On Ng^α-Homeomorphisms in Nano Topological Spaces

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Abstract: In this paper we introduce and study new class of homeomorphisms called Ng^α-homeomorphisms (nano g^α-homeomorphismss) in nano topological spaces and certain properties are investigated. We also investigate the concept of nano g^α- homeomorphisms and discussed their relationships with other forms of nano sets. Further, we show that the set of all Ngα-homeomorphisms form a group under the operation composition of mappings.

Keywords: Nano Closed map, Nano Open map, Ng^α- irresolute, Ng^α-Closed map, Ng^α- Open map, Ng^α-homeomorphisms.

1. Introduction

In 1933, Maki et.al [9] introduced the concept of g-homeomorphisms and gc-homeomorphisms in topological spaces. The notion of nano topology was introduced by LellisThivagar [7,8] which was defined in terms of approximations and boundary region of a subset of an universe using an equivalence relation on it. He also established and analyzed the nano forms of weakly open sets such as nano α-open sets, nano semi-open sets and nano pre-open sets. Bhuvaneswari and Mythili Gnanapriya [1] introduced and studied the concepts of Nano generalized-closed sets and Nano generalized α-closed sets.

The structure of this manuscript is as follows. In section 2, we recall some fundamental definitions and results which are useful to prove our main results. In section 3 we define and study the notion of Ng ^α-homeomorphisms in nano topological spaces.

2. Preliminaries

Definition 2.1 [7] Let U be a non-empty finite set of objects called the universe R be an equivalence relation on U named as the indiscerniblity relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$.

1. The Lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$. That is, $L_R(X) = \{\bigcup_{x\in U} \{R(x): R(x) \subseteq X\}\}$, where $R(x)$ denotes the equivalence class determined by x.
2. The Upper approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X) = \{\bigcup_{x\in U} \{R(x): R(x) \cap X \neq \phi\}\}$
3. The Boundary region of X with respect to R is the set of all objects which can be classified as neither as X nor as not X with respect to R and it is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X)$-$L_R(X)$.

Definition 2.2 [7] If (U, R) is an approximation space and $X, Y \subseteq U$, then

1. $L_R(X) \subseteq X \subseteq U_R(X)$.
2. $L_R(\phi) = U_R(\phi) = \phi$ and $L_R(U) = U_R(U) = U$.
3. $U_R(X \cup Y) = U_R(X) \cup U_R(Y)$
4. $U_R(X \cap Y) \subseteq U_R(X) \cap U_R(Y)$
5. $U_R(X \cup Y) \supseteq U_R(X) \cup U_R(Y)$
6. $U_R(X \cap Y) = U_R(X) \cap U_R(Y)$
7. $L_R(X) \subseteq L_R(Y)$ and $U_R(X) \subseteq U_R(Y)$ whenever $X \subseteq Y$

8. $U_R(X^c) = [L_R(X)]^c$ and $L_R(X^c) = [U_R(X)]^c$
9. $U_R(U_R(X)) = L_R(U_R(X)) = U_R(X)$
10. $L_R(L_R(X)) = U_R(L_R(X)) = L_R(X)$

Definition 2.3 [7] Let U be the universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$. $\tau_R(X)$ satisfies the following axioms:

1. U and $\phi \in \tau_R(X)$
2. The union of elements of any subcollection of $\tau_R(X)$ is in $\tau_R(X)$.
3. The intersection of the elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$

That is, $\tau_R(X)$ forms a topology on U is called the nano topology on U with respect to X. We call $\{U, \tau_R(X)\}$ is called the nano topological space.

Definition 2.4 [7] If $(U,\tau_R(X))$ is a nano toplogical space with respect to X where $X \subseteq U$ and if $A \subseteq U$. Then

• The nano interior of the set A is defined as the union of all nano open subsets contained in A and is denoted by Nint(A). Nint(A) is the largest nano open subset of A.

• The nano closure of the set A is defined as the intersection of all nano closed sets containing A and is denoted by Ncl(A). Ncl(A) is the smallest nano closed set containing A.

