Research Article

Applications Of Generalized Hypergeometric Analysis Function Of Second Order Differential Subordination

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Abstract: We present some findings for second order differential subordination in the open unit disk involving generalized hypergeometric function using the convolution operator.

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1. Introduction

Let $\mathcal{L} = \{w \in \mathbb{C} : |w| < 1\}$ be an open unit disc in $\mathbb{C}$. Let $H(\mathcal{L})$ be the analytic functions class in $\mathcal{L}$ and let $\mathcal{L}[a, \varepsilon]$ be the subclass of $H(\mathcal{L})$ of the form

$$g(w) = a + a_\iota w^\iota + a_{\iota+1} w^{\iota+1} + \cdots ,$$

where $a \in \mathbb{C}$ and $\iota \in \mathbb{N} = \{1,2, \dots\}$ with $H_0 \equiv H[0,1]$ and $H \equiv H[1,1]$. Let $g(w)$ be an analytic function an open unit disc. If the equation $v = g(w)$ has never more than p-solutions in $\mathcal{L} = \{w \in \mathbb{C} : |w| < 1\}$, then $g(w)$ is said to be p-valent in $\mathcal{L}$. The class of all analytic p-valent functions is denoted by $\mathcal{A}_p$, where g is expressed of the forms

$$g(w) = w^p + \sum_{\iota=p+\varepsilon}^{\infty} a_\iota w^\iota, \quad (p, \iota \in \mathbb{N} = \{1,2,3, \dots\}, w \in \mathcal{L}). \tag{1}$$

The Hadamard product for two functions in $\mathcal{A}_p$, such that

$$k(w) = w^p + \sum_{\iota=p+\varepsilon}^{\infty} c_\iota w^\iota, \quad (w \in \mathcal{L}) \tag{2}$$

is given by

$$g(w) * k(w) = w^p + \sum_{\iota=p+\varepsilon}^{\infty} a_\iota c_\iota w^\iota. \quad (w \in \mathcal{L}) \tag{3}$$

If g and k are members of $H(\mathcal{L})$, we can assume that a function g is subordinate to a function k or k is said to be superordinate to g if there exists a Schwarz function $l(w)$ which is analytic in $\mathcal{L}$ and $|l(w)| < 1, (w \in \mathcal{L})$, such that $g(w) = k(l(w))$. The term this subordination is used to describe this relationship

$$g(w) \prec k(w) \; or \; g \prec k .$$

Moreover, if the function k is univalent in $\mathcal{L}$, then we have the following equivalence [1,6,7,11]

$$g(w) \prec k(w) \Leftrightarrow g(0) = k(0) \; and \; g(\mathcal{L}) \subset k(\mathcal{L}) .$$

The class V is normalized convex functions in $\mathcal{L}$, we define for from

$$V = \left\{ g \in A : \Re e \left(1 + \frac{w g''(w)}{g'(w)} \right) > 0 , (w \in \mathcal{L}) \right\}.$$

Miller and Mocanu proposed the differential subordinations approach in 1978 [12,16], and the theory began to evolve in 1981 [10]. Miller and Mocanu compiled all of the information in a book published in 2000 [11,15]. If p is analytic in $\mathcal{L}$ and meets the second-order differential subordination condition, then

$$\Psi(p(w), wp'(w), wp''(w); w) \prec h(w), \tag{4}$$

p is known as a differential subordination solution. If $p \prec q$ for all p satisfying, the univalent function q is considered a dominant of the solutions of the differential subordination or simply a dominant (4). The best dominant of all is a dominant q that satisfies $\tilde{q} \prec$ q for all dominants (4).

See [3,4,5] for the use of generalized hypergeometric functions and Wright's generalized hypergeometric functions in geometric function theory. For the purposes of this paper, we define a linear operator in terms of Wright's generalized hypergeometric function.

$$\Omega_p^t[(\alpha_n, A_n)1, q; (\beta_n, B_n)1, s]: A_P^t \to A_P^t,$$

Dziok and Raina [2,8] looked into it recently. For a function g of the form(1), the following can be seen:

$$\Omega_p^t[(\alpha_n, A_n)1, q; (\beta_n, B_n)1, s](g * k)(w) = w^P + \sum_{n=p+1}^{\infty} \chi_n(\alpha_1)\, a_n\, b_n w^n, \tag{5}$$

where

$$\chi_n(\alpha_1) = \pi\, \frac{\Gamma(\beta_1 + B_1(n-p)) \dots \Gamma(\beta_S + B_S(n-p))(n-p)!}{\Gamma(\alpha_1 + A_1(n-p)) \dots \Gamma\left(\alpha_q + A_q(n-p)\right)}\ , \pi = \left(\prod_{n=1}^{q} \Gamma(\alpha_n\right)^{-1} \left(\prod_{n=1}^{s} \Gamma(\beta_n\right),$$

we have it for the sake of convenience

$$\Omega_p^t[\alpha_1\,](g * k)(w) = \Omega_p^t\left[(\alpha_1, A_1), \dots, (\alpha_q, A_q); (\beta_1\,, B_1), \dots, (\beta_s\,, B_s)\,\right](g * k)(w)$$

