on labelling.v

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