Research Article

Fixed Point Property Of Lebesgue Fuzzy Metric Spaces With Contraction Conditions

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AbstractIn this paper, we using $(\alpha-\psi)-\delta-$contraction fuctions in complete fuzzy metric space and establish sequential characterization properties of Lebesgue fuzzy metric space. We prove the existence of common fixed point theorems for $(\alpha-\psi)-\delta-$ contractions mapping in fuzzy metric space using the property of Lebesgue fuzzy metric space and give a few models on the side of our outcomes. We also inaugurate some motivating results based on Lebesgue complete fuzzy $(\alpha-\psi)-\delta-$contractive mappings.

Keywords: Fuzzy metric space, Contraction mapping, Fixed point, Lebesgue property, $(\alpha-\psi)-\delta-$contraction fuctions.

1. Introduction

Topology is the study of geometric properties that does not depend only on the exact shape of the objects, but rather it acts on how the points are connected to each other. Infact, topology deals with those properties that remain invariant under the continuous transformation of a map. Zadeh (15) presented and examined the idea of a fuzzy set in his fundamental paper. The investigation of fuzzy sets started a broad fuzzy of a few numerical ideas and has applications to different parts of applied sciences. The idea of fuzzy measurement spaces was presented at first by Kramasll and Michalek (10). Banach constriction standard is unquestionably an old style aftereffect of current examination. Specifically, Mihet, (13) presented the ideas of fuzzy $\psi-$contractive mappings which grow the class of fuzzy compressions in Gregori and Sapena (6) and numerous creators abbas. Samet and vetro (14) presented the idea of $\alpha-\psi-$contractive mappings and used similar ideas to make a few intriguing fixed statement hypotheses in setting of metric spaces. Thereafter unique fixed point issues for $\alpha-\psi-$constrictions in fuzzy measurement spaces were talked about quickly by different authors $(see$ [1,2,8,11,12,13,]).

In this paper utilizing the notion of lebesgue property in unique fixed point theorem for $(\alpha-\psi)-\delta-$ contractions mapping in fuzzy metric space we prove a results which improves the recent works of Abbas et al. [1], Arora and Kumar [2], Samet et al. [14].

2. Preliminaries

Definition 2.1. The 3-tuple $(E,d,*)$ is called a fuzzy metric space if X is an arbitrary non-empty set, * is a continuous t-norm and d is a fuzzy metric in $X^2\times[0,\infty]\to[0,1]$, satisfying the following conditions: for all $x,y,z\in X$ and $t,s>0$.

$[\boldsymbol{FM.1}]$ $M(x,y,0)=0$
$[\boldsymbol{FM.2}]$ $M(x,y,t)=1\ \forall\ t>0\ iff\ \ x=y$.
$[\boldsymbol{FM.3}]$ $M(x,y,t)=M(x,y,t)$
$[\boldsymbol{FM.4}]$ $M(x,y,t)*M(x,z,s)\leq M(x,z,t+s)$
$[\boldsymbol{FM.5}]$ $M(x,y,.):[0,\infty]\to[0,1]$, Is left continuous
$[\boldsymbol{FM.6}]$ $\lim_{t\to\infty} M\,(x,y,t)=1$.

Definition 2.2. Let $(E,d,*)$ be a fuzzy metric space and let a sequence X_n in x is said to be converge to $x\in X$ if $\lim_{n\to\infty} M\,(X_n,x,t)=1$, for each $t>0$.

Definition 2.3. let (E,d) be a metric space and $T:X\to X$ be a given mapping. We say that T is an $\alpha-\psi-$contractive mapping if there exists two functions $a:X\times X\to[0,+\infty)$ and $\psi\in\Psi$: $a(x,y)d(Tx,Ty)\leq\psi\big(d(x,y)\big)$, For all $x,y\in X$.

Definition 2.4. Let $T:X\to X$ and $a:X\times X\times[0,\infty)\to[0,1]$, we say that T is $\alpha-$admissible if
$x,y\in X,\alpha(X,y,t)\leq 1\ \Rightarrow\alpha(Tx,Ty,t)\leq 1$.

3. Main Results

Theorem 3. 1. Let (E, d) be a complete Fuzzy metric space. Let (T, φ) be a fuzzy $\alpha - \psi$ −contractive mapping from (E, d) into itself satisfying the following:

(i) (T, φ) is a fuzzy α −admissible,

(ii) There exists ${f_{e_0}}^0 FC(E)$ such that $(\alpha, \phi)M\left({f_{e_0}}^0, (T, \varphi){f_{e_0}}^0\right) \leq 1$,

(iii) (T, φ) is Fuzzy continuous.

