Analytical Solution Of Time Fractional Nonlinear Schrodinger Equation By Homotopy Analysis Method

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Abstract: In this research paper, we have applied Homotopy Analysis Approach to the "time fractional nonlinear Schrodinger equation" to find its analytical periodic and solitary wave solution. Presence of convergence control parameter in this method guarantee the solution of time fractional differential equation in the form of rapidly convergent series. Obtained analytical solution has been compared and found in good agreement. This work demonstrates reliability and potential of HAM to study the time fractional partial differential equation.

Keywords: Time Fractional Nonlinear Schrodinger (TFNS) Equation, Homotopy Analysis Method (HAM), Analytical Solution, Convergence Control Parameter, Fractional Differential Equations (FDEs)

1. Introduction

Fractional calculus literature is as ancient as classical calculus. Recently, the field of Fractional differential equations (FDEs) has attracted considerable interest from the physical as well as mathematical perspective in nonlinear phenomena. The main cause of increasing attention is due to the precise interpretation of many concepts in fluid mechanics, engineering, physics, and biology which have been characterized by fractional ordered nonlinear equations [1–6]. Studies of FDEs are also utilized to form innovative challenges in study of neurons, geology, image processing, finance and hydrology etc.

There are many descriptions about fractional derivatives which are described in Podlubny [7] like R-L derivative, Caputo derivative etc. All descriptions have their own benefit but still these definitions challenge one another. Oldham [8] recognized that generalization of these definitions have been a topic of attention in mathematics.. Debnath [9] illustrated the capabilities of fractional calculus. In recent decades, Researchers have noticed that models of fractional order promote control theory more conveniently than the classical one.

To find the solutions of nonlinear partial differential equations, physical science has developed a number of analytical methods and many efforts have been put forwarded till now. It's not that much easy to get the exact solution of nonlinear differential equations consisting of a large number of various characteristics. Consequently, the analysis of FDEs has been hindered due to inadequacy of well defined analytical methods to work with them. Rather than finding their exact solution, a few researchers have been suceesful to derive their solutions in closed or explicit form.

Homotopy Analysis Method (HAM) [10-14] is one of the recently discovered approaches, which is hybrid of the perturbation method and Homotopy, a concept in topology. It derives analytic and approximate solutions for linear as well as for nonlinear problems. Initial form of the HAM is explained by Liao [10] in his dissertation. He [11] presented an auxiliary variable $c_0 \neq 0$ in the zeroth order deformation equation to modify and monitor the convergence region and solution rate. He [12] launched an auxiliary function $H(x,t) \neq 0$ to extend more the zeroth order deformation equation. Privileges of HAM are that .it does not require discretization, small parameter, weak non-linearity assumptions and linear term in the equation. As related to other techniques, HAM offers an appropriate

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path to regulate and customize the convergence area of the series solution. The utilization of HAM has been illustrated to a number of challenges emerging from engineering and science for all kinds of initial and boundary conditions represented by nonlinear equations containing derivatives of fractional and integer order [6,10–22].

However, HAM's applications for extracting exact or approximating solution of nonlinear FDEs have not been broadly demonstrated. It has been noticed that the dense mathematical structures [23-26] are admitted by nonlinear Schrödinger (NLS) equation. Therefore, it is of great significance to study whether or not the fractional form of the above equation preserve its mathematical characteristics, which is given below:

$$\begin{aligned} & i\frac{\partial^\beta v}{\partial t^\beta}+\frac{\partial^2 v}{\partial x^2}+2\left(|v|^2+|w|^2\right)v=0, \\ & i\frac{\partial^\beta w}{\partial t^\beta}+\frac{\partial^2 w}{\partial x^2}+2\left(|v|^2+|w|^2\right)w=0, 0<\beta\leq 1,\ i=\sqrt{-1}, \end{aligned} \tag{1}$$

In the recent years, a great effort has been made in finding the exact solution of nonlinear differential equations to understand the most nonlinear physical phenomena. The fractional model of NLS equation is one of the most efficient universal models which describe various physical nonlinear systems. For example, NLS equation is appeared in study of small amplitude gravity waves on the surface of deep inviscid water. Additionally, NLSE has also appeared in the propagation of plane-diffracted wave beams in the focusing regions of the ionosphere. various applications of NLS are: dynamics in particle accelerators [27], non-uniform dielectric media, solitary waves in piezoelectric semiconductors, hydrodynamics and plasma waves, nonlinear optical waves, quantum condensates [28-31].

