$\mathcal{N}s\hat{g}$-continuous functions in Nano Topological Spaces

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Abstract: This paper focuses on $\mathcal{N}s\hat{g}$-continuous functions (nano semi $\hat{g}$-continuous functions) in nano topological spaces and certain properties are investigated. We also investigate the concept of $\mathcal{N}s\hat{g}$-continuous functions and discussed their relationships with other forms of nano continuous functions. Further, we have given an appropriate examples to understand the abstract concepts clearly.

Keywords: $\mathcal{N}s\hat{g}$-closed sets, $\mathcal{N}s\hat{g}$-continuous functions.

1. Introduction

Topology is a branch of Mathematics through which we elucidate and investigate the ideas of continuity , within the framework of Mathematics. The study of topological spaces, their continuous mappings and general properties make up one branch of topologies known as general topology. In 1970, Levine [10] introduced the concept of generalized closed sets in topological spaces. This concept was found to be useful to develop many results in general topology. In 1991, Balachandran et.al [1] introduced and investigated the notion of generalized continuous functions in topological spaces. In 2008, Jafari et.al [6] introduced $\hat{g}$-closed sets in topological spaces. The notion of nano topology was introduced by Lellis Thivagar [8] which was defined in terms of approximations and boundary region of a subset of an universe using an equivalence relation on it. He also established and analyzed the nano forms of weakly open sets such as nano α-open sets, nano semi-open sets and nano pre-open sets. Bhuvaneswari and Mythili Gnanapriya [4], introduced and studied the concept of Nano generalized-closed sets in nano topological spaces. In 2017, Lalitha [7] defined the concept of $\mathcal{N}\hat{g}$ closed and open sets in nano topological spaces.

2 Preliminaries

Definition 2.1 *[8]* Let U be a non-empty finite set of objects called the universe R be an equivalence relation on U named as the indiscerniblity relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$.

1. The Lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$. That is, $L_R(X) = \{\bigcup_{x\in U} \{R(x): R(x) \subseteq X\}\}$, where $R(x)$ denotes the equivalence class determined by x.

2. The Upper approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $U_R(X)$.

   That is, $U_R(X) = \{\bigcup_{x\in U} \{R(x): R(x) \cap X \neq \phi\}\}$

3. The Boundary region of X with respect to R is the set of all objects which can be classified as neither as X nor as not X with respect to R and it is denoted by $B_R(X)$.

   That is, $B_R(X) = U_R(X)$-$L_R(X)$

Definition 2.2 *[8]* Let U be the universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$. $\tau_R(X)$ satisfies the following axioms:

1. U and $\phi \in \tau_R(X)$
2. The union of elements of any subcollection of $\tau_R(X)$ is in $\tau_R(X)$.
3. The intersection of the elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$

That is, $\tau_R(X)$ forms a topology on U is called the nano topology on U with respect to X. We call $\{U, \tau_R(X)\}$ is called the nano topological space.

Definition 2.3 *[8]* If $(U, \tau_R(X))$ is Nano topological space with respect to X where $X \subseteq U$ and if $A \subseteq U$,then

1.The nano interior of the set A is defined as the union of all nano open subsets contained in A and is denoted by $\mathcal{N}int(A)$. $\mathcal{N}int(A)$ is the largest nano open subset of A.

2. The nano closure of the set A is defined as the intersection of all nano closed sets containing A and is denoted by $\mathcal{N}cl(A)$. $\mathcal{N}cl(A)$ is the smallest nano closed set containing A.

Definition 2.4 Let $(U, \tau_R(X))$ be a nano topological space and $A \subseteq U$. Then A is said to be ,