Then (T, φ) has a unique fixed point, that is, there exists $f_e \in FC(E)$ such that

$$(T, \varphi)f_e = f_e$$

Proof: Let ${f_{e_0}}^0 \in$ such that $(\alpha, \phi)M\left({f_{e_0}}^0, (T, \varphi){f_{e_0}}^0\right) \leq 1$.

Define the sequence $\{{f_{e_n}}^n\}$ in (E, d) by

$${f_{e_{n+1}}}^{n+1} = (T, \varphi){f_{e_n}}^n, \forall n \in \mathbb{N}.$$

If ${f_{e_n}}^n = {f_{e_{n+1}}}^{n+1}$, for some $n \in \mathbb{N}$, then $f_e = {f_{e_{n+1}}}^{n+1}$ is a unique fixed point of (T, φ).

Assume that ${f_{e_n}}^n \neq {f_{e_{n+1}}}^{n+1}$, $\forall\, n \in \mathbb{N}$.

Since T is a fuzzy α −admissible,

we have $(\alpha, \emptyset)M\left({f_{e_0}}^0, {f_{e_n}}^n\right) = (\alpha, \emptyset)\left({f_{e_0}}^0, (T, \varphi){f_{e_0}}^0\right) \leq 1$.

By induction, we get

$$(\alpha, \emptyset)\left({f_{e_0}}^0, {f_{e_{n+1}}}^{n+1}\right) \leq 1, \forall\, n \in \mathbb{N}. \qquad (3.1.1)$$

Applying the inequality (3.1.1) with $f_e = {f_{e_{n+1}}}^{n+1}$ and $g_e = {f_{e_n}}^n$, and using (3.1.1), we obtain

$$d\left({f_{e_n}}^n, {f_{e_{n+1}}}^{n+1}\right) = d\left((T, \varphi){f_{e_{n-1}}}^{n-1}, (T, \varphi){f_{e_{n-2}}}^{n-2}, (T, \varphi){f_{e_n}}^n\right)$$
$$\leq (\alpha, \emptyset)({f_{e_{n-1}}}^{n-1})\,(\alpha, \emptyset)({f_{e_{n-2}}}^{n-2})d\left((T, \varphi){f_{e_{n-1}}}^{n-1}, (T, \varphi){f_{e_n}}^n\right)$$
$$\leq (\alpha, \emptyset)({f_{e_{n-1}}}^{n-1})d\left((T, \varphi){f_{e_{n-1}}}^{n-1}, (T, \varphi){f_{e_n}}^n\right)$$
$$\leq (\alpha, \emptyset)d\left((T, \varphi){f_{e_n}}^n, (T, \varphi){f_{e_n}}^n\right)$$
$$\leq \psi\left(d({f_{e_{n-1}}}^{n-1}, {f_{e_n}}^n)\right)$$

By induction, we get

$$d\left({f_{e_n}}^n, \widetilde{u_{e_{n+1}}{}^{n+1}}\right) \leq \psi^n\left(d\left({f_{e_0}}^0, {f_{e_1}}^1\right)\right), \psi^n\left(d\left({f_{e_1}}^1, {f_{e_2}}^2\right)\right) \ldots., \forall n \in \mathbb{N} \qquad (3.1.2)$$

From the inequality(3.1.2) and using the triangular inequality and for $n, m \in \mathbb{N}$ with $m > n$,

$$d\left({f_{e_n}}^n, {f_{e_m}}^m\right) \leq d\left({f_{e_n}}^n, {f_{e_{n+1}}}^{n+1}\right) + d\left({f_{e_{n+1}}}^{n+1}, {f_{e_{n+2}}}^{n+2}\right) + \cdots d\left({f_{e_{m-1}}}^{m-1}, {f_{e_m}}^m\right)$$
$$= \sum_{k=n}^{m-1} d\left({f_{e_k}}^k, {f_{e_{k+1}}}^{k+1}\right)$$
$$\leq \sum_{k=n}^{m-1} \psi^k\left({f_{e_k}}^k, {f_{e_{k+1}}}^{k+1}\right)$$
$$\leq \sum_{k=n}^{m-1} \psi^k\left(\tilde{d}\left({f_{e_0}}^0, {f_{e_1}}^1\right)\right)$$
$$\leq \sum_{k=n}^{+\infty} \psi^k\left(d\left({f_{e_0}}^0, {f_{e_1}}^1\right)\right),$$