Definition 2.5 Let $(U, \tau_R(X))$ be a nano topological space and $A \subseteq U$. Then A is said to be

1. Nano g-closed [1], if Ncl(A)⊆V whenever A⊆V and V is nano-open in U.
2. Nano gα-closed[2], if Nαcl(A)⊆V whenever A⊆V and V is nano α-open in U.
3. Nano αg-closed [2], if Nαcl(A)⊆V whenever A⊆V and V is nano-open in U.
4. Nano gp-closed [4], if Npcl(A)⊆V whenever A⊆V and V is nano-open in U.
5. Nano gpr-closed [10], if Npcl(A)⊆V whenever A⊆V and V is nano regular open in U.

3. Ng^α-Homeomorphisms in Nano Topological Spaces

In this section, we define and study the concept of Ng ^α -homeomorphisms (briefly, Nano g ^α -homeomorphisms) sets in nano topological spaces and obtain some of its properties.

Definition 3.1 A function f : $(U,\tau_R(X)) \rightarrow (V,\tau_R(Y))$ is said to be Ng^α-homeomorphisms if

• f is one to one & onto

• f is Ng^α-continuous

• f is Ng^α-open

Theorem 3.2 Let f : $(U,\tau_R(X)) \rightarrow (V,\tau_R(Y))$ be an one to one onto mapping. Then f is Ng^α-homeomorphisms if and only if f is Ng^α-closed and Ng^α-continuous.

Proof: Let f:$(U,\tau_R(X)) \rightarrow (V,\tau_R(Y))$ be a Ng^α-homeomorphisms. Then f is Ng^α-continuous. Let A be an arbitrary nano closed set in $(U,\tau_R(X))$. Then U-A is nano open. Since f is Ng^α-open, f(U-A) is Ng^α-open in $(V,\tau_R(Y))$. That is, V-f(A) is Ng^α-open in $(V,\tau_R(Y))$ for every nano closed set A in $(U,\tau_R(X))$ implies that f(A) is Ng^α-Closed in $(V,\tau_R(Y)$. Hence f : $(U,\tau_R(X)) \rightarrow (V,\tau_R(Y))$ is Ng^α-closed. Conversely, let f be Ng^α-closed and Ng^α-continuous function. Let G be a nano open set in $(U,\tau_R(X))$. Then U-G is nano closed in $(U,\tau_R(X))$. Since f is Ng^α-closed, f(U-G) is Ng^α-closed in $(V,\tau_R(Y))$. That is, f(U-G)=V-f(G) is Ng^α-closed in $(V,\tau_R(Y))$. Hence f(G) is Ng^α-open in $(V,\tau_R(Y))$ for every nano open set G in $(U,\tau_R(X))$. Thus f is One to one and Onto and hence f:$(U,\tau_R(X)) \rightarrow (V,\tau_R(Y))$ is Ng^α-homeomorphisms.

Theorem 3.3 A one to one map f from $(U,\tau_R(X))$ onto $(V,\tau_R(Y))$ is a Ng^α-homeomorphisms if and only if f(Ngα-Cl(A)=NCl(f(A)) for every subset A of $(U,\tau_R(X))$.

Proof: If f:(U,τ_R(X)) → (V,τ_R(Y)) is Ngα-homeomorphisms, then f is Ng^α-continuous and Ng^α-closed. If A ⊆ U, it follows that f(Ng^αCl(A))⊆NCl(f(A)). Since f is Ng^α-continuous. Since Ng^αCl(A) is nano closed in (U,τ_R(X)) and f is Ng ^α -closed function, f(Ng ^α Cl(A)) is Ng ^α -closed in (V, τ_R (Y)). Also Ng ^α Cl(f(Ng ^α Cl(A)))=f(Ng ^α Cl(f(A))). Since A ⊆ Ng ^α Cl(A), f(A) ⊆ f(Ng ^α Cl(A)) and hence NCl(f(A))⊆NCl(f(Ng^αCl(A)))=f(Ng^αCl(A)). Thus NCl(f(A))⊆f(Ng^αCl(A)). Therefore, f(Ng^αCl(A)=NCl(f(A)) if f is Ng^α-homeomorphisms. Conversely, if f(Ng^αCl(A)=NCl(f(A)) for every subset A of (U,τ_R(X)), then f is Ng^α-continuous. If A is nano closed in (U,τ_R(X)), then A is Ng^α-closed in (U,τ_R(X)). Then Ng^αCl(A))=A which implies f(Ng^αCl(A))=f(A). Hence, by the given hypothesis, it follows that NCl(f(A))=f(A). Thus f(A) is nano closed in (V,τ_R(Y)) and hence Ng^α-closed in (V,τ_R(Y)) for every nano closed set A in (U,τ_R(X)). That is, f is Ng^α-closed. Thus f:(U,τ_R(X)) → (V,τ_R(Y)) is Ng^α-homeomorphisms.