Schrödinger fractional model solved by various methods [32-33], among them, homotopy perturbation method [34-36], Adomian decomposition method [35,37], two dimensional differential transform methods [38], fractional Riccati expansion method [39], differential transform method [40], variational iteration method [41]. Wang and Xu [42] applied integral transforms technique to answer the space time fractional Schrodinger equation. Split-step finite difference method is employed by Wang [43] to accomplish the nonlinear Schrödinger equations. A substantial work has been done by Masemola et al. [44] who envisaged conservation laws and optical solitons for generated nonlinear Schrödinger's equation with detuning and linear attenuation. Recently, the fractional model of coupled nonlinear Schrödinger's equation has been solved by Jacobi spectral collocation method by Bhrawy et al. [45], linearly implicit Conservative difference scheme by Wang et al. [46] and Kudryashov method by Eslami [47]. The paper focuses on to find analytical solution of TFNS equation.

2. Preliminaries

2.1 Caputo fractional derivative

$$D_t^\beta\left(h(t)\right)=\frac{1}{\Gamma(g-\beta)}\int_0^t (t-\xi)^{g-\beta-1}h^g(\xi)d\xi \ for\, g-1<\beta\leq g, t>0, g\in N \tag{2}$$

2.2 R-L fractional derivative

$$D_t^\beta\left(h(t)\right)=\frac{1}{\Gamma(g-\beta)}\frac{d^g}{dt^g}\int_0^t (t-\xi)^{g-\beta-1}h(\xi)d\xi \ for\, g-1<\beta\leq g, t>0, g\in N \tag{3}$$

2.3 R-L fractional partial derivative of order beta for the function $v(x,t)$ w.r.t. t

This is the modification of above definition, which holds for the function of two variables and β is order of fractional derivative.

$$D_t^{\beta}\left(v(x,t)\right)=\frac{1}{\Gamma(g-\beta)}\frac{\partial^g}{\partial t^g}\int_0^t (t-\xi)^{g-\beta-1}v(x,\xi)d\xi \ \ for\ g-1<\beta\le g, t>0, g\in N$$
$$\frac{\partial^g v}{\partial t^g}\ when\ \beta=g \tag{4}$$

2.4 The Leibnitz rule for R-L fractional derivatives

We know Leibnitz rule is defined for the product of two functions. Hence, below is the definition of Leibnitz rule for fractional derivative of the product of two functions. $r(x,t)$ and $s(x,t)$ are function of two variable such that they are differentiable and integrable. β is order of fractional derivative.

$$D_t^{\beta}\left(r(x,t).s(x,t)\right)=\sum_{k=0}^{\infty}\binom{\beta}{k}D_t^{\beta-k}\left(r(x,t)\right).D_t^{k}\left(s(x,t)\right),\ \beta>0,$$
$$where\binom{\beta}{k}=\frac{(-1)^k\beta.\Gamma(k-\beta)}{\Gamma(1-\beta).\Gamma(k+1)} \tag{5}$$

3. Introduction to HAM

Consider the system of time FDEs.

$$\Lambda_i[v_i(z,t)]=0 \tag{6}$$

Where Λ_i time fractional differential operator, z and t are independent variables and $v_i(z,t)$ are unknown functions. Zeroth-order deformation equation constructed by Liao by means of generalizing the traditional homotopy method is given by

$$(1-q_i)L[\eta_i(z,t;q_i)-v_{i,0}(z,t)]=q_ih_iH_i(z,t)\Lambda_i[\eta_i(z,t;q_i)], \tag{7}$$

Where $q_i\in[0,1]$ is embedding parameter, $h\neq 0$ and $H\neq 0$ are controlling auxiliary parameter and auxiliary function respectively. It is important to have freedom to choose auxiliary parameter and functions. L is linear fractional auxiliary operator with the following property $L[\eta_i(z,t)]=0$ when $\eta_i(z,t)=0$. $\eta_i(z,t;q_i)$ are unknown functions and $v_{i,0}(z,t)$ are initial guess of $v_i(z,t)$. $\eta_i(z,t;0)=v_{i,0}(z,t)$ and $\eta_i(z,t;1)=v_i(z,t)$ holds when $q_i=0$ and $q_i=1$ respectively. Thus, as q_i varies from 0 to 1, the solution $\eta_i(z,t;q_i)$ varies from the initial guess $v_{i,0}(z,t)$ to the solution $v_i(z,t)$. Expanding $\eta_i(z,t;q_i)$ in Taylor series with respect to q_i, we get

