Research Article

Invariance Analysis, Explicit Solution and Numerical Exact solution of Time Fractional Partial Differential Equation

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Abstract: In this research article, we have performed invariance analysis to time fraction partial differential equation (FPDE) by Lie symmetry reduction and converted the fractional order system into fractional ordinary differential equation (FODE) with the application of Erdyli-Kober (E-K) differ-integral operators in Riemann-Liouville (R-L) sense. Exact solutions are being established by power series technique and numerical solutions have been verified with application of the Fractional Reduced Differential Transforms and Homotop Analysis Methods.

Keywords: Time fractional partial differential equation, Reduced Differential Transforms Method, Homotopy Analysis Method.

1.Introduction:

The seed of fractional calculus were sowed over 32 decades ago as the generalization of integral order classical calculus. The idea of derivative and integration (not an integer) was not acceptable by the physical sciences due to lack of physical and geometrical interpretations for more than 30 decades but during last three decades Oldham (1974), Podlubny (1999), Debnath (2003), Kilbas & Srivastva (2006) have applied the fractional calculus in field of applied science and promoted the work that variation in real order of derivatives influenced the geometrical interpretation. Podlubny (2001) described geometrical interpretation in left and right handed Riemann-Liouville fractional integral as shadow of walls. Tavassoli (2013) performed the relationship between generalized derivative of power function at tangent points and the order of fractional derivative.

In present, the utilization of fractional calculus is rapidly growing field of research to physical and biological sciences. The exact solution of fractional systems is challenging and interesting topic of research for applied mathematicians. Wazwaz (2007) , Jafari (2013) and Lin (2016) implemented some semi-analytic, analytic, iterative schemes on concerning linear and nonlinear classical and fractional PDEs. The FPDEs have enriched more attention in fluid flow, travelling wave models, signal processing, identification of systems, optimization, finance, biological modeling and fractional dynamics.

Lie Group symmetries and applications has been studied successfully by Olver (2002). Biswas (2012 & 2014)
provided the study of shock waves, bifurcation and conservation laws analysis of Boussinesq equation with non-linear power laws of waves and also attempted quasi-solitons by symmetry analysis.Huang & Zahdanov (2014), Bakkayraj & Sahdevan (2015) applied group formalism approach and found the solution of PDEs. Burguess (1997), Iyiola (2014), Lonescu (2017) and Gandhi (2020) illustrated the applications of biological mathematical modeling on distinct cancer tumor growth with the use of fractional and classical PDEs. Singla (2016), Wang (2013 & 2017) and Zhang (2015 & 2017) has explained about the invariance analysis, extended infinitesimals and explicit solutions with one independent variable of time fractional KdVs. Gandhi (2020) considered the generalized fourth order and fifth order KdVs, converted the FPDEs into FODEs by extended use of Erdyli-Kober operators, which has already been suggested by Sneddon (1975) and exact solution generated by power series solution along with graphical interpretation. Conservation laws with symmetry reduction of Boussinesq-Burgers fractional system is provoked by Shi (2019). Jena (2020) reported the solitary wave solution of system of Hirota-Satsoma coupled KdV and mKdV time fractional PDEs.

Iyiola (2014) presented the cancer mathematical model and experienced the solution of time fractional PDEs by q-HAM and discussed the applicability and ideality of the model on Burgess equation given below:.

$$\frac{\partial^{\mu}\omega(x,t)}{\partial t^{\mu}}-\frac{\partial^{2}\omega(x,t)}{\partial x^{2}}+K(x,t))\omega(x,t)=0 \text{ where } 0<\mu<1 \quad (1)$$

In our present work, we have analyzed equation (1) by Lie symmetry analysis with power series method and applied Reduced Differential Transforms Method (RDTM) and Homotop Analysis Method(HAM) for numerical solutions.. Consider the FPDE with fractional order $0<\mu<1$.

$$\partial_t^{\mu}\omega(x,t)=\omega_{xx}-2x^{-2}\omega(x,t) \quad (2)$$

The proposed work sequentially arranged as in section 2: some facts and prelims explained, in section 3: Lie symmetry, Reduced Differential Transforms and Homotop Analysis Methods elaborated. Application of these techniques has been done successfully in section 4 and ended with remarks and conclusions.

2. Preliminaries

2.1 Definition: The R-L fractional (non integer) derivative is explained as

$$D_t^{\mu}(f(t))=\frac{1}{\Gamma(\lambda-\mu)}\frac{d^{\lambda}}{dt^{\lambda}}\int_0^t(t-\psi)^{\lambda-\mu-1}f(\psi)d\psi;\ t>0,\ \lambda-1<\mu\le\lambda,\ \lambda\in N. \quad (3)$$

2.2 Definition: The R-L fractional (non integer) order partial derivative for function $\omega(x,t)$ with variable'‘t' is

$$\partial_t^{\mu}(\omega(x,t))=\begin{cases}\dfrac{1}{\Gamma(\lambda-\mu)}\dfrac{\partial^{\lambda}}{\partial t^{\lambda}}\displaystyle\int_0^t(t-\psi)^{\lambda-\mu-1}\omega(\psi,x)d\psi;t>0,\lambda-1<\mu<\lambda,\lambda\in N\\ \dfrac{\partial^{\lambda}u}{\partial t^{\lambda}}\quad for\ \mu=\lambda\end{cases} \quad (4)$$