1. Nano semi-closed [8], if $\mathcal{N}int(\mathcal{N}cl(A)) \subseteq A$.
2. $\mathcal{N}g$-closed [2], if $\mathcal{N}cl(A) \subseteq G$ whenever $A \subseteq G$ and G is nano open.
3. $\mathcal{N}gs$-closed [3], if $\mathcal{N}scl(A) \subseteq G$ whenever $A \subseteq G$ and G is nano open.
4. $\mathcal{N}gp$-closed [3], if $\mathcal{N}pcl(A) \subseteq G$ whenever $A \subseteq G$ and G is nano open.
5. $\mathcal{N}gsp$-closed [11], if $\mathcal{N}spcl(A) \subseteq G$ whenever $A \subseteq G$ and G is nano open.
6. $\mathcal{N}\hat{g}$-closed [7], if $\mathcal{N}cl(A) \subseteq G$ whenever $A \subseteq G$ and G is nano semi-open.
7. $\mathcal{N}\alpha g$ closed [4], if $\mathcal{N}\alpha cl(A) \subseteq G$ whenever $A \subseteq G$ and G is nano open.
8. $\mathcal{N}sg$-closed [5] , if $\mathcal{N}scl(A) \subseteq G$, whenever $A \subseteq G$ and G is nano semi open.
9. $\mathcal{N}s\hat{g}$-closed [12], if $\mathcal{N}cl(A) \subseteq G$ whenever $A \subseteq G$ and G is nano sg open.

Definition 2.5 Let $(U, \tau_R(X))$ and $(V, \tau_{R'}(Y))$ be a nano topological spaces. Then the function $f: (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ is nano continuous on U, if the inverse image of every nano open set in V is nano open set in U.

3.Nano Semi $\hat{g}$ - continuous function

In this section, we define and study the new class of functions, namely nano semi $\hat{g}$-continuous function (briefly, $\mathcal{N}s\hat{g}$-continuous function) in nano topological spaces and obtain some of its properties. Also we investigate the relationships between the other existing continuous functions.

Definition 3.1

1. Let $(U, \tau_R(X))$ and $(V, \tau_{R'}(Y))$ be two Nano topological spaces. Then a mapping $f: (U, \tau_R(X)) \to ((V, \tau_{R'}(Y)))$ is $\mathcal{N}s\hat{g}$- continuous on U if the inverse image of every nano closed set V is $\mathcal{N}s\hat{g}$ closed in U.
2. A space $(U, \tau_R(X))$ is called a $\mathcal{N}s\hat{g}$-space if every $\mathcal{N}s\hat{g}$-closed set in it is nano-closed

Theorem 3.2 *If* $f: (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ is nano continuous then it is $\mathcal{N}s\hat{g}$-continuous but not conversely.

Proof: Let $f: U \to V$ be nano continuous. Let A be a closed set in V, then $f^{-1}(A)$ is closed set in U. since every nano closed set is $\mathcal{N}s\hat{g}$ closed. Hence $f^{-1}(A)$ is $\mathcal{N}s\hat{g}$ closed. Therefore f is $\mathcal{N}s\hat{g}$-continuous.

Example 3.3 Let $U = \{a, b, c, d\}$, $X = \{a, b\}$ with $U/R = \{\{b\}, \{, c\}, \{a, d\}\}$, then $\tau_R(X) = \{U, \phi, \{b\},$ $\{a, b, d\},\{a, d\}\}$ and $\tau_R^c(X) = \{U, \phi, \{a, c, d\},\{c\},\{b, c\}\}$ which are nano closed sets,

Let $V = \{a, b, c, d\}$, $Y = \{c, d\}$ with $V/R = \{\{a, b, d\}, \{c\}\}$, then $\tau_{R^1}(Y) = \{V, \phi, \{c\}, \{a, b, d\}\}$

but $f(a) = b$, $f(b) = a$, $f(c) = d$, $f(d) = c$, is nano continuous but not $\mathcal{N}s\hat{g}$-continuous.

Theorem 3.4 If $f: (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ then the following holds

1. Every $\mathcal{N}s\hat{g}$-continuous is nano αg continuous
2. Every $\mathcal{N}s\hat{g}$-continuous is $\mathcal{N}g$ continuous.
3. Every $\mathcal{N}s\hat{g}$-continuous is $\mathcal{N}sg$ continuous.
4. Every $\mathcal{N}s\hat{g}$-continuous is $\mathcal{N}gs$ continuous.
5. Every $\mathcal{N}s\hat{g}$-continuous is $\mathcal{N}gsp$ continuous

