Combinatorial Construction of Second Order Rotatable Designs

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Abstract: In this paper, we constructed a new series for the construction of Second Order Rotatable Design using Partially Balanced Incomplete Block Designs (PBIBD).

Keywords: Partially Balanced Incomplete Block Design, Second Order Rotatable Designs, Group Divisible PBIBD

1. Introduction

The primary goal of the research on rotatable designs was to estimate the response and its accuracy. Estimating the difference between responses at two points in the space dimension will also be important. The local slope (change rate) of the response surface should be calculated if the difference occurs at two points close together.

When a design is rotatable, then the estimate of Y gives all information about the responses with the same precision at all points which are equidistant from the coded origin of the design. In other way of saying this is that the contours of variance of estimated response, the variance of the predicted value will be spherical about the design origin. In any experimental design it is not essential that the design should be exact rotatable but the knowledge of how to obtain the design is useful in producing approximate rotatability while perhaps attaining other desirable design characteristics.

The variance of predicted response $\hat{Y}$ of the design Second Order Response Surface Model satisfying the property that at any particular point in a design, is a function of the distance from that design point to the origin, more specifically, all the rotatable designs are spherical or nearly spherical variance function. When c = 3, the $V(\hat{Y})$ can be expressed in the form of a function of ρ^2 as

$$V(\hat{Y}_u) = A\,\rho^4 + B\,\rho^2 + \mathbf{C} \qquad (1)$$

where $A = \frac{\sigma^2}{N\Delta}\left[\frac{\Delta-\lambda_2^2}{\lambda_4(c-1)}\right]$; $B = \frac{\sigma^2}{N\Delta}\left[\frac{\Delta-2\lambda_2^2}{\lambda_2}\right]$; $\mathbf{C} = \frac{\sigma^2}{N\Delta}\left[\Delta+v\lambda_2^2\right]$

and $\Delta = [\,\lambda_4\,(c+v-1) - v\lambda_2^2]$.

2. Construction Of New Series Of Second Order Rotatable Designs

In this section, the constructions are illustrated with suitable examples.

Method 2.1: Consider a Group Divisible PBIBD with parameters v = mn, b, r, k, λ_1, λ_2, n_1 and n_2. Identify the first and second associates for each treatment. Construct a design of order v x v corresponds to each pair of treatments. Place $\pm\,\alpha$, if the pair of treatments belongs to the first associate class and place $\pm\,\beta$, if the pair of treatments belongs to the second associate class, otherwise put '0' and choosing appropriate fraction of factorials for v factors, with levels ±1 (let 2^{k1} is that the suitable fraction of 2^v). Complete the design by taking n_0 central points if necessary, the unknown levels 'α' and 'β' can be chosen so that they satisfy the rotatable condition is $\Sigma\, x^4_{ui} = 3.\Sigma\, x^2_{ui}\, x^2_{uj}$. The resulting design 'D' is a v-dimensional SORD with five levels.

Theorem 2.1: A new series of SORD with five levels can be obtained based on the group divisible PBIBD with two – associate pairs of factors, the parameters v = mn, b, r, k, λ_1, λ_2, n_1 and n_2.

Proof: Consider the parameters v= mn, b, r, k, λ_1, λ_2 and n_1 and n_2 of group divisible PBIBD(2) and assume that T_i, T_j and T_k be any 3 factors with pairs (T_i, T_j) being first associates and the pair (T_i, T_k) being second associates of the original group divisible design. Place $\pm\alpha$, if the pair of treatments belong to first associate class and Place $\pm\beta$ if the pair of treatments belong to the second associates otherwise put '0'. Complete the design by taking n_0 central points if necessary, the unknown levels 'α' and 'β' can be chosen so that they satisfy the rotatable condition is $\Sigma x^4_{ui} = 3.\Sigma x^2_{ui} x^2_{uj}$.

For a group divisible PBIBD design with two association classes. Let S_1 be the set of pairs of treatment which occur to get in $\lambda_1$blocks. The number of pairs in $N_1$is $vn_1/2$. Let the remaining pairs of treatments belong to S_2 where each at the pair will occur together in λ_2 blocks and number of such pair is $vn_2/2$, We shall call two PBIBD with similar association a scheme if the sets S_1 and S_2 remain unaltered but the values of λ`s may be different. Now, if we take the incident matrix of another PBIB design similar to the first one with values of λ as λ_1^2 and λ_2^2. replacing 1 by β, we shall get another set of N_2 points by multiplying with suitable unaffected set of combinations. The totality of (N_1+N_2) design points will satisfy the following conditions;

$\Sigma x^4_{ui} = r_1 2^{k1}\alpha^4 + r_2 2^{k2}\beta^4 = C(N_1+N_2)\lambda_4$ = constant

$\Sigma x^2_{ui} x^2_{uj} = \lambda`_1 2^{k1}\alpha^4 + \lambda_1^2 2^{k2}\beta^4$ for $(i,j) \in S_1$

$\Sigma x^2_{ui} x^2_{uj} = \lambda`_2 2^{k1}\alpha^4 + \lambda_2^2 2^{k2}\beta^4$ for $(i,j) \in S_2$

Where, λ_1^2 and λ_2^2, are the parameters of second PBIB design. Now, if α and β are chosen such that

$$\lambda`_1 2^{k1}\alpha^4 + \lambda_1^2 2^{k2}\beta^4 = \lambda`_2 2^{k1}\alpha^4 + \lambda_2^2 2^{k2}\beta^4 \quad (2)$$

then we get $\Sigma x^2_{ui} x^2_{uj}$ = constant. The unknown levels 'α' and 'β' can be chosen so that they satisfy the rotatable condition $\Sigma x^4_{ui} = 3.\Sigma x^2_{ui} x^2_{uj}$. Choose the real positive values for α and β so that the design exist. The resulting design D provides a v-dimensional SORD in five levels. The above new class of combinatorial construction of SORD is illustrated with suitable example using a Group Divisible PBIBD parameters.