2.3 Definition: The Leibnitz rule in R-L fractional derivatives sense established the relation

$$D_t^{\mu}(\omega_1(x,t),\omega_2(x,t))=\sum_{\lambda=0}^{\infty}\binom{\mu}{\lambda}D_t^{\mu-\lambda}(\omega_1(x,t)).D_t^{\lambda}(\omega_2(x,t))\ ;\mu>0,\binom{\mu}{\lambda}=\frac{(-1)^n\mu\Gamma(\lambda-\mu)}{\Gamma(1-\mu)\Gamma(\lambda+1)} \quad (5)$$

2.4 Definition: For $D_t^{\alpha}(t-p)^{\mu}=\dfrac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}(t-p)^{\mu-\alpha};t\in(p,b],\alpha\ge0,\mu>0.$ (6)

2.5 Definition: The R-L fractional integration operator I^{μ} of order ‘μ’ is defined as

$$I_t^{\mu}[\phi(t)]=\frac{1}{\Gamma(\mu)}\int_0^t(t-\rho)^{\mu-1}\phi(t)(\rho)d\rho;\quad \mu>0 \quad (7)$$

Along with the following manipulated results

$$\begin{aligned}&(i)\ I^0\phi(t)=\phi(t)\\&(ii)\ I^{\mu}I^{\beta}(\phi)=I^{\beta}I^{\mu}(\phi)=I^{\mu+\beta}(\phi)\\&(iii)\ I^{\mu}(t-a)^m=\frac{\Gamma(m+1)}{\Gamma(\mu+m+1)}(t-a)^{\mu+m};\mu\ge0\ and\ m>-1\end{aligned} \quad (8)$$

3. Methodologies

3.1 Lie symmetry analysis

Here, the basic terminology concerned with Lie symmetry analysis proposed. Mathematicians has been applied the lie symmetry approach in classical and fractional order PDEs and we illustrated the important terms and steps of this exclusive approach on time fractional PDE.

Consider a time fractional PDE as

$$\partial_t^{\mu}\omega=H(x,t,\omega,\omega_x,\omega_{xx},........);\ 0<\mu<1 \quad (9)$$

Infinitesimal transformations of Lie algebra with single parameter ‘ε’ given to be

$$\bar{t}=t(x,t,\omega;\rho);\ \bar{x}=x(x,t,\omega;\rho);\ \bar{\omega}=\omega(x,t,\omega;\rho) \quad (10)$$

Vector field generated by infinitesimals is

$$S = \tau\,\partial_t + \xi\,\partial_x + \eta\,\partial_\omega \; with \; \tau = \left.\frac{d\bar{t}}{d\rho}\right|_{\rho=0},\; \xi = \left.\frac{d\bar{x}}{d\rho}\right|_{\rho=0},\; \eta = \left.\frac{d\bar{\omega}}{d\rho}\right|_{\rho=0} \tag{11}$$

Apply prolongation operator to FPDE (9)

$$\Pr^{(\mu,2)}[X(\partial_t^\mu \omega - H)|_{\Delta=0}] = 0 \; where \; \Pr^{(\mu,2)} = S + \eta^{\mu,t}\frac{\partial}{\partial(\partial_t^\mu \omega)} + \eta^x \frac{\partial}{\partial \omega_x} + \eta^{xx}\frac{\partial}{\partial \omega_{xx}} \tag{12}$$

Where extended infinitesimals are derived as

$$\begin{aligned} \eta^{\mu,t} &= D_t^\mu(\eta) - D_t^\mu(\xi\,\omega_x) + \xi\,D_t^\mu(\omega_x) + D_t^\mu(u\omega D_t(\tau)) + \tau\,D_t^{\mu+1}(\omega) - D_t^{\mu+1}(\tau\,\omega);\\ \eta^x &= D_x(\eta) - \omega_t\,D_x(\tau) - \omega_x\,D_x(\xi);\\ \eta^{xx} &= D_x(\eta^x) - \omega_{tx}\,D_x(\tau) - \omega_x\,D_{xx}(\xi) \end{aligned} \tag{13}$$

Here D_t and D_x defines total derivatives with respect to time ‘t’and space ‘x’ variables, respectively

$$\begin{aligned} D_t &= \partial_t + \omega_t\frac{\partial}{\partial\omega} + \omega_{tt}\frac{\partial}{\partial\omega_t} + \omega_{xt}\frac{\partial}{\partial\omega_x} + ...;\\ D_x &= \partial_x + \omega_x\frac{\partial}{\partial\omega} + \omega_{tx}\frac{\partial}{\partial\omega_t} + \omega_{xx}\frac{\partial}{\partial\omega_x} + ... \end{aligned} \tag{14}$$