Remark 3.5 The converse of the above theorem is need not be true in general

Example 3.6 Let $U = V = \{a, b, c, d\}$, $X = \{a, b\}$ with $U/R = \{\{b\}, \{, c\}, \{a, d\}\}$, then $\tau_R(X) = \{U, \phi, \{b\}, \{a, b, d\}, \{a, d\}\}$ *and* $\tau_R^c(X) = \{U, \phi, \{a, c, d\}, \{c\}, \{b, c\}\}$ which are nano closed sets and $Y = \{b, c\}$, $V/R = \{\{a\}, \{b\}, \{c, d\}\}$ *then* $\tau_{R'}(Y) = \{V, \phi, \{b\}, \{b, c, d\}, \{c, d\}\}$. *Let* $f(a) = b$, $f(b) = a$, $f(c) = d$, $f(d) = c$, is nano αg continuous but not $\mathcal{N}s\hat{g}$-continuous.

Example 3.7 *Let* $U = \{a, b, c, d\}$, $X = \{a, b\}$ *with* $U/R = \{\{a\}, \{c\}, \{b, d\}\}$, *then* $\tau_R(X) = \{U, \phi, \{a\},$ $\{a, b, d\},\{b, d\}\}$ and $V = \{a, b, c, d\}$, $Y = \{a, b\}$ with $V/R = \{\{a, d\},\{b\},\{c\}\}$, then $\tau_R(Y) = \{V, \phi, \{b\},\{a, b, d\},\{a, d\}\}$. Let $f(a) = b$, $f(b) = a$, $f(c) = d$, $f(d) = c$, let $A = \{a, b, c\}$ is $\mathcal{N}gsp$-continuous but not $\mathcal{N}s\hat{g}$ continuous.

Theorem 3.8 A function $f: (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ is $\mathcal{N}s\hat{g}$-continuous if and only if $f^{-1}(K)$ is $\mathcal{N}s\hat{g}$-open in $(U, \tau_R(X))$ for every nano-open set K in $(V, \tau_{R'}(Y))$

Proof Let $f: (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ be $\mathcal{N}s\hat{g}$-continuous and K be an nano-open set in $(V, \tau_{R'}(Y))$. Then K^c is nano-closed in $(V, \tau_{R'}(Y))$ and since f is $\mathcal{N}s\hat{g}$-continuous, $f^{-1}(K)^c$ is $\mathcal{N}s\hat{g}$-closed in $(U, \tau_R(X))$. But $f^{-1}(K^c) = (f^{-1}(K))^c$ and so $f^{-1}(K)$ is $\mathcal{N}s\hat{g}$-open in $(U, \tau_R(X))$.

Conversely, assume that $f^{-1}(K)$ is $\mathcal{N}s\hat{g}$-open in $(U, \tau_R(X))$ for each nano-open set K in $(V, \tau_{R'}(Y))$. Let F be a nano-closed set in $(V, \tau_{R'}(Y))$. Then F^c is nano-open in $(V, \tau_{R'}(Y))$ and by assumption, $f^{-1}(F^c)$ is $\mathcal{N}s\hat{g}$-open in $(U, \tau_R(X))$. Since $f^{-1}(F^c) = (f^{-1}(F))^c$, we have $f^{-1}(F)$ is $\mathcal{N}s\hat{g}$-closed in $(U, \tau_R(X))$ and so f is $\mathcal{N}s\hat{g}$-continuous.

Theorem 3.9 If $f: (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ is $\mathcal{N}s\hat{g}$-continuous and $g: (V, \tau_{R'}(Y)) \to (W, \tau_{R''}(Z))$ is nano-continuous, then their composition $g \circ f: (U, \tau_R(X)) \to (W, \tau_{R''}(Z))$ is $\mathcal{N}s\hat{g}$-continuous.

Proof Let F be any nano-closed set in $(W, \tau_{R''}(Z))$. Since $g: (V, \tau_{R'}(Y)) \to (W, \tau_{R''}(Z))$ is nano-continuous, $g^{-1}(F)$ is nano-closed in $(V, \tau_{R'}(Y))$. Since $f: (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ is $\mathcal{N}s\hat{g}$-continuous, $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is $\mathcal{N}s\hat{g}$-closed in $(U, \tau_R(X))$ and so $g \circ f$ is $\mathcal{N}s\hat{g}$-continuous.