Example 3.4 Let U = {a,b,c,d} with X = {a,b} and U/R = {{a},{b,c},{d}}. Then the nano closed sets are τ_R^C(X)={U,ϕ,{a,c,d},{a},{a,b}}. Also let V = {x,y,z,w} with Y = {x,y} and V/R = {{x},{z},{y,w}}. Then τ_R^C(X)={V,ϕ,{y,z,w},{z},{x,z}}. Define f:(U,τ_R(X)) → (V,τ_R(Y)) as f(a) = x, f(b) = y, f(c) = z, f(d) = w. Then f is bijective , Ng ^α -continuous and Ng ^α -open and so the function f:(U, τ_R (X)) → (V, τ_R (Y)) is Ng ^α - homeomorphisms.

Theorem 3.5 If a function f:(U,τ_R(X)) → (V,τ_R(Y)) is Nano(Nα-homeomorphisms respectively) homeomorphisms, then it is Ng^α-homeomorphisms but not conversely.

Proof: Let the function f:(U,τ_R(X)) → (V,τ_R(Y)) is Nano(Nα-homeomorphisms) homeomorphisms, by the definition, f is bijective, Nano(Nα-continuous) continuous and Nano(Nα-closed) closed. Hence f:(U,τ_R(X)) → (V,τ_R(Y)) is Ng^α-continuous. Since f:(U,τ_R(X)) → (V,τ_R(Y)) is Nano(Nα-closed) closed, the image of every Nano(Nα-closed) closed set in (U,τ_R(X)) is Nano(Nα-closed) closed in (V,τ_R(Y)) and hence Ng^α-closed in (V,τ_R(Y)). Every Nano(Nα-closed) closed set is Ngα-closed. Thus the function f:(U,τ_R(X)) → (V,τ_R(Y)) is Ng^α-closed. Therefore, every Nano(Nα-homeomorphisms) homeomorphisms f:(U,τ_R(X)) → (V,τ_R(Y)) is Ng^α-homeomorphisms. The converse of the above theorem need not be true as seen from the following example.

Example 3.6 From the above example, the function f is Ng^α-homeomorphisms. Now $f^{-1}(v)$ = U,f^{-1}(ϕ)=ϕ, $f^{-1}(y, z, w)$ = {b,c,d},$f^{-1}(z)$ = {c},$f^{-1}(x, z)$ = {a,c}. Hence the inverse image of nano open sets in (V,τ_R(Y)) are not nano open in (U,τ_R(X)) and hence f is not nano continuous function. Then the function f:(U,τ_R(X)) → (V,τ_R(Y)) is not nano homeomorphisms .

Example 3.7 Let U = {a,b,c,d} with X = {b,d} and U/R = {{a},{b},{c,d}}. Then the Nano closed sets are τ_R^C(X)={U,ϕ,{a,c,d},{a},{a,b}} and Ng^α-closed sets are {U,ϕ,{a},{a,b},{a,c},{a,d},{a,c,d},{a,b,c},{a,b,d}}. Also let V = {a,b,c,d} with Y = {a,b} and V/R = {{a},{c},{b,d}}. Then τ_R^C(Y)={V,ϕ,{b,c,d},{c},{a,c}} and Nα-closed sets are {V,ϕ,{b,c,d},{c},{a,c}},{a,b,d}}. Define a f:(U,τ_R(X)) → (V,τ_R(Y)) as f(a) = d, f(b) = c, f(c) = b, f(d) = a. Here f is not Ng^α-homeomorphisms. Since the inverse image of closed set {a} in (V,τ_R(Y)) is {d} which is not Nα-closed in (U,τ_R(X)). However, f is Ng^α-homeomorphisms.