Liebnitz rule of fractional order in R-L sense is taken to be

$$D_t^\mu(\phi\psi) = \sum_{\lambda=0}^{\infty}\binom{\mu}{n} D_t^\lambda(\phi).D_t^{\mu-\lambda}(\psi) \; where \; D_t^0(\phi) = \phi,\; D_t^{\lambda+1}(\psi) = D_t(D_t^\lambda(\psi)) \tag{15}$$

Applying minor calculations by Leibnitz rule we obtain

$$\xi\,D_t^\mu(\omega_x) - D_t^\mu(\xi\,\omega_x) = -\sum_{\lambda=1}^{\infty}\binom{\mu}{\lambda} D_t^{\mu-\lambda}(\omega_x)\,D_t^\lambda(\xi) \tag{16}$$

$$\tau\,D_t^{\mu+1}(\omega) - D_t^{\mu+1}(\tau\,\omega) + D_t^\mu(\omega\,D_t(\tau)) + \mu\,D_t(\tau)\,D_t^\mu\omega = -\sum_{\lambda=1}^{\infty}\binom{\mu}{\lambda+1} D_t^{\mu-\lambda}(\omega), D_t^{\lambda+1}(\tau) \tag{17}$$

Fractional order chain rule for composition of functions represented as

$$\frac{d^\mu}{dt^\mu}\big(\phi(\psi(t))\big) = \sum_{\lambda=0}^{\infty}\frac{U_\lambda}{\lambda!}\frac{d^\lambda\phi(z)}{dz^\lambda} \; where\, U_\lambda = \sum_{k=0}^{\lambda}(-1)^\lambda\binom{\lambda}{k}.\psi^k(t).\partial_t^\mu(\psi^{\lambda-k}(t))\;;z = \psi(t) \tag{18}$$

Using generalized Leibnitz rule (15-18), we have

$$D_t^\psi(\eta) = \partial_t^\psi(\eta) - \omega\partial_t^\psi(\eta_\omega) + \eta_\omega\partial_t^\psi(\omega) + \sum_{\lambda=1}^{\infty}\binom{\psi}{\lambda}\partial_t^\lambda(\eta_\omega)\partial_t^{\psi-\lambda}(\omega) + \sigma \tag{19}$$

where

$$\sigma = \sum_{\lambda=2}^{\infty}\sum_{n=2}^{\lambda}\sum_{k=2}^{n}\binom{\mu}{\lambda}\binom{\lambda}{n}\frac{t^{\lambda-\mu}U_k}{k!\Gamma(\lambda+1-\mu)}\frac{\partial^{\lambda-n+k}\eta}{\partial t^{\lambda-n}\partial\omega^k}\;;\sigma = 0\;\; for\; \eta\; is\, linearly\; dependent\; on\;\; \omega \tag{20}$$

we obtain

$$\begin{aligned} \eta^{\mu,t} &= D_t^\mu(\eta) - D_t^\mu(\xi\,\omega_x) + \xi\,D_t^\mu(\omega_x) + D_t^\mu(\omega\,D_t(\tau)) + \tau\,D_t^{\mu+1}(\omega) - D_t^{\mu+1}(\tau\,\omega)\\ &= \partial_t^\mu(\eta) - \omega\,\partial_t^\mu(\eta_\omega) + (\eta_\omega - \mu\,D_t(\tau))\partial_t^\mu(\omega) + \sum_{\lambda=1}^{\infty}\left[\binom{\mu}{\lambda}\partial_t^n(\eta_\omega) - \binom{\mu}{\lambda+1}D_t^\lambda(\xi)\,\partial_t^{\mu-\lambda}(\omega)\right]\partial_t^{\mu-\lambda}\\ &\quad - \sum_{\lambda=1}^{\infty}\binom{\mu}{\lambda}D_t^\lambda(\xi)\,\partial_t^{\mu-\lambda}(\omega_x) + \sigma \end{aligned} \tag{21}$$

We use above equations in prolonged equation (12), Make the coefficients of ω_x, ω_{xx} equals to zero and solve the obtained set of system of PDEs and FPDEs to get the infinitesimals and explicit solutions.

3.2 Fractional reduced differential transform method:

Consider a function $\omega(t)$ is m-times differentiable and continuous with time ‘*t’* analytic in the domain as

$$W_m = \frac{1}{\Gamma(1+m\mu)}\big(D_t^m\omega(t)\big)_{t=0} \tag{22}$$

Here, the mapping $W_m(t)$ is the transformed function of $\omega(t)$ .So, $\omega(t)$ will be inverse differential transform of $W_m(x)$ is defined as

$$\omega(t) = \sum_{m=0}^{\infty} W_m (t - t_0)^{m\mu} \tag{23}$$

From above equations, it found to be

$$\omega(t) = \sum_{m=0}^{\infty} \frac{1}{\Gamma(m\mu + 1)} (D_t^m \omega(t))_{t=0} (t - t_0)^{m\mu} \tag{24}$$

It can be observed that the expansion of the fractional reduced differential transform method is originated from the Taylor's series expansion with the initial restriction $W_0 = \omega(0)$. To apply this methodology; one should take the reduced differential transformation of the FPDE to be solved after obtained iteration formula for W_m .
Inverse differential transform found the approximate solution.

$$\omega_m(t) = \sum_{m=0}^{n} W_m (t - t_0)^{m\mu} \tag{25}$$

Therefore, the solution *ω(t)* is given by

$$\omega(t) = \lim_{m \to \infty} \omega_m(t) \tag{26}$$

for *t=0*, equation (24) becomes

$$\omega(t) = \sum_{m=0}^{\infty} \frac{1}{\Gamma(m\mu + 1)} (D_t^m \omega(t))_{t=0} t^{m\mu} \tag{27}$$

Some fractional reduced transforms of functions are given under, to solve the system of FPDEs by this methodology.