Using the relationship (5), it is clear that

$$wA_1\left(\Omega_p^t[\alpha_1](g * k)(w)\right)' = (\alpha_1 - pA_1)\Omega_p^t[\alpha_1\,](g * k)(w) + \alpha_1 \Omega_p^t[\alpha_1 + 1](g * k)(w). \tag{6}$$

For $t \in \mathbb{N}_0\,, p \geq 0$, we let $\Re_{p,t}(\lambda)$ be the class of functions $g \in A$ satisfying

$$\Re e\{(\Omega_p^t[\alpha_1\,](g * k)(w))'\} \leq \lambda\,, (0 \leq \lambda < 1, w \in \mathcal{L}). \tag{7}$$

The following lemmas will be used to obtain our key results.

Lemma 1.1 ([13,9]). Let k be a convex function in $\mathcal{L}$ and let $h(\mathcal{L}) = k(w) + n\beta wk'(w)$, where $\beta > 0$ and $n \in \mathbb{N}$. If $p(w) = k(0) + p_n w^n + p_{n+1}w^{n+1} + \cdots$, is holomorphic in $\mathcal{L}$ and

$$p(w) + \beta wp'(w) \prec h(w),$$

then

$$p(w) \prec k(w).$$

Lemma 1.2 ([14]). Let $\Re e\{\tau\} > 0, n \in \mathbb{N}$, and let $M = \frac{n^2+|\tau|^2 - |n^2-\tau^2|}{4nRe\{\tau\}}$. Let h be an analytic function in $\mathcal{L}$ with $k(0) = 1$, and $\Re e\left\{1 + \frac{wh''(w)}{h'(w)}\right\} > -\mathrm{M}$. If $p(w) = 1 + p_n w^n + p_{n+1}w^{n+1} + \cdots$, is analytic in $\mathcal{L}$ and $p(w) + \frac{1}{\tau}wp'(w) \prec h(w)$, we get $p(w) \prec q(w)$, where q is the differential equation's solution

$$q(w) + \frac{n}{\tau}wq'(w) = h(w), \qquad q(0) = 1,$$

then

$$q(w) = \frac{\tau}{nw^{\tau/n}} \int_0^w t^{(\tau/n)-1}\, h(t)dt\,, \quad (w \in \mathcal{L}).$$

2. Main results

Theorem 2.1. Let q be convex function in $\mathcal{L}$ with $q(0) = 1$ and let $h(w) = q(w) + \frac{1}{\mu+1}\, wq'(w)$, where $\mu \in \mathbb{C}$, and $\Re e\{\mu\} > -1$. If $g \in \Re_{p,t}(\beta)$, $\xi = \gamma\mu\,(g * k)$, where

$$\xi\,(w) = \gamma\mu\,(g * k)(w) = \frac{\mu+1}{w^{\mu}} \int_0^w t^{\mu-1}\,(g * k)(t)dt, \tag{7}$$

then

$$(\Omega_p^t[\alpha_1](g * k)(w))' \prec h(w). \tag{8}$$

It imply

$$\left(\Omega_p^t[\alpha_1]\xi(w)\right)' \prec q(w).$$

Proof. We can deduce the following from the equality (7):

$$w\mu\, \xi(w) = (\mu+1)\int_0^w t^{\mu-1}(g * k)(t)dt. \tag{9}$$

When we differentiate the equality (9) in terms of w, we get

$$(\mu)\xi(w) + w\xi'(w) = (\mu+1)(g * k)(w),$$

then, we obtain

$$(\mu)\Omega_p^t[\alpha_1]\xi(w) + w(\Omega_p^t[\alpha_1]\xi(w))' = (\mu+1)\Omega_p^t[\alpha_1](g * k)(w). \tag{10}$$

When we differentiate (8) in terms of w, we get

$$(\Omega_p^t[\alpha_1]\xi(w))' + \frac{1}{\mu+1}w((\Omega_p^t[\alpha_1]\xi(w))'' = ((\Omega_p^t[\alpha_1]g(w))'. \tag{11}$$