$$\eta_i(z,t;q_i)=v_{i,0}(z,t)+\sum_{n=1}^{\infty}v_{i,n}(z,t)q_i^n \tag{8}$$

Where

$$v_{i,n}(z,t)=\frac{1}{n!}\frac{\partial^n\eta_i(z,t;q_i)}{\partial q_i^n}\bigg|_{q_i=0}$$

If the auxiliary linear operator, parameter, functions and initial guess are chosen properly, then the series converges at $q_i=1$ and we have

$$v_i(z,t)=v_{i,0}(z,t)+\sum_{n=1}^{\infty}v_{i,n}(z,t) \tag{9}$$

Differentiating equation n time w.r.t. q_i and then putting $q_i=0$ and dividing by $n!$, we get nth-order deformation equation.

$$L[v_{i,n}(z,t)-\chi_n v_{i,n-1}(z,t)]=h_iH_i(z,t)R_{i,n}(v_{i,n-1}) \tag{10}$$

Where

$$R_{i,n}(v_{i,n-1}) = \frac{1}{(n-1)!}\frac{\partial^{n-1}N[\eta_i(z,t;q_i)]}{\partial q_i^{n-1}} \text{ and } \chi_n = \begin{cases} 0, n \le 1, \\ 1, n > 1, \end{cases}$$

4. Application of HAM on TFNS

Equation (1) can be written in the form given below

$$\frac{\partial^\beta v}{\partial t^\beta} = i\left(\frac{\partial^2 v}{\partial x^2} + 2\left(|v|^2 + |w|^2\right)v\right).$$
$$\frac{\partial^\beta w}{\partial t^\beta} = i\left(\frac{\partial^2 v}{\partial x^2} + 2\left(|v|^2 + |w|^2\right)w\right). \qquad (11)$$

And so we define the nonlinear operators as

$$N_v(\varphi,\psi) = \frac{\partial^\beta \varphi}{\partial t^\beta} - i\left(\frac{\partial^2 \varphi}{\partial x^2} + 2\left(|\varphi|^2 + |\psi|^2\right)\varphi\right),$$
$$N_w(\varphi,\psi) = \frac{\partial^\beta \psi}{\partial t^\beta} - i\left(\frac{\partial^2 \psi}{\partial x^2} + 2\left(|\varphi|^2 + |\psi|^2\right)\psi\right), \qquad (12)$$

After following the described process in the section 3, we find the recurrence relation for the components $v_n(x,t)$ and $w_n(x,t)$

$$v_n(x,t) = (\chi_n + c_0)\left(v_{n-1}(x,t) - v_{n-1}(x,0)\right) - ic_0 J_t^\beta\left[\frac{\partial^2 v_{n-1}}{\partial x^2} + 2\sum_{j=0}^{n-1}\sum_{k=0}^{n-j-1}\left[v_j\overline{v_k} + w_j\overline{w_k}\right]v_{n-j-k-1}\right] \qquad (13)$$

$$w_n(x,t) = (\chi_n + c_0)\left(w_{n-1}(x,t) - w_{n-1}(x,0)\right) - ic_0 J_t^\beta\left[\frac{\partial^2 w_{n-1}}{\partial x^2} + 2\sum_{j=0}^{n-1}\sum_{k=0}^{n-j-1}\left[v_j\overline{v_k} + w_j\overline{w_k}\right]w_{n-j-k-1}\right] \qquad (14)$$

Equations (13) & (14) yields the terms of the infinite series solution of equation (1) as below:

$$v_1 = -ic_0 J_t^\beta\left[\frac{\partial^2 v_0}{\partial x^2} + 2\left[v_0^2\overline{v_0} + w_0\overline{w_0}v_0\right]\right], \qquad (15)$$

$$w_1 = -ic_0 J_t^\beta\left[\frac{\partial^2 w_0}{\partial x^2} + 2\left[v_0\overline{v_0}w_0 + w_0^2\overline{w_0}\right]\right], \qquad (16)$$