Functions	Reduced differential transforms
$\omega(t)$	$W_k = \frac{1}{\Gamma(m\mu+1)} \left(D_t^m \omega(t)\right)_{t=0}$
$\omega(t) = u_1(t) \pm u_2(t)$	$W_k = U_{1k} \pm U_{2k}$
$\omega(t) = c.u(t)$	$W_k(x) = c.U_k(x)$
$\omega(t) = \frac{d}{dt}\left(u(t)\right)$	$W_k = \frac{d}{dt}\left(U_k\right)$
$\omega(t) = u(t)v(t)$	$W_k = \sum_{m=0}^{k} V_m U_{k-m} = \sum_{m=0}^{k} U_m V_{k-m}$
$\omega(t) = D_t^{n\mu}(u(t))$	$W_k = \frac{\Gamma(1+k\mu+n\mu)}{\Gamma(1+k\mu)} U_{k+\mu}$

4.3 Homotopy analysis method:

The general idea of HAM is presented by Liao in 1992 and extended concept is considering on a FPDE in the form

$$\mathrm{N}[D_t^{\mu} \omega(x,t)] - f(x,t) = 0 \tag{28}$$

Where 'N' is non linear operator, D_t^{α} denotes the Caputo fractional derivative, *(x, t)* are independent variables, f is known function and 'ω' is an unknown function. To generalize the original homotopy method, the zeroth-order deformation equation is constructed as

$$(1-q)L(\Phi(x,t;q) - \omega_0(x,t)) = qhH(x,t)(N[D_t^{\mu}\Phi(x,t;q)] - f(x,t)) \tag{29}$$

Where *n ≥ 1, q ϵ [0,1]* denotes embedded parameter, L is auxiliary linear operator, *h ≠ 0* is auxiliary parameter, *H (x, t)* is non-zero auxiliary function. It is clearly seen that when *q = 0 and q = 1,* above equation becomes

$$\Phi(x,t;0) = \omega_0(x,t) \; and \; \Phi(x,t;1) = \omega(x,t) \tag{30}$$

So as '*q*' increases from *0 to 1*, the solution *Φ(x,t;q)* varies from the initial guess $\omega_0(x, t)$ to solution ω*(x, t).* If we choose ω_0, *L, h, H(x, t)* appropriately, solution *Φ(x,t;q)* of (28) exists for *q ϵ [0,1]*. Expansion of *Φ(x,t;q)* in Taylor's series represents

$$\Phi(x,t;q) = \omega_0(x,t) + \sum_{m=1}^{\infty} \omega_m(x,t) q^m \tag{31}$$

Where $\omega_m(x,t) = \dfrac{1}{m!}\dfrac{\partial^m \Phi(x,t;q)}{\partial q^m}\bigg|_{q=0}$

Let the vector ω_n be define as follows:

$$\overrightarrow{\omega_n} = \{\omega_0(x,t), \omega_1(x,t)........\omega_n(x,t)\} \tag{32}$$

Differentiating equation (29) m-times with respect to the parameter q, then evaluating at *'q = 0'* and finally dividing them by m!, we have, what is known as m[th] order deformation equation as

$$L[\omega_m(x,t) - \chi_m \omega_{m-1}(x,t)] = hH(x,t)R_m(\overrightarrow{\omega}_{m-1}) \tag{33}$$

With some initial conditions $\omega_m^{(k)}(x,0) = 0,\ k = 0,1,2,.......m-1$

$$\text{Where } R_m(\overrightarrow{\omega}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}(N[D_t^\alpha \Phi(x,t;q)] - f(x,t))}{\partial q^{m-1}}\bigg|_{q=0} \text{ and } \chi_m = \begin{cases} 0 & m \le 1 \\ 1 & otherwise \end{cases} \tag{34}$$