Theorem 3.10 A function $\mathrm{f}: (\mathrm{U}, \tau_{\mathrm{R}}(\mathrm{X})) \rightarrow (\mathrm{V}, \tau_{\mathrm{R}'}(\mathrm{Y}))$ is $\mathcal{N}$s$\hat{\mathrm{g}}$-continuous if and only if $\mathrm{f}(\mathcal{N}\mathrm{s}\hat{\mathrm{g}}\mathrm{cl}(\mathrm{A}) \subseteq \mathcal{N}\mathrm{clf}(\mathrm{A})$ or every subset A of $(\mathrm{U}, \tau_{\mathrm{R}}(\mathrm{X}))$.

Proof: Let $f: (U, \tau_R(X)) \rightarrow (V, \tau_{R'}(Y))$ be $\mathcal{N}s\hat{g}$-continuous and $A \subseteq U$,Then $f(A) \subseteq V$. Hence $\mathcal{N}clf(A)$ is nano closed in V. Since $f^{-1}\mathcal{N}clf(A)$ is also $\mathcal{N}s\hat{g}$-closed in $(U, \tau_R(X))$. Since $f(A) \subseteq \mathcal{N}clf(A)$ we have $A \subseteq f^{-1}\mathcal{N}clf(A)$. Thus $f^{-1}(\mathcal{N}clf(A))$ is a $\mathcal{N}s\hat{g}$-closed set containing A. But $\mathcal{N}s\hat{g}cl(A) \subseteq f^{-1}(\mathcal{N}clf(A)$ which implies $f(\mathcal{N}s\hat{g}cl(A)) \subseteq f^{-1}(\mathcal{N}clf(A)$ which implies $f(\mathcal{N}s\hat{g}cl(A)) \subseteq \mathcal{N}clf(A)$.

Conversely, Let $f(\mathcal{N}s\hat{g}cl(A)) \subseteq \mathcal{N}clf(A)$ for every subset A of $(U, \tau_R(X))$. Let F be a nano closed set in $(V, \sigma_R(Y))$. Now $f^{-1}(F) \subseteq U$. Hence $f(\mathcal{N}s\hat{g}cl(f^{-1}(F)) \subseteq f^{-1}(\mathcal{N}cl(F)) = f^{-1}(F)$ as F is nano closed. Hence $f(\mathcal{N}s\hat{g}cl(f^{-1}(F)) \subseteq f^{-1}(F) \subseteq \mathcal{N}sgcl(f^{-1}(F))$. Thus we have $\mathcal{N}sgcl(f^{-1}(F)) = (f^{-1}(F))$, which implies that $f^{-1}(F)$ is $\mathcal{N}s\hat{g}$-closed in U for every nano closed set F in V. That is f is $\mathcal{N}s\hat{g}$ continuous.

Remark 3.11 Let $\mathrm{f}: (\mathrm{U}, \tau_{\mathrm{R}}(\mathrm{X})) \rightarrow (\mathrm{V}, \tau_{\mathrm{R}'}(\mathrm{Y}))$ be $\mathcal{N}$s$\hat{\mathrm{g}}$-continuous then $\mathrm{f}(\mathcal{N}\mathrm{s}\hat{\mathrm{g}}\ \mathrm{cl}(\mathrm{A}))$ is not necessarily equal to $\mathcal{N}\mathrm{clf}(\mathrm{A})$ where $\mathrm{A} \subseteq \mathrm{U}$.

Theorem 3.12 Let $\mathrm{f}: (\mathrm{U}, \tau_{\mathrm{R}}(\mathrm{X})) \rightarrow (\mathrm{V}, \tau_{\mathrm{R}'}(\mathrm{Y}))$ is $\mathcal{N}$s$\hat{\mathrm{g}}$-continuous if and only if $\mathcal{N}\mathrm{s}\hat{\mathrm{g}}\mathrm{cl}(\mathrm{f}^{-1}(\mathrm{B})) \subseteq \mathrm{f}^{-1}(\mathcal{N}\mathrm{cl}(\mathrm{B}))$ for every subset B of V.