In the equality problem, use differential subordination (8). (11), we obtain

$$(\Omega_p^t[\alpha_1]\xi(w))' + \frac{1}{\mu+1}w((\Omega_p^t[\alpha_1]\xi(w))'' \prec h(w). \tag{12}$$

Now, let us define

$$p(w) = (\Omega_p^t[\alpha_1]\xi(w))'. \tag{13}$$

Then, with a quick calculation,

$$p(w) = \left[w + \sum_{n=2}^{\infty}\chi_n(\alpha_1)\frac{\mu+1}{\mu+n}a_n b_n w^n\right]' = 1 + p_1 z + p_2 z + \ldots, \quad (p \in H[1,1]).$$

In the equality problem, use differential subordination (12). (13), we have,

$$p(w) + \frac{1}{\mu+1}wp'(w) \prec h(w) = q(w) + \frac{1}{\mu+1}wq'(w).$$

Making use of Lemma 1.2, we obtain

$$p(w) \prec q(w).$$

Theorem 2.2. Let $\Re e\{\mu\} > -1$ and let $M = \frac{1+|\mu+1|^2-|\mu^2+2\mu|}{4Re\{\mu+1\}}$. Let h be an analytic function in $\mathcal{L}$ with $h(0) = 1$ and suppose that $\Re e\left\{1 + \frac{wh''(w)}{h'(w)}\right\} > -\text{E}$. If $(g * k) \in \Re_{p,t}(\beta)$ and $\xi = \gamma_\mu^\lambda(g * k)$, where ξ is defined by (10), then

$$(\Omega_p^t[\alpha_1](g * k)(w))' \prec h(w) \tag{14}$$

It imply

$$(\Omega_p^t[\alpha_1]\xi(w))' \prec q(w),$$

where q is the differential equation's solution

$$h(w) = q(w) + \frac{1}{\mu+1}wq'(w), \quad q(0) = 1,$$

given by

$$q(w) = \frac{\mu+1}{w^{\mu+1}}\int_0^z t^\mu (g * k)(t)dt.$$

Proof. If we use $n = 1$ and $\gamma = \mu + 1$ in Lemma 1.2, then the proof is straightforward using the proof of Theorem 2.2.

$$h(w) = \frac{1 + (2\beta - 1)w}{1 + w}, \quad 0 \le \beta < 1,$$

we get the following result from Theorem 2.2.

Corollary 2.3. If $0 \le \beta < 1, 0 \le \zeta < 1, p \ge 0, \Re e\{\mu\} > -1$ and $\xi = \gamma\mu\,(g * k)$ is defined by the equation $\Re e\{\Omega_p^t[\alpha_1]h(w))'\} > \beta$, then, we have $\gamma_\mu(\Re_{p,t}(\beta)) \subset \Re_{p,t}(\zeta)$, where $\zeta = \min_{|w|=1} \Re e\{q(w)\} = \zeta(\mu, \beta)$. Also,

$$\zeta = \zeta(\mu, \beta) = (2\beta - 1) + 2(\mu + 1)(1 - \beta)\tau(\mu), \tag{15}$$

where

$$\tau(\mu) = \int_0^1 \frac{t^\mu}{1+t}\, dt. \tag{16}$$

Proof. Let $f \in \Re_{p,t}(\beta)$. By from (7), we get

$$\Re e\{(\Omega_p^t[\alpha_1](g * k)(w))'\} > \beta$$

this is the same as

$$(\Omega_p^t[\alpha_1](g * k)(w))' \prec h(z).$$

We obtain by applying Theorem 2.1.

$$(\Omega_p^t[\alpha_1\,]\xi(z))' \prec q(z).$$

If we consider

$$h(w) = \frac{1 + (2\beta - 1)w}{1 + w}, \quad 0 \le \beta < 1\,.$$

Then h is convex , and we have by Theorem 2.2

$$\left(\Omega_p^t[\alpha_1\,]\xi(w)\right)' \prec q(w) = \frac{\mu+1}{w^{\mu+1}} \int_0^w t^\mu \frac{1 + (2\beta - 1)t}{1+t}\, dt = (2\beta - 1) + 2\frac{(1-\beta)(\mu+1)}{w^{\mu+1}} \int_0^w \frac{t^\mu}{1+t}\, dt.$$

If $\Re e\{\mu\} > -1$, and $q(\mathcal{L})$ is symmetric with respect to the real axis because of its convexity, we obtain

$$\Re e\{(\Omega_p^t[\alpha_1]\xi(w))'\} \ge \min_{|w|=1} \Re e\{q(w)\} = \Re e\{q(1)\} = \zeta(\mu, \beta) = (2\beta - 1) + 2(\mu + 1)(1 - \beta)\tau(\mu), \tag{17}$$

where $\tau(\mu)$ is the value of (16). We have inequity (17) as a result of injustice

$$\gamma_\mu(\Re_{p,t}(\beta)) \subset \Re_{p,t}(\zeta),$$

where ζ is given by (15).