Letting $k \to \infty$, we obtain $\{{f_{e_k}}^k\}$ is a Cauchy sequence in Fuzzy metric space in (E, d). since (E, d) is complete, there exists ${f_{e_k}}^k \in FC(E)$ *such that* ${f_{e_k}}^k \to f_e$ *as* $n \to \infty$. from the fuzzy continuity of (T, φ), it follows that

$${f_{e_{k+1}}}^{k+1} = (T, \varphi){f_{e_k}}^k \to (T, \varphi)f_e$$

As $n \to \infty$. by the uniqueness of the limit, we get

$$f_e = (T, \varphi)f_e$$

Definition 3. 2. Let (E, d) be a fuzzy metric space and $(T, \varphi) : (E, d) \to (E, d)$ be a given fuzzy mapping. Then we say that (T, φ) is fuzzy (α, β) −Banach contractive mapping, if there exists two fuzzy functions $(\alpha, \psi), (\beta, \alpha) : \mathrm{FC}(E) \to \mathcal{R}(E^*)$ and $0 \leq r < 1$ such that

$$(\alpha, \psi)(f_e)(\beta, \phi)(g_{e'})\left((T, \varphi)f_e, (T, \varphi)f_e, (T, \varphi)g_{e'}\right) \leq r.\, d(f_e, g_{e'}), \quad \forall f_e, g_{e'} \in FC(E).$$

Definition 3. 3. A fuzzy metric space (E, d) is said to have the lebesgue property if given an open cover $\mathcal{G}$ *of* (E, τ_d), there exist $r \in (0,1), t > 0$ such that $\{FC(x, r, t) \colon x \in X\}$ refines $\mathcal{G}$. we call such fuzzy metric spaces lebesgue.

Proposition 3. 4. Let (E, d) be a metric space. Then (E, d) is lebesgue if and only if (E, τ_d) is lebesgue.

Definition 3. 5. τ_d is called the lebesgue topology induced by (E, d).

Theorem 3. 6. Let (E, d) be a complete fuzzy metric space. Let (T, φ) be a Lebesgue fuzzy $(\alpha - \psi) - \delta$ −contractive mapping from (E, d) into itself satisfying the following:

(i) (T, φ) is a Lebesgue fuzzy type α −admissible,

(ii) There exists $f_{e_0}{}^0 \in FC(E)$ such that $(\alpha,\psi)M\left(f_{e_0}{}^0,(T,\varphi)f_{e_0}{}^0\right) \le 1$,

(iii) If $\{f_{e_n}{}^n\}$ is a sequence in (E,d) such that $(\alpha,\psi)M\left(f_{e_n}{}^n, f_{e_{n+1}}{}^{n+1}\right) \le 1 \ \forall\ n \in \mathbb{N}$.

And $f_{e_n}{}^n \to f_e \ as\ n \to +\infty$, then $(\alpha,\phi)M\left(f_{e_n}{}^n, f_e\right) \le 1 \ \forall\ n \in \mathbb{N}$.

(a) (T,φ) is fuzzy continuous,

(b) If $\{f_{e_n}{}^n\}$ is a sequence in (E,d) such that $\left\{f_{e_n}{}^n\right\} \to f_e FC(E)\ as\ n \to \infty$ and $(\alpha,\phi)M\left(f_{e_n}{}^n\right) \le 1 \ \forall\ n \in \mathbb{N}, then\ (\alpha,\phi)M(f_e) \le 1$.

Then (T,φ) has a unique fixed point. Furthermore, if $(\alpha,\psi)f_e \le 1$ and $(\beta,\emptyset)f_e \le 1$, for all fixed point $f_e FLC(E)$, then (T,φ) has a unique fixed point.

Proof: Let $f_{e_0}{}^0 \in FLC(E)$ such that $(\alpha,\psi)f_{e_0}{}^0 \le 1$ and $(\beta,\emptyset)f_{e_0}{}^0 \le 1$. we will construct the iterative sequence $\left\{f_{e_n}{}^n\right\}$, where $f_{e_n}{}^n = (T,\varphi)f_{e_{n-1}}{}^{n-1}, for\ all\ n \in N$. Since (T,φ) is a lebesgue fuzzy sets (α,β) −admissible mapping, we have

$$(\alpha,\psi)Mf_{e_0}{}^0 \le 1 \Rightarrow (\beta,\emptyset)f_{e_1}{}^1 = (\beta,\emptyset)((T,\varphi)f_{e_0}{}^0) \le 1$$

And

$$(\beta,\emptyset)Mf_{e_0}{}^0 \le 1 \Rightarrow (\alpha,\psi)f_{e_1}{}^1 = (\alpha,\psi)((T,\varphi)f_{e_0}{}^0) \le 1$$

By similar method, we get

$$(\alpha,\psi)Mf_{e_0}{}^0 \le 1\ and\ (\beta,\emptyset)f_{e_n}{}^n \le 1, \forall\ n \in N$$