4. Applications

4.1 Invariant Analysis by Lie symmetry analysis on FPDE

Applying Lie symmetry prolongation on (2) as discussed above, we obtain

$$\eta_t^\mu - 4x^{-3}\omega\xi + 2x^{-2}\eta - \eta^{xx} = 0 \tag{35}$$

Substituting eqns. (13-21) in (35), it reduces to

$$\begin{aligned} &\partial_t^\mu(\eta) - \omega\partial_t^\mu(\eta_\omega) + (\eta_\omega - \mu D_t(\tau))\partial_t^\mu(\omega) + \sum_{\lambda=1}^{\infty}\left[\binom{\mu}{\lambda}\partial_t^\lambda(\eta_\omega) - \binom{\mu}{\lambda+1}D_t^\lambda(\xi)\partial_t^{\mu-\lambda}(\omega)\right]\partial_t^{\mu-\lambda} \\ &-\sum_{\lambda=1}^{\infty}\binom{\mu}{\lambda}D_t^\lambda(\xi)\partial_t^{\mu-\lambda}(\omega_x) - 4x^{-3}\omega\xi + 2x^{-2}\eta - [\eta_{xx} - (\xi_{xx} - 2\eta_{x\omega})\omega_x - \omega_t\tau_{xx} - (2\xi_{x\omega} - \eta_{\omega\omega})\omega_x{}^2 - \\ &2\tau_{xx}\omega_x\omega_t - \xi_{\omega\omega}\omega_x{}^3 - \tau_{\omega\omega}\omega_x{}^2\omega_t + 2\tau_x\omega_{xt}\omega_{xx}(2\xi_\omega - \eta_\omega) - \tau_\omega\omega_t\omega_{xx} - 2\tau_\omega\omega_x\omega_{xt} - 3\omega_x\omega_{xx}\xi_\omega] = 0 \end{aligned} \tag{36}$$

Now set of equations formed by equating coefficients of linearly independent derivatives zero.

$$\sum_{\lambda=1}^{\infty}\left[\binom{\mu}{\lambda}\partial_t^\lambda(\eta_\omega) - \binom{\mu}{\lambda+1}D_t^\lambda(\xi)\partial_t^{\mu-\lambda}(\omega)\right] = 0 \tag{37}$$

$$\sum_{\lambda=1}^{\infty}\binom{\mu}{\lambda}D_t^\lambda(\xi) = 0 \tag{38}$$

$$\begin{aligned} &\tau_x = 0;\ \tau_{xx} = 0; \tau_\omega = 0;\ \tau_{\omega\omega} = 0; \\ &\tau_{\omega x} = 0; \xi_\omega = 0;\ \xi_{\omega\omega} = 0; \\ &\eta_\omega - \mu D_t(\tau) = 0; \\ &\partial_t^\mu\eta - u\partial_t^\mu(\eta_\omega) - 4x^{-3}\omega\xi + 2x^{-2}\eta - \eta_{xx} = 0 \\ &2\tau_{x\omega} - \xi_{xx} = 0; \\ &\eta_{\omega\omega} - 2\xi_{x\omega} = 0 \end{aligned} \tag{39}$$

Solving set of PDEs (29-31) to get infinitesimals

$$\tau = tc_1 + c_2;\ \xi = \left(\frac{\mu x c_1}{2}\right);\ \eta = \mu c_1\omega \tag{40}$$

Infinitesimal generator S of lie group is given by

$$S = (tc_1 + c_2)\frac{\partial}{\partial t} + (\mu c_1 x/2)\frac{\partial}{\partial x} + (\mu c_1\omega)\frac{\partial}{\partial\omega} \tag{41}$$

Infinitesimal symmetries related to S are

$$S_1 = t\frac{\partial}{\partial t} + (\mu x/2)\frac{\partial}{\partial x} + \mu\omega\frac{\partial}{\partial\omega};\ S_2 = \frac{\partial}{\partial t} \tag{42}$$

These infinitesimal symmetries follows Lie algebra and found to be skew symmetric *$[S_2,S_1] = -S_2$ and $[S_1,S_2] = S_2$, with $[S_1,S_1] = 0$, $[S_2,S_2]=0$.*

Characteristic equations formed by (42)

$$\frac{d\omega}{\omega}=\frac{2dx}{\mu x}=\frac{dt}{t}\ and\ \frac{du}{0}=\frac{dx}{0}=\frac{dt}{1} \tag{43}$$

Similarity solutions obtained as

$$\omega=t^{-\mu}F(\upsilon)\,and\ \upsilon=x.t^{-\mu/2} \tag{44}$$

$$\omega=F(x) \tag{45}$$

Where $F(\upsilon)$ and $F(x)$ are similarity function for (44) and (45) respectively.

Using (44) and (2) we obtain FODE

$$\partial_t^{\mu}\omega=t^{-2\mu}[F''(\upsilon)-2x^{-2}F(\upsilon)] \tag{46}$$

We use E- K fractional derivative operator to solve its L.H.S defined as

$$\left(E_{\partial}^{\ \mu,\tau}\Psi\right)(\upsilon)=\prod_{k=0}^{m-1}\left(\tau+k-\frac{1}{\partial}\upsilon\frac{d}{d\upsilon}\right)(K^{\tau+\mu,m-\mu}\Psi)(\upsilon)\ with\ \partial>0,\upsilon>0 and\ \mu>0;$$
$$m=\begin{cases}[\mu]+1, & \mu\notin N\\ \mu\ , & \mu\in N\end{cases} \tag{47}$$

And

$$(K_{\partial}^{\ \mu,\tau}\Psi)(\upsilon)=\begin{cases}\dfrac{1}{\Gamma(\mu)}\displaystyle\int_1^{\infty}(v-1)^{\mu-1}v^{-(\tau+\mu)}\Psi(\upsilon.v^{1/\partial})dv, & \mu>0\\ g(\upsilon) & ,\quad \mu=0\end{cases} \tag{48}$$

Which is Erdelyi -Kober defined fractional integral operator.