Theorem 2.4. If q be a convex function and $q(0) = 1$. Let h a function such that $h(w) = q(w) + wq'(w)$, and $k \in \mathbb{N}_0$, $p \ge 0$, $g \in A$, such that

$$(\Omega_p^t[\alpha_1](g * k)(w))' \prec h(w) = q(w) + wq'(w), \tag{18}$$

then

$$\frac{\Omega_p^t[\alpha_1](g * k)(w)}{w} \prec q(w).$$

Proof. Let

$$p(w) = \frac{\Omega_p^t[\alpha_1](g * k)(w)}{w}. \tag{19}$$

We have (19) as a differentiator.

$$\Omega_p^t[\alpha_1](g * k)(w))' = p(w) + wp'(w). \ (w \in \mathcal{L})$$

When you use (18), you get

$$p(w) + wp'(w) \prec h(w) = q(w) + wq'(w),$$

we can use Lemma 1.1 to solve this problem

$$p(w) \prec q(w).$$

Then, we obtain

$$\frac{\Omega_p^t[\alpha_1](\mathscr{g} * k)(w)}{w} \prec q(w).$$

Theorem 2.5. If q be a convex function and $q(0) = 1$. Let h the function $h(w) = q(w) + wq'(w)$, and $k \in \mathbb{N}_0,\ p \geq 0, \mathscr{g} \in A$, such that

$$\left(\frac{\Omega_p^t[\alpha_1 + 1](\mathscr{g} * k)(w)}{\Omega_p^t[\alpha_1](\mathscr{g} * k)(w)}\right)' \prec h(w), \tag{20}$$

then

$$\frac{\Omega_p^t[\alpha_1 + 1](\mathscr{g} * k)(w)}{\Omega_p^t[\alpha_1](\mathscr{g} * k)(w)} \prec q(w).$$

Proof. In the case of the function $\mathscr{g} \in A$, which is given by the equation (1), we get

$$\Omega_p^t[(\alpha_n,\ A_n)1, q; (\beta_n,\ B_n)1, s](\mathscr{g} * k)(w) = w + \sum_{n=2}^{\infty} \chi_n\,(\alpha_1)\,a_n\,b_n w^n = \Omega_p^t[\alpha_1](\mathscr{g} * k)(w).$$

Hence

$$p(w) = \frac{\Omega_p^t[\alpha_1 + 1\,](\mathscr{g} * k)(w)}{\Omega_p^t[\alpha_1](\mathscr{g} * k)(w)} = \frac{w\ +\ \sum_{n=2}^{\infty} \chi_n\,(\alpha_1 + 1)\ \dfrac{\mu+1}{\mu+n}\,a_n\,b_n w^n}{w\ +\ \sum_{n=2}^{\infty} \chi_n\,(\alpha_1)\ \dfrac{\mu+1}{\mu+n}\ a_n\,b_n w^n}$$

$$= \frac{1\ +\ \sum_{n=2}^{\infty} \chi_n\,(\alpha_{1+1})\ \dfrac{\mu+1}{\mu+n}\,a_n\,b_n w^{n-1}}{1\ +\ \sum_{n=2}^{\infty} \chi_n\,(\alpha_1)\ \dfrac{\mu+1}{\mu+n}\,a_n\,b_n w^{n-1}},$$

then

$$\big(p(w)\big)' = \frac{\left(\Omega_p^t[\alpha_1 + 1](\mathscr{g} * k)(w)\right)'}{\Omega_p^t[\alpha_1](\mathscr{g} * k)(w)} - p(w)\frac{\left(\Omega_p^t[\alpha_1](\mathscr{g} * k)(w)\right)'}{\Omega_p^t[\alpha_1](\mathscr{g} * k)(w)},$$

we obtain

$$p(w) + wp'(w) = \frac{\left(w\Omega_p^t[\alpha_1 + 1\,](\mathscr{g} * k)(w)\right)'}{\Omega_p^t[\alpha_1\,](\mathscr{g} * k)(w)}.$$

As a result of the relationship (20),

$$p(w) + wp'(w) \prec h(w) = q(w) + wq'(w),$$

We can use Lemma 1.1 to solve this problem

$$p(w) \prec q(w).$$

Therefor

$$\frac{\Omega_p^t[\alpha_1](\mathscr{g} * k)(w)}{w} \prec q(w).$$

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