$$v_2 = (1+c_0)v_1 - ic_0 J_t^\beta\left[\frac{\partial^2 v_1}{\partial x^2} + 2\left[v_0^2\overline{v_1} + 2|v_0|^2 v_1 + w_0\overline{w_1}v_0 + |w_0|^2 v_1 + w_1\overline{w_0}v_0\right]\right], \qquad (17)$$

$$w_2 = (1+c_0)w_1 - ic_0 J_t^\beta\left[\frac{\partial^2 w_1}{\partial x^2} + 2\left[w_0\left(\overline{v_1}v_0 + \overline{v_0}v_1 + \overline{w_1}w_0\right) + \left(|v_0|^2 + 2|w_0|^2\right)w_1\right]\right], \qquad (18)$$

$$v_3 = (1+c_0)v_2 - ic_0 J_t^\beta\left[\frac{\partial^2 v_2}{\partial x^2} + 2\left[\begin{array}{l} v_0^2\overline{v_2} + 2|v_0|^2 v_2 + 2|v_1|^2 v_0 + 2w_0\overline{w_1}v_0 \\ +|w_0|^2 v_2 + |w_1|^2 v_0 + w_1\overline{w_0}v_1 + w_2\overline{w_0}v_0 \end{array}\right]\right],$$

(19)

$$w_3 = (1+c_0)w_2 - ic_0 J_t^\beta\left[\frac{\partial^2 w_2}{\partial x^2} + 2\left[\begin{array}{l} \overline{v_2}v_0w_0 + \overline{v_0}v_2w_0 + w_0^2\overline{w_2} + \overline{v_0}v_1w_1 + 2|w_0|^2 w_2 + \\ 2|w_1|^2 w_0 + \overline{w_0}w_1^2 + \overline{v_1}\left(v_1w_0 + v_0w_1\right) + |v_0|^2 w_2 \end{array}\right]\right], \qquad (20)$$

4.1 Periodic wave solution

Suppose $v(x,0)=a_1e^{ik_1x}$ and $w(x,0)=a_2e^{ik_2x}$, where a_1, a_2, k_1 and k_2 are real constants. Similarly following the described process in the section 3, we calculated the terms $v_n(x,t), w_n(x,t)$ of the infinite series solution of (1) as follows

$$v_0 = a_1e^{ik_1x}, \tag{21}$$

$$w_0 = a_2e^{ik_2x}, \tag{22}$$

$$v_1 = -a_1e^{ik_1x}\,\frac{iC_1c_0t^{\beta}}{\Gamma(1+\beta)}, \tag{23}$$

$$w_1 = -a_2e^{ik_2x}\,\frac{iC_2c_0t^{\beta}}{\Gamma(1+\beta)}, \tag{24}$$

$$v_2 = a_1e^{ik_1x}\left[\frac{-i(1+c_0)C_1c_0t^{\beta}}{\Gamma(1+\beta)}+\frac{(iC_1)^2c_0^2t^{2\beta}}{\Gamma(1+2\beta)}\right], \tag{25}$$

$$w_2 = a_2e^{ik_2x}\left[\frac{-i(1+c_0)C_2c_0t^{\beta}}{\Gamma(1+\beta)}+\frac{(iC_2)^2c_0^2t^{2\beta}}{\Gamma(1+2\beta)}\right], \tag{26}$$

Where $C_j = 2(a_1^2+a_2^2)-k_j^2,\ j=1,2.$ In the same way, we compute $v_{3,} w_3.....$ and so the infinite series solution of the Equation (1) is presented by the below equations

$$v(x,t) = a_1e^{ik_1x}\left[1-\frac{i(2+c_0)C_1c_0t^{\beta}}{\Gamma(1+\beta)}+\frac{(iC_1)^2c_0^2t^{2\beta}}{\Gamma(1+2\beta)}+....\right], \tag{27}$$

$$w(x,t) = a_2e^{ik_2x}\left[1-\frac{i(2+c_0)C_2c_0t^{\beta}}{\Gamma(1+\beta)}+\frac{(iC_2)^2c_0^2t^{2\beta}}{\Gamma(1+2\beta)}+....\right], \tag{28}$$