Here we need to find $\partial_t^{\mu}\omega\ where\ \lambda-1<\mu<\lambda$

The R-L fractional derivative for reduced similarity transformation (36) is

$$\frac{\partial^{\mu}\omega}{\partial t^{\mu}}=\frac{\partial^{\lambda}}{\partial t^{\lambda}}\left[\frac{1}{\Gamma(\lambda-\mu)}\int_0^t(t-s)^{\lambda-\mu-1}s^{-\mu}F\left(xs^{-\mu/2}\right)ds\right] \tag{49}$$

Assume $p=t/s$, it reduces to

$$\frac{\partial^{\mu}\omega}{\partial t^{\mu}}=\frac{\partial^{\lambda}}{\partial t^{\lambda}}\left[\frac{t^{\lambda-2\mu}}{\Gamma(\lambda-\mu)}\int_1^{\infty}(p-1)^{\lambda-\mu-1}p^{-\lambda+2\mu-1}F(\upsilon\,p^{\mu/2})dv\right]$$
$$=\frac{\partial^{\lambda}}{\partial t^{\lambda}}\left[t^{\lambda-2\mu}.(K_{\frac{2}{\mu}}^{\ 1-\mu,\lambda-\mu}f)(\upsilon)\right] \tag{50}$$

For further reduction, consider $\psi(\upsilon)\in C'(0,\infty)\ for\,\upsilon=xt^{-\mu/2}$

$$t\frac{\partial}{\partial t}\psi(\upsilon)=t.\left(-\frac{\mu}{2}xt^{\mu/2-1}\right)\psi'(\upsilon)=-\upsilon\frac{\mu}{2}\psi'(\upsilon) \tag{51}$$

Now above expression takes the form

$$\frac{\partial^{\lambda}}{\partial t^{\lambda}}\left[t^{\lambda-2\mu}.(K_{\frac{2}{\mu}}^{\ 1-\mu,\lambda-\mu}f)(\upsilon)\right]=\frac{\partial^{\lambda-1}}{\partial t^{\lambda-1}}\left[\left(t^{\lambda-2v-1}\right)\left(\lambda-2\mu-\upsilon\frac{\mu}{2}\frac{d}{d\upsilon}\right)\left[(K_{\frac{2}{\mu}}^{\ 1-\mu,\lambda-\mu}f)(\upsilon)\right]\right] \tag{52}$$

Repeating (λ-1) times to obtain

$$\frac{\partial^{\lambda}}{\partial t^{\lambda}}\left[t^{\lambda-2\mu}.(K_{\frac{2}{\mu}}^{\ 1-\mu,\lambda-\mu}f)(\upsilon)\right]=\left(t^{\lambda-2\mu-1}\right)\prod_{j=0}^{n-1}\left[\left(1+j-2\mu-\upsilon\frac{\mu}{2}\frac{d}{d\upsilon}\right)\left[(K_{\frac{2}{\mu}}^{\ 1-\mu,\lambda-\mu}f)(\upsilon)\right]\right]$$
$$=\left(t^{\lambda-2\mu-1}\right)\left(P_{2/\mu}^{1-2\mu,\lambda-\mu}f\right)(\upsilon) \tag{53}$$

But in our model $\lambda=1$, so

$$\partial_t^{\mu}\omega=t^{-2\mu}\left(P_{\frac{2}{\mu}}^{\ 1-2\mu,1-\mu}f\right)(\upsilon) \tag{54}$$

Using above expressions to form the FODE as

$$\left(P_{\frac{2}{\mu}}^{1-2\mu,1-\mu} f\right)(z) = F^{''}(\upsilon) + 2x^{-2}F(\upsilon) \tag{55}$$

Equation (55) is the reduced FODE of (2)

Using (45) and (2), we obtain an ODE

$$F''(x) - 2x^{-2}F(x) = 0 \tag{56}$$

To solve this ODE

Let

$$F(x) = \frac{a_1}{x^2} + \frac{a_2}{x} + a_0 + \beta_1 x + \beta_2 x^2 \tag{57}$$

$$F'(x) = \frac{-2a_1}{x^3} - \frac{a_2}{x^2} + \beta_1 + 2\beta_2 x \tag{58}$$

$$F''(x) = \frac{6a_1}{x^4} + \frac{2a_2}{x^3} + 2\beta_2 \tag{59}$$

Using (56-59), to obtain

$$a_0 = a_1 = \beta_1 = 0 \; and \; a_2 \neq 0, \beta_2 \neq 0 (say) \tag{60}$$

Finally, the exact solution is

$$\omega(x,t) = F(x) = \left(\frac{a_2}{x} + \beta_2 x^2\right) \tag{61}$$

Where a_2 and β_2 are arbitrary constants.

4.2 Application of fractional reduced differential transform method:

$$\begin{cases} \dfrac{\partial^\mu \omega(x,t)}{\partial t^\mu} = \dfrac{\partial^\mu \omega(x,t)}{\partial x^2} - \dfrac{2}{x^2}\omega(x,t) \\ \omega(x,0) = \dfrac{a}{x} + bx^2 \end{cases} \tag{62}$$

Where *a and b* are arbitrary constants.