Proof: Let $B \subset V$ and $f: (U, \tau_R(X)) \rightarrow (V, \tau_{R'}(Y))$ be nano $s\hat{g}$-continuous. Then $\mathcal{N}cl(B)$ is nano closed in $(V, \tau_{R'}(Y))$ and hence $f^{-1}(\mathcal{N}cl(B))$ is nano $s\hat{g}$-closed in $\left(U, \tau_R(X)\right)$. Therefore $\mathcal{N}s\hat{g}cl\left(f^{-1}(B)\right) = f^{-1}\left(\mathcal{N}cl(B)\right)$. Since $B \subseteq (\mathcal{N}cl(B))$, then $f^{-1}B) \subseteq f^{-1}\mathcal{N}cl(B)$ (i.e) $\mathcal{N}s\hat{g}cl(f^{-1}(B)) \subseteq \mathcal{N}s\hat{g}cl(f^{-1}cl(B)) = f^{-1}(B)$. Hence $(\mathcal{N}s\hat{g}cl(f^{-1}(B)) \subseteq f^{-1}\mathcal{N}cl(B))$.

Conversely, let $\mathcal{N}s\hat{g}cl(f^{-1}(B)) \subseteq f^{-1}(\mathcal{N}cl(B))$ for every subset $B \subseteq V$. Now let B be nano closed in $(V, \tau_{R'}(Y))$, then $\mathcal{N}cl(B) = B$. Given $\mathcal{N}s\hat{g}cl(f^{-1}(B)) \subseteq f^{-1}(\mathcal{N}cl(B))$. Hence $\mathcal{N}s\hat{g}cl(f^{-1}(B)) \subseteq (f^{-1}(B))$. Thus $f^{-1}(B)$ is nano $s\hat{g}$-closed set in $(U, \tau_R(X))$ for every nano closed set B in $(V, \tau_{R'}(Y))$. Hence $f: (U, \tau_R(X)) \rightarrow (V, \tau_{R'}(Y))$ is $\mathcal{N}s\hat{g}$-continuous.

Theorem 3.13 A function $\mathrm{f}: (\mathrm{U}, \tau_{\mathrm{R}}(\mathrm{X})) \rightarrow (\mathrm{V}, \tau_{\mathrm{R}'}(\mathrm{Y}))$ is $\mathcal{N}$s$\hat{\mathrm{g}}$-continuous and only if $\mathrm{f}^{-1}(\mathcal{N}\mathrm{int}(\mathrm{B})) \subseteq \mathcal{N}\mathrm{s}\hat{\mathrm{g}}\mathrm{int}(\mathrm{f}^{-1}(\mathrm{B}))$ for every subset B of $(\mathrm{V}, \tau_{\mathrm{R}'}(\mathrm{Y}))$.

Proof: Let $f: (U, \tau_R(X)) \rightarrow (V, \tau_{R'}(Y))$ is $\mathcal{N}s\hat{g}$-continuous and $B \subseteq V$. Then $\mathcal{N}int(B)$ is nano open in V. Now $f^{-1}(\mathcal{N}int(B))$ is nano $s\hat{g}$-open in $(U, \tau_R(X))$ i.e $\mathcal{N}s\hat{g}int(f^{-1}(\mathcal{N}int(B)) = f^{-1}(\mathcal{N}int(B))$. Also for $B \subseteq V$, $\mathcal{N}int(B) \subseteq B$ always, Then $f^{-1}(\mathcal{N}int(B)) \subseteq f^{-1}(B)$.