In general, this solution may not lead to closed form but if we choose $c_0=-1$ and $\beta \to 1$ then Equations from (21)-(26) become

$$v_0 = a_1e^{ik_1x}, \tag{29}$$

$$w_0 = a_2e^{ik_2x}, \tag{30}$$

$$v_1 = a_1e^{ik_1x}\,\frac{iC_1t}{1!}, \tag{31}$$

$$w_1 = a_2e^{ik_2x}\,\frac{iC_2t}{1!}, \tag{32}$$

$$v_2 = a_1e^{ik_1x}\,\frac{(iC_1)^2t^2}{2!}, \tag{33}$$

$$w_2 = a_2e^{ik_2x}\,\frac{(iC_2)^2t^2}{2!}, \tag{34}$$

Similarly we will find the remaining terms and exact periodic wave solutions is given by the below equations.

$$v(x,t) = a_1e^{i(k_1x+C_1t)} = a_1e^{i\left(k_1x+\left(2\left(a_1^2+a_2^2\right)-k_1^2\right)t\right)} \tag{35}$$

$$w(x,t) = a_2e^{i(k_2x+C_2t)} = a_2e^{i\left(k_2x+\left(2\left(a_1^2+a_2^2\right)-k_2^2\right)t\right)} \tag{36}$$

These are exactly same as given by Tan et al. [48].

4.2 Solitary wave solution

Suppose $v(x,0) = a_1 \in e^{ib_1 x} \sec h(a_1 x)$ and $w(x,0) = a_1 \in e^{c_1 + i(b_1 x + d_1)} \sec h(a_1 x)$

Where $\in = \dfrac{1}{\sqrt{1+e^{2c_1}}}$, a_1, b_1, c_1 and d_1 are real constants. Similarly following the described process in the section 3, we calculated the components $v_n(x,t), w_n(x,t)$ of the solution of (1).

$$v_0 = a_1 \in e^{ib_1 x} \sec h(a_1 x) \tag{37}$$

$$w_0 = a_1 \in e^{c_1 + i(b_1 x + d_1)} \sec h(a_1 x) \tag{38}$$

$$v_1 = \frac{-a_1 \in e^{ib_1 x} \sec h(a_1 x)[iB + 2a_1 b_1 \tanh(a_1 x)]c_0 t^{\beta}}{\Gamma(1+\beta)}, \tag{39}$$

$$w_1 = \frac{-a_1 \in e^{c_1 + i(b_1 x + d_1)} \sec h(a_1 x)[iB + 2a_1 b_1 \tanh(a_1 x)]c_0 t^{\beta}}{\Gamma(1+\beta)}, \tag{40}$$

$$v_2 = -a_1 \in e^{ib_1 x} \sec h(a_1 x)\left[B^2 - 4ia_1 b_1 B \tanh(a_1 x) - 8a_1^2 b_1^2 \tanh^2(a_1 x)\right]\frac{t^2}{2!}, \tag{41}$$

$$w_2 = -a_1 \in e^{c_1 + i(b_1 x + d_1)} \sec h(a_1 x)\left[(a_1^2 + b_1^2)^2 - 4ia_1 b_1 B \tanh(a_1 x) - 8a_1^2 b_1^2 \tanh^2(a_1 x)\right]\frac{t^2}{2!}, \tag{42}$$

Similarly the remaining terms and exact solitary wave solutions are given by the below equations.

$$v(x,t) = a_1 \in e^{i(b_1 x + Bt)} \sec h(a_1(x - 2b_1 t)) = \frac{a_1 e^{i\left(b_1 x + \left(a_1^2 - b_1^2\right)t\right)} \sec h(a_1(x - 2b_1 t))}{\sqrt{1+e^{2c_1}}}, \tag{43}$$

$$w(x,t) = a_1 \in e^{c_1 + id_1} e^{i(b_1 x + Bt)} \sec h(a_1(x - 2b_1 t)) = \frac{a_1 e^{c_1 + id_1} e^{i\left(b_1 x + \left(a_1^2 - b_1^2\right)t\right)} \sec h(a_1(x - 2b_1 t))}{\sqrt{1+e^{2c_1}}}, \tag{44}$$

The above results obtained ,are same as derived by using method called Hirota bilinearisation [49].

5. Conclusion

In this paper, we explained about HAM method and applied it on time fractional NLS equation. We derived their exact periodic wave solution in general form and particularly found in agreement with solution by Tan et. al. Then we derived analytical solitary wave solution and again found in agreement with solution given by Hirota Bilinearisation Method. Hence, we observe that Ham can be extended to time fractional equation successfully and particular solution for different conditions can be obtained from general solution.

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