In order to solve this problem applying reduced differential transforms to get

$$\frac{\Gamma(k\mu + \mu + 1)}{\Gamma(k\mu + 1)} W_{k+1} = \frac{\partial^2 W_k}{\partial x^2} - \frac{2}{x^2} W_k(x) \tag{63}$$

Substitute *k = 0 in* (63)

$$\frac{\Gamma(\mu + 1)}{\Gamma(1)} W_1 = \frac{\partial^2 W_0}{\partial x^2} - \frac{2}{x^2} W_0(x) \tag{64}$$

$$W_1 = \frac{1}{\Gamma(\mu+1)}\left[\frac{\partial^2}{\partial x^2}\left(\frac{a}{x} + bx^2\right) - \frac{2}{x^2}\left(\frac{a}{x} + bx^2\right)\right] = 0 \tag{65}$$

As $W_1 = 0$; we get a sequence of functions $\langle W_m(x) \rangle$ which are zero.

There is only exact solution $\omega(x,t) = \frac{a}{x} + bx^2$ for FPDE (2) which is independent of time also.

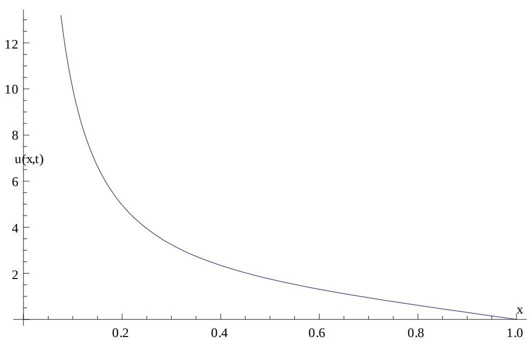

Figure2: when a=1, b= -1 and solution is independent of time

4.3 Application of homotopy analysis method

In order to solve this problem applying HAM on equation (2), we choose linear operator

$$L[\Phi(x,t;q)] = D_t^{\alpha}\Phi(x,t;q) \tag{58}$$

With $L[c_1] = 0$, c_1 is constant. We use initial approximation

$$\omega_0(x,t) = \omega(x,0) = \frac{a}{x} + bx^2 \ . \tag{66}$$

Define nonlinear operator as

$$N[\Phi(x,t;q)] = D_t^{\alpha}\Phi(x,t;q) - \Phi_{xx}(x,t;q) + \frac{2}{x^2}\Phi(x,t;q) \tag{67}$$

We construct the zeroth order deformation equation

$$(1-q)L[\Phi(x,t;q) - \omega_0(x,t)] = qhH(x,t)N[\Phi(x,t;q)] \tag{68}$$

We choose $H(x,t) = 1$ to obtain the mth order deformation equation to be

$$L[\omega_m(x,t) - \chi_m\omega_{m-1}(x,t)] = hR_m(\vec{\omega}_{m-1}) \tag{69}$$

With initial condition for $m \geq 1$, $\omega_m(x,0) = 0, \chi_m$ is defined in (34) and

$$R_m(\vec{\omega}_{m-1}) = D_t^{\alpha}\omega_{m-1} - \omega_{xx(m-1)} + \frac{2}{x^2}\omega_{m-1} \tag{70}$$

So the solution of equation (2) for m ≥ 1 becomes

$$\omega_m(x,t) = \chi_m\omega_{m-1} + hI_t^{\alpha}[R_m(\vec{\omega}_{m-1})] \tag{71}$$

From above we obtain the components of series solution by HAM successively, which is given by

$$\begin{aligned} \omega_1(x,t) &= \chi_1\omega_0 + hI_t^{\alpha}[D_t^{\alpha}\omega_0 - \omega_{0xx} + 2x^{-2}\omega_0] = 0 \\ \omega_2(x,t) &= \chi_2\omega_0 + hI_t^{\alpha}[\omega_{1t} - \omega_{1xx} + 2x^{-2}\omega_1] = 0 \end{aligned} \tag{72}$$

In same way $\omega_m(x,t) = 0$ for m = 3, 4, 5……..

Then the series solution is given by

$$\omega(x,t;h) = \frac{a}{x} + bx^2 + \sum_{i=1}^{M}\omega_i(x,t;h) = \frac{a}{x} + bx^2 \tag{73}$$

Conclusions:

In this work, the Lie symmetry technique have been used for obtaining the invariance of Burgess nonlinear model. Due to formation of two infinitesimals, we have reduced the system into two ODEs. The obtained ODEs from fractional order differential equations have been solved by power series solution and attained exact solution of FPDE (2).The proposed analysis is powerful and efficient tool in finding the solution of nonlinear time FPDEs. It is clear that this technique avoids unrealistic suppositions and liberalization. The numerical solutions found with the applications of two different methodologies FRDTM and HAM, are same as obtained by applying symmetry reduction. Finally, it is concluded that the nonlinear time FPDE can be used further in mathematical modeling or physical phenomenon in future works.