Therefore $\mathcal{N}s\hat{g}int(f^{-1}(\mathcal{N}int(B))) \subseteq \mathcal{N}s\hat{g}$ $\mathrm{int}(f^{-1}(B))$ conversely, Let $f^{-1}(\mathcal{N}int(B)) \subseteq \mathcal{N}s\hat{g}int(f^{-1}(B))$ for every subset B of V. Let B be nano open in V and hence $\mathcal{N}int(B) = B$. Given $f^{-1}(\mathcal{N}int(B)) \subseteq \mathcal{N}s\hat{g}int(f^{-1}(B))$ i.e ,$f^{-1}(B) \subset \mathcal{N}s\hat{g}int(f^{-1}(B))$. Also $\mathcal{N}s\hat{g}int(f^{-1}(B)) \subseteq f^{-1}(B)$. Hence $f^{-1}(B) = \mathcal{N}s\hat{g}int(f^{-1}(B))$ which implies that $f^{-1}(B)$ is implies that $f^{-1}(B)$ is $\mathcal{N}s\hat{g}$ open in U for every nano-open set B of V. Therefore $f: (U, \tau_R(X)) \rightarrow (V, \tau_{R'}(Y))$ is $\mathcal{N}s\hat{g}$-continuous.

4 Weakly $\mathcal{N}s\hat{g}$-continuous function

Definition 4.1 Let $(\mathrm{U}, \tau_{\mathrm{R}}(\mathrm{X}))$ be a nano topological space. Let x be a point of U and G be a subset of U Then G is called an $\mathcal{N}$s$\hat{\mathrm{g}}$-neighborhood of x (briefly, $\mathcal{N}$s$\hat{\mathrm{g}}$-nbhd of x) in U if there exists an $\mathcal{N}$s$\hat{\mathrm{g}}$-open set L of U such that $\mathrm{x} \in \mathrm{L} \subseteq \mathrm{G}$.

Theorem 4.2 Let A be a subset of $(\mathrm{U}, \tau_{\mathrm{R}}(\mathrm{X}))$. Then $\mathrm{x} \in \mathcal{N}\mathrm{s}\hat{\mathrm{g}}\text{-}\mathrm{cl}(\mathrm{A})$ if and only if for any $\mathcal{N}$s$\hat{\mathrm{g}}$-nbhd $\mathrm{G_x}$ of x in $(\mathrm{U}, \tau_{\mathrm{R}}(\mathrm{X}))$, $\mathrm{A} \cap \mathrm{G_x} \neq \emptyset$.

Proof Necessity. Assume $x \in \mathcal{N}s\hat{g}\text{-}cl(A)$. Suppose that there is an $\mathcal{N}s\hat{g}$-nbhd G of the point x in $(U, \tau_R(X))$ such that $G \cap A = \emptyset$. Since G is $\mathcal{N}s\hat{g}$-nbhd of x in $(U, \tau_R(X))$, by Definition, there exists a $\mathcal{N}s\hat{g}$-open set L_x such that $x \in L_x \subseteq G$. Therefore, we have $L_x \cap A = \emptyset$ and so $A \subseteq (L_x)^c$. Since $(L_x)^c$ is a $\mathcal{N}s\hat{g}$-closed set containing A, we have $\mathcal{N}s\hat{g}cl(A) \subseteq (L_x)^c$ and therefore $x \not\in \mathcal{N}s\hat{g}\text{-}cl(A)$, which is a contradiction.

Sufficiency. Assume for each $\mathcal{N}s\hat{g}$-nbhd G_x of x in $(U, \tau_R(X))$, $A \cap G_x \neq \emptyset$. Suppose that $x \not\in \mathcal{N}s\hat{g}\text{-}cl(A)$. Then there exists a $\mathcal{N}s\hat{g}$-closed set F of $(U, \tau_R(X))$ such that $A \subseteq F$ and $x \not\in F$. Thus $x \in F^c$ and F^c is $\mathcal{N}s\hat{g}$-open in $(U, \tau_R(X))$ and hence F^c is an $\mathcal{N}s\hat{g}$-nbhd of x in $(U, \tau_R(X))$. But $A \cap F^c = \emptyset$, which is a contradiction.

Definition 4.3 A function $f: (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ is said to be weakly $\mathcal{N}s\hat{g}$-continuous if for each $x \in U$ and each $Y \in V$ containing $f(x)$, there exists $X \in \mathcal{N}s\hat{g}$-open containing x such that $f(X) \subset Ncl(Y)$.

Remark 4.4 Every weakly continuous function is weakly $\mathcal{N}s\hat{g}$-continuous-continuous, but the converse is not true.