Theorem 3.8 If a function f:(U, τ_R (X)) → (V, τ_R (Y)) is Ng ^α -homeomorphisms , then it is N α g-homeomorphisms(Ngpr-homeomorphisms respectively) but not conversely.

Proof: Let the function f:(U,τ_R(X)) → (V,τ_R(Y)) is Ng^α-homeomorphisms, by the definition, f is bijective, Nαg-continuous(Ngpr-continuous) and Nαg-closed(Ngpr-closed). Hence f:(U,τ_R(X)) → (V,τ_R(Y)) is Ng^α-continuous. Since f:(U,τ_R(X)) → (V,τ_R(Y)) is Nαg-closed(Ngpr-closed), the image of every Nαg-closed(Ngpr-closed) set in (U,τ_R(X)) is Nαg-closed(Ngpr-closed) in (V,τ_R(Y)) and hence Ng^α-closed in (V,τ_R(Y)). Every Nαg-closed(Ngpr-closed) closed set is Ng^α-closed. Thus the function f:(U,τ_R(X)) → (V,τ_R(Y)) is Ng^α-closed. Therefore, every Nαg-homeomorphisms(Ngpr-homeomorphisms) homeomorphisms f:(U,τ_R(X)) → (V,τ_R(Y)) is Ng^α-homeomorphisms. The converse of the above theorem need not be true as seen from the following example.

Example 3.9 Let U = {a,b,c,d} with X = {b,d} ⊆ U and U/R = {{a},{b},{c,d}}. Then the Nano closed sets are τ_R^C(X)={U,ϕ,{a,c,d},{a},{a,b}} and Ng^α-closed sets are {U,ϕ,{a},{a,b},{a,c},{a,d},{a,c,d},{a,b,c},{a,b,d}}. Also let V =

{a,b,c,d} with Y = {a,b} ⊆ U and V/R = {{a},{c},{b,d}}. Then τ_R^C(Y)={V,ϕ,{b,c,d},{c},{a,c}} and Nα-closed sets are {V,ϕ,{b,c,d},{c},{a,c}},{a,b,d}}. Define a function f:(U,τ_R(X)) → (V,τ_R(Y)) as f(a) = d, f(b) = c, f(c) = a, f(d) = b. Then f is bijective, Ng^α-Continuous and Ng^α-open and so the function f:(U,τ_R(X)) → (V,τ_R(Y)) is Ng^α-homeomorphisms. Now $f^{-1}(v) = U$, $f^{-1}(\phi)=\phi$, $f^{-1}(b,c,d)$ = {a,b,d}, $f^{-1}(c)$ = {b}, $f^{-1}(a,c)$ = {b,c}. Hence the inverse image of nano open sets in (V,τ_R(Y)) are not nano open in (U,τ_R(X))and hence f is not Nano continuous . Thus the function f:(U,τ_R(X)) → (V,τ_R(Y)) is not Nano homeomorphisms.

Example 3.10 Let U = {a,b,c,d} with X = {b,d} and U/R = {{a},{b},{c,d}}. Then the Nano closed sets are τ_R^C(X) = {U,ϕ,{a,c,d},{a},{a,b}} and Ng^α-closed sets are {U,ϕ,{a},{a,b},{a,c},{a,d},{a,c,d},{a,b,c},{a,b,d}}. Also let V = {a,b,c,d} with Y = {a,b} ⊆ V and V/R = {{a},{c},{b,d}}. Then τ_R^C(Y)={V,ϕ,{b,c,d},{c},{a,c}} and Nα-closed sets are {V,ϕ,{b,c,d},{c},{a,c}},{a,b,d}}. Define a function f:(U,τ_R(X)) → (V,τ_R(Y)) as c. Here f is not Ng^α-homeomorphisms. Since the inverse image of closed set {b,c,d} in (V,τ_R(Y)) is {b,c,d} which is not Ngpr-closed in (U,τ_R(X)). However, f is Ng^α-homeomorphisms.