References:

1. Biswas, A., Song, M., Triki, H., Kara, A.H., Ahmad, A.H., Strong, A. & Hama, A. (2014). Solitons, shock waves, conservation laws and bifurcation analysis of boussinesq equation with power law non linearity and dual dispersion. *Appl. Math. Inf. Sci.*, 3, 949-957.
2. Biswas, A., & Khalique, C.M. (2012). Optical quasi-solitons by lie symmetry analysis. *Jour. King saud university-sci.* 24, 271-276.
3. Bakkyaraj, T., Sahdevan, R. (2015). Group formalism of lie transformations to time fractional partial differential equations. *Parmana-journal of physics*, 85, 849-860.
4. Burgess, P.K., & Kulesa, P.M.(1997). The interaction of growth rates and diffusion coefficients in three dimensional mathematical model of gliomsa. *Journal of neuropath Exp. Neur*, 56, 704-713.
5. Debnath, L. (2003). Recent applications of fractional calculus to science and enginnering. *International journal of mathematics and mathematical science*, 54, 3413-3442.
6. Gandhi, H., Tomar, A., & Singh, D. (2020). A predicted mathematical cancer tumor growth model of brain and its analytical solution by reduced differential transform method. *TCCE 2019, AISC, Springer* 1169, 203-213.
7. Gandhi, H., Singh, D., Tomar, A. (2020). Explicit solution to general fourth order time fractional KdV equation by Lie symmetry analysis. *AIP* 2253, 020012.
8. Gandhi, H., Tomar, A., & Singh, D. (2020). Lie symmetry analysis to general fifth order time fractional Korteweg-de-Vries equation and Its explicit solution, *TCCE 2019, AISC, Springer* 1169,189-201.

9. Huang, Q., & Zhdanov, R. (2014). The efficiency of lie group approach analysis of Harry- dym equation with Riemann-Liouvelle derivative. *physica A*, 209, 110-118.
10. Iyiola, O.S., & Zaman, F.D. (2014). A fractional diffusion equation model for cancer tumor. *AIP Adv*., 4, 107-121.
11. Jafari, H., Nazarib, M., Baleanuc, D., & Khaliquea, C.M. (2013). A new approach for solving a system of fractional partial differential equations. *Journal of computers and mathematics with Applications,* 66, 838–843.
12. Jena,R.M., Chakeraverty, S., & Baleanu, D. (2020). Solitary wave solution for a generalized Hirota-Satsuma coupled KdV and MKdV equations. *Alexandria Engg. Journal*, 2877-2889.
13. Kilbas, & Srivastva, H.M. (2006). Theory and applications of fractional differential equations, *Elesvier*, 204.
14. Lin, L., Zhu, S., Xu, Y., & Shi, Y. (2016). Exact solutions of Gardner equations through tanh-coth Method, *Applied Mathematics*, 7, 2374-2381.
15. Lonescu, C., Lopes, A., Copot, D., Machado, J.A.T., & Bates, J.H.T. (2017).The role of fractional calculus in modeling biological phenomena: *A review.,Commun. Nonlinear Sci NumerStimilat.* 141-159.
16. Olver, P.J. (2002). Application of Lie group symmetries to differential equations, *Graduated text in Mathematics, Springer-Verlag,*107.
17. Oldham , K.B., Spanial, J. (1974). The Fractional Calculus. *Academic press, New York*.
18. Podlubny, I. (1999). Fractional differential equations: An introduction to fractional derivatives. *Academic press, San Diego, CA*.
19. Podlubny, I. (2001). Geometric and physical interpretation of fractional integration and fractional differentiation, *Report TUKE-10-2001*, 1-17.
20. Singla, K., & Gupta, R.K. (2016). On invariant analysis of some time fractional non linear systems of partial differential equations. *Journal of mathematical physics*,57, 101504.
21. Sneddon,I.N. (1975). The use in mathematical physics of Erdelyi-kober operators and some of their generalizations. *Lecture notes, Springer New York*, 457, 37-79.
22. Shi, D., Zhang, Y., Liu, W & Liu, J. (2019). Some exact solutions and conservation laws of the coupled time fractional Boussinesq-Burgers system, *Symmetry,*11,77.
23. Tavassoli, M. H., Tavassoli, A., & Ostad Rahimi, M.R.(2013). The geometric and physical interpretation of fractional order derivatives of polynomial functions, *Diff. Geom.-Dynamical Systems,* 15, 93-104.
24. Wazwaz, A.M. (2007). The variational iteration method for solving linear and non linear systems of PDEs. *Journal computer mathematics and applications,* 54, 895-902.
25. Wang, G. (2013). Symmetry properties and explicit solutions of the non linear time fractional kdv equation, *Springer* 1, 232-245.
26. Wang, G.W., & Hashemi, M.S. (2017). Lie symmetry analysis and soliton solutions of time fractional K(m,n) equation, *Pramana-journal of physics,* 88-95.
27. Zhang, Y. (2015). Lie symmetry analysis to general time fractional korteweg-de-varies equation. *Fractional differential calculus,* 5, 125-135.
28. Zhang, Y. (2017). Lie symmetry analysis and exact solutions of general time fractional fifth order KdV equation. *Int. Journal of applied mathematics,* 47,1-10.