Lemma 4.5 For a function $f: (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ the following are equivalent

1. f is weakly $\mathcal{N}s\hat{g}$-continuous function.
2. $f: (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ is weakly continuous.
3. $f^{-1}(Y) \subset \mathcal{N}s\hat{g}int(f^{-1}(Ncl(Y)))$ for every $Y \in V$. s
4. $\mathcal{N}s\hat{g}cl(f^{-1}(Y)) \subset f^{-1}(Ncl(Y))$ for every $Y \in V$.

Lemma 4.6 Let A be a subset of a space $(U, \tau_R(X))$ then the following hold

1. $\mathcal{N}s\hat{g}cl(A) = A \cup \mathcal{N}s\hat{g}cl(\mathcal{N}s\hat{g}int(\mathcal{N}s\hat{g}cl(A)))$
2. $\mathcal{N}s\hat{g}int(A) = A \cap \mathcal{N}s\hat{g}int(\mathcal{N}s\hat{g}cl(\mathcal{N}s\hat{g}int(A)))$.

The following theorem is very useful in the sequel.

Theorem 4.7 For a function $f: (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$, the following are equivalent

1. f is weakly $\mathcal{N}s\hat{g}$-continuous
2. $f^{-1}(Y) \subset \mathcal{N}s\hat{g}(f^{-1}(cl(Y)))$ for every $Y \in V$.

Theorem 4.8 If $f: (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ is weakly $\mathcal{N}s\hat{g}$-continuous, and $g: (V, \tau_{R'}(Y)) \to (W, \tau_{R''}(Z))$ is nano continuous, then the composition $(g \circ f): (U, \tau_R(X)) \to (W, \tau_{R''}(Z))$ is weakly $\mathcal{N}s\hat{g}$-continuous.

Proof Since f is weakly $\mathcal{N}s\hat{g}$-continuous, by Lemma 5.3 $f: (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ is weakly continuous and hence $(g \circ f): (U, \tau_R(X)) \to (W, \tau_{R''}(Z))$ is weakly continuous. Therefore, $(g \circ f): (U, \tau_R(X)) \to (W, \tau_{R''}(Z))$ is weakly $\mathcal{N}s\hat{g}$-continuous, by Lemma 4.5

Theorem 4.9 If $f: (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ be an nano open continuous surjection. Then a function $g: (V, \tau_{R'}(Y)) \to (W, \tau_{R''}(Z))$ is weakly $\mathcal{N}s\hat{g}$-continuous, if and only if $(g \circ f): (U, \tau_R(X)) \to (W, \tau_{R''}(Z))$ is weakly $\mathcal{N}s\hat{g}$-continuous.

Proof Necesseity. Suppose that g is weakly $\mathcal{N}s\hat{g}$-continuous. Let K be any nano open set of $(W, \tau_{R''}(Z))$. $g^{-1}(K) \subset \mathcal{N}s\hat{g}(g^{-1}(cl(K)))$. Since f is nano open and nano continuous, we have $f^{-1}(\mathcal{N}s\hat{g}(B)) \subset \mathcal{N}s\hat{g}(f^{-1}(B))$ for every subset B of V. Therefore, we obtain $(g \circ f)^{-1}(K) \subset \mathcal{N}s\hat{g}((g \circ f)^{-1}(cl(K)))$. It is clear that $(g \circ f)$ is weakly $\mathcal{N}s\hat{g}$-continuous.

Sufficiency. Suppose that $(g \circ f)$ is weakly $\mathcal{N}s\hat{g}$-continuous. Let K be any nano open set of $(W, \tau_{R''}(Z))$. $(g \circ f)^{-1}(K) \subset \mathcal{N}s\hat{g}(g \circ f)^{-1}(cl(K))$. Since f is nano open and nano continuous, we have $f(\mathcal{N}s\hat{g}(A)) \subset$

$\mathcal{N}s\hat{g}(f(A))$ for every subset A of U. Moreover, since f is surjective, we obtain $g^{-1}(K) \subset \mathcal{N}s\hat{g}(g^{-1}(cl(K)))$. It is clear that g is weakly $\mathcal{N}s\hat{g}$-continuous.