Theorem 3.11 A function f:(U,τ_R(X)) → (V,τ_R(Y)) is said to be Ng^α-Continuous Function. Then the following statements are equivalent.

- f is Ng^α-open function
- f is Ng^α-homeomorphisms
- f is Ng^α-closed function.

Proof: (i)⇒(ii) By the definition, The function f:(U,τ_R(X)) → (V,τ_R(Y)) is bijective, Ng^α-cotinuous and Ng^α-open. Hence the function f:(U,τ_R(X)) → (V,τ_R(Y)) is Ng^α-homeomorphisms.

(ii)⇒(iii) By the definition, The function f:(U,τ_R(X)) → (V,τ_R(Y)) is Ng^α-homeomorphisms and hence Ng^α-open. Let A be the Nano closed set in (U,τ_R(X)). Then A^C is Nano open in (U,τ_R(X)).

By assumption, f(A^C) is Ng^α-open in (V,τ_R (Y)). i.e.,f(A^C)=(f(A))^C is Ng^α-open in (V,τR(Y)) and hence f(A) is Ng^α-closed in (V,τ_R (Y))for every Nano closed set A in (U,τ_R (X)). Hence the function f:(U,τ_R (X))→(V,τ_R (Y)) is Ng^α- closed function.

(iii)⇒(i) Let F be a Nano open set in (U,_R(X)). Then F^C is Nano closed set in (U,_R(X)). By the definition, f(F^C) is Ng^α-closed in (V,τ_R (Y)). Now, f(F^C) = $(f(F))^C$ is Ng^α-closed, i.e., f(F) is Ng^α-open in (V,_R(Y)) for every Nano open set F in (U,_R(X)). Hence f:(U,_R(X))_→ (V,_R(Y)) is Ng^α-open function.

The composition of two Ng^α-homeomorphisms need not always be a Ng^α-homeomorphisms as seen from the following example.

Example3.12 Let(U,τ_R (X)),(V,τ_R (Y)) and (W,τ_R (Z)) be three Nano topological spaces and Let U=V=W={a,b,c,d}, then the Nano closed sets are τ_R (X)={U,ϕ,{b,c,d},{d},{a,d}}, τ_R (Y) ={V,ϕ,{a,c,d},{c},{b,c}}and τ_R(Z)={W,ϕ,{b,c,d},{c},{a,c}}. Define two functions f:(U,τ_R (X))→(V,τ_R (Y)) and g:(V,τ_R (Y))→(W,τ_R (Z)) as f(a)=b, f(b)=a, f(c)=d, f(d)=c and g(a)=a, g(b)=c, g(c)=d, g(d)=b.Here the functions f and g are Ng^α-continuous and bijective. Also the image of every nano open set in (U,_R(X)) is Ng^α-open in (V,τ_R (Y)). i.e.,f^{-1}(b,c,d)={a,c,d},f^{-1}(d)={c},f^{-1} (a,d)={a,d}.Thus the function f:(U,τ_R (X))→(V,τ_R (Y)) is Ng-open and thus Ng^α-homeomorphisms.The function g:(V,τ_R (Y))→(W,τ_R (Z)) is also Ng^α-continuous, bijective and Ng^α-open. Hence g is also Ng^α-homeomorphisms. But their composition gof: (U,_R(X))(W,_R(Z)) is not a Ng^α-homeomorphis m because for the nano open set F={a,c} in (W,τ_R (Z)), $(gof)^{-1}$(F)=f^{-1}(g^{-1}({a,c}))=f^{-1}({a,c})={a,b} is not Ng^α-open in (U,_R(X)). Hence the composition gof : (U,_R(X))→(W,_R(Z)) is not a Ng^α-continuous and thus not a Ng^α-homeomorphisms. Thus the composition of two Ng^α-homeomorphismss need not be a Ng^α-homeomorphisms.

4. Conclusion and Future Work

In this paper, introduced the concept of nano topology and nano homeomorphisms in nano topological spaces. Some of its properties have been discussed. Further, Continuity of nano regular, nano normal and applications of nano topological spaces may be studied.

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