Example 2.1: Suppose the parameters v=8, b=2, r =1, k=4, λ_1=0, λ_2=1, n_1 = 3, n_2 =4, m=2 and n=4 of a group divisible PBIBD and let T_1, T_2, T_3, T_4, T_5, T_6, T_7 and T_8 be the eight treatments. The two blocks of GDPBIBD are (T_1, T_3, T_5, T_7) & (T_2, T_4, T_6, T_8). The below are the treatments of the association schemes are:

Treatments →	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
Second Associate Treatments	T_2	T_1	T_2	T_1	T_2	T_1	T_2	T_1
	T_4	T_3	T_4	T_3	T_4	T_3	T_4	T_3
	T_6	T_5	T_6	T_5	T_6	T_5	T_6	T_5
	T_8	T_7	T_8	T_7	T_8	T_7	T_8	T_7
First Associate Treatments	T_3	T_4	T_1	T_2	T_1	T_2	T_1	T_2
	T_5	T_6	T_5	T_6	T_3	T_4	T_3	T_4
	T_7	T_8	T_7	T_8	T_7	T_8	T_5	T_6

Let S_1 be the set of pairs of treatment which occur to get in $\lambda_1$blocks. The number of pairs in N_1 is $vn_1/2$. Let the remaining pairs of treatments belong to S_2 where each at the pair will occur together in λ_2 blocks and number of such pair is $vn_2/2$

$$S_1 = \begin{Bmatrix} T_1T_2 & T_1T_4 & T_1T_6 & T_1T_8 \\ T_3T_2 & T_3T_4 & T_3T_6 & T_3T_8 \\ T_5T_2 & T_3T_6 & T_5T_6 & T_5T_8 \\ T_7T_2 & T_3T_8 & T_7T_6 & T_7T_8 \end{Bmatrix} \qquad S_2 = \begin{Bmatrix} T_1T_3 & T_1T_5 & T_1T_7 \\ T_3T_5 & T_3T_7 & T_5T_7 \\ T_2T_4 & T_2T_6 & T_2T_8 \\ T_4T_6 & T_4T_8 & T_6T_8 \end{Bmatrix}$$

The resultant second order rotatable design is:

$$\begin{bmatrix}
\pm\alpha & \pm\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\
\pm\alpha & 0 & 0 & \pm\alpha & 0 & 0 & 0 & 0 \\
\pm\alpha & 0 & 0 & 0 & 0 & \pm\alpha & 0 & 0 \\
\pm\alpha & 0 & 0 & 0 & 0 & 0 & 0 & \pm\alpha \\
0 & \pm\alpha & \pm\alpha & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \pm\alpha & \pm\alpha & 0 & 0 & 0 & 0 \\
0 & 0 & \pm\alpha & 0 & 0 & \pm\alpha & 0 & 0 \\
0 & 0 & \pm\alpha & 0 & 0 & 0 & 0 & \pm\alpha \\
0 & \pm\alpha & 0 & 0 & \pm\alpha & 0 & 0 & 0 \\
0 & 0 & 0 & \pm\alpha & \pm\alpha & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \pm\alpha & \pm\alpha & 0 & 0 \\
0 & 0 & 0 & 0 & \pm\alpha & 0 & 0 & \pm\alpha \\
0 & \pm\alpha & 0 & 0 & 0 & 0 & \pm\alpha & 0 \\
0 & 0 & 0 & \pm\alpha & 0 & 0 & \pm\alpha & 0 \\
0 & 0 & 0 & 0 & 0 & \pm\alpha & \pm\alpha & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \pm\alpha & \pm\alpha \\
\pm\beta & 0 & \pm\beta & 0 & 0 & 0 & 0 & 0 \\
\pm\beta & 0 & 0 & 0 & \pm\beta & 0 & 0 & 0 \\
\pm\beta & 0 & 0 & 0 & 0 & 0 & \pm\beta & 0 \\
0 & 0 & \pm\beta & 0 & \pm\beta & 0 & 0 & 0 \\
0 & 0 & \pm\beta & 0 & 0 & 0 & \pm\beta & 0 \\
0 & 0 & 0 & 0 & \pm\beta & 0 & \pm\beta & 0 \\
0 & \pm\beta & 0 & \pm\beta & 0 & 0 & 0 & 0 \\
0 & \pm\beta & 0 & 0 & 0 & \pm\beta & 0 & 0 \\
0 & \pm\beta & 0 & 0 & 0 & 0 & 0 & \pm\beta \\
0 & 0 & 0 & \pm\beta & 0 & \pm\beta & 0 & 0 \\
0 & 0 & 0 & \pm\beta & 0 & 0 & 0 & \pm\beta \\
0 & 0 & 0 & 0 & 0 & \pm\beta & 0 & \pm\beta
\end{bmatrix}$$

Let us consider two PBIBD designs with v=8 and other parameters are: b_1 =16, r_1 = 4, k_1=2, λ_1^1=1, λ_2^1==0, b_2=12, r_2=3, k_2=2, λ_1^2=0 , λ_2^2=1 We have n_1= 3 and n_2= 4; N_1=12 and N_2= 16.

$\Sigma\, x^4_{ui} = 4\alpha^4 + 3\beta^4$ $\qquad$ $\Sigma\, x^2_{ui}\, x^2_{uj} = 16\alpha^4 + 12\beta^4$

For the value of $\alpha^2/\beta^2 = \frac{1}{3}$, the 28 design points for 8 factors satisfy all the conditions of rotatability.

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