5 Almost $\mathcal{N}s\hat{g}$-continuous function

Definition 5.1 A mapping $f: (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ is said to be almost $\mathcal{N}s\hat{g}$-continuous if the inverse image of every nano regular open set of V is an $\mathcal{N}s\hat{g}$-open in U.

Theorem 5.2 Let $f: (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ be a mapping. Then the following conditions are equivalent

1. f is almost $\mathcal{N}s\hat{g}$-continuous
2. for each $x \in U$ and for each nano regular-open set O in V such that $F(x) \in O$, there exists an $\mathcal{N}s\hat{g}$-open set A in U such that $x \in A$ and $f(A) \subset O$.
3. the inverse image of every nano regular-closed set of V is a $\mathcal{N}s\hat{g}$-closed set of U.

Proof (1) → (2). Let O be nano regular open in V and $f(x) \in O$. Then $x \in f^{-1}(O)$ and $f^{-1}(O)$ is an $\mathcal{N}s\hat{g}$-open set. Let $A = f^{-1}(O)$. Thus $x \in A$ and $f(A) \subset O$.

(2) → (3). Let O be nano-regular open in V and $x \in f^{-1}(O)$. Then $f(x) \in O$. By (2) there is an $\mathcal{N}s\hat{g}$-open set A_x, in U such that $x \in A_x$ and $f(A_x) \subset O$. And so $x \in A_x \subset f^{-1}(O)$. Therefore $f^{-1}(O)$ is union of $\mathcal{N}s\hat{g}$-open sets is an $\mathcal{N}s\hat{g}$-open set in U. Hence f is almost $\mathcal{N}s\hat{g}$-continuous.

Definition 5.3 . A space U is nano semi-regular if for each point x of U and each nano-open set K containing x, there is an nano-open set L such that $x \in L \subset \text{Nint}(\text{Ncl}(L) \subset K$.

Theorem 5.4 An almost $\mathcal{N}s\hat{g}$-continuous mapping $f: (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ is $\mathcal{N}s\hat{g}$-continuous if V is nano semi-regular.

Proof Let $x \in U$ and let A be an nano-open set containing $f(x)$. Since V is nano-semi regular there is an nano-open set M in V such that $f(x) \in M \subset Nint(Ncl(M)) \subset A$. Since $Nint(Ncl(M))$ is nano-regular open in V and f is almost $\mathcal{N}s\hat{g}$-continuous, there is a set $K \in \mathcal{N}s\hat{g}(U)$ containing x such that $f(x) \in f(K) \subset Nint(Ncl(M))$. Thus K is an $\mathcal{N}s\hat{g}$-open set containing x and $f(K) \subset A$. Hence f is $\mathcal{N}s\hat{g}$-continuous.

Remark 5.5 Composition of two almost $\mathcal{N}s\hat{g}$ continuous mappings need not be almost $\mathcal{N}s\hat{g}$-continuous.

Lemma 5.6 *If $f: (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ is an nano-open and nano-continuous mapping then $f^{-1}(B) \in \mathcal{N}s\hat{g}(U)$ for every $B \in \mathcal{N}s\hat{g}(V)$.*

Theorem 5.7 , If $f: (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ is nano-open and continuous, $g: (V, \tau_{R'}(Y)) \to (W, \tau_{R''}(Z))$ is almost $\mathcal{N}s\hat{g}$-continuous, then $g \circ f: (U, \tau_R(X)) \to (W, \tau_{R''}(Z))$ is almost $\mathcal{N}s\hat{g}$-continuous.

Proof Suppose K is nano-regular open set in W. Then $g^{-1}(K)$ is an $\mathcal{N}s\hat{g}$-open set in V because g is almost $\mathcal{N}s\hat{g}$-continuous. And so f being nano-open and nano-continuous by above Lemma, $f^{-1}(g^{-1}K)) \in \mathcal{N}s\hat{g}(U)$. Consequently $g \circ f: (U, \tau_R(X)) \to (W, \tau_{R''}(Z))$ is almost $\mathcal{N}s\hat{g}$-continuous.

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