From the fuzzy set (α,β) −lebesgue condition of (T,φ), we have

$$d\left(f_{e_n}{}^n, f_{e_{n+1}}{}^{n+1}\right) = M\left((T,\varphi)f_{e_{n-1}}{}^{n-1},(T,\varphi)f_{e_n}{}^n\right)$$
$$\le (\alpha,\psi)f_{e_{n-1}}{}^{n-1},(\beta,\emptyset)f_{e_n}{}^n.M(T,\varphi)f_{e_{n-1}}{}^{n-1},(T,\varphi)f_{e_n}{}^n$$
$$\le r.M\left(f_{e_{n-1}}{}^{n-1}, f_{e_n}{}^n\right) \text{ for all } n \in N,$$

Where

$$d\left(f_{e_{n-1}}{}^{n-1}, f_{e_n}{}^n\right) = \max\{M\left(f_{e_n}{}^n, f_{e_{n-1}}{}^{n-1}\right), M((T,\varphi)f_{e_{n-1}}{}^{n-1}, f_{e_{n-1}}{}^{n-1}), M(T,\varphi)f_{e_n}{}^n, f_{e_n}{}^n)$$
$$\frac{1}{2}[\left(M(T,\varphi)f_{e_{n-1}}{}^{n-1}, f_{e_n}{}^n\right) + M(f_{e_{n-1}}{}^{n-1},(T,\varphi)f_{e_n}{}^n)]\}$$
$$= \max\{M\left(f_{e_n}{}^n, f_{e_{n-1}}{}^{n-1}\right), M(f_{e_n}{}^n, f_{e_{n-1}}{}^{n-1}), M(f_{e_{n+1}}{}^{n+1}, f_{e_n}{}^n)$$
$$\frac{1}{2}[\left(M(f_{e_n}{}^n, f_{e_n}{}^n\right) + M(f_{e_{n-1}}{}^{n-1}, f_{e_{n+1}}{}^{n+1})]\}$$
$$\le \max\{M\left(f_{e_n}{}^n, f_{e_{n-1}}{}^{n-1}\right), M\left(f_{e_{n-1}}{}^{n-1}, f_{e_{n+1}}{}^{n+1}\right)\}.$$

Thus $M\left(f_{e_n}{}^n, f_{e_{n+1}}{}^{n+1}\right) \le. \max\{M\left(f_{e_n}{}^n, f_{e_{n-1}}{}^{n-1}\right), M\left(f_{e_n}{}^n, f_{e_{n+1}}{}^{n+1}\right), M(f_{e_{n-1}}{}^{n-1}, f_{e_{n+1}}{}^{n+1})\}$

Suppose $\left(f_{e_{n-1}}{}^{n-1}, f_{e_{n+1}}{}^{n+1}\right)$ is maximum. Then

$$M\left(f_{e_n}{}^n, f_{e_{n+1}}{}^{n+1}\right) \le. M\left(f_{e_n}{}^n, f_{e_{n+1}}{}^{n+1}\right) \le M\left(f_{e_n}{}^n, f_{e_{n+1}}{}^{n+1}\right)$$

$\le M\left(f_{e_n}{}^n, f_{e_{n+1}}{}^{n+1}\right)$ is a contradiction.

Let $n, m \in N$ such that $n > m$. Then we get

$$M\left(f_{e_m}{}^m, f_{e_n}{}^n\right) \le M\left(f_{e_m}{}^m, f_{e_{m+1}}{}^{m+1}\right) + M\left(f_{e_{m+1}}{}^{m+1}, f_{e_{m+2}}{}^{m+2}\right) + \cdots + M\left(f_{e_{n-1}}{}^{n-1}, f_{e_n}{}^n\right)$$
$$\le (r^m + r^{m+1} + \cdots + r^{n-1}).M\left(f_{e_0}{}^0, f_{e_1}{}^1\right)$$
$$\le \frac{r^m}{1-r}.M(f_{e_0}{}^0, f_{e_1}{}^1)$$

Thus this implies

$M\left(f_{e_m}{}^m, f_{e_n}{}^n\right) \to 0\ as\ (m,n \to \infty)$.

So $\{f_{e_m}{}^m\}$ is a fuzzy Cauchy sequence, by the completeness of (E,d), there is a fuzzy point $f_e \in FLC(E)$ such that $f_{e_n}{}^n \to f_e\ as\ (n \to \infty)$.

Now we assume that (T,φ) is fuzzy lebesgue continuous. Then, we obtain

$$f_e = \lim_{n\to\infty} f_{e_{n+1}}{}^{n+1} = \lim_{n\to\infty}(T,\varphi)\, f_{e_n}{}^n = (T,\varphi)\left(\lim_{n\to\infty} f_{e_n}{}^n\right) = (T,\varphi)f_e$$

Now we will assume that the condition (**b**) holds. Then $(\beta,\emptyset)f_{e_n}{}^n \le 1$. thus we have for each $n \in N$,

$$M\left((T,\varphi)f_e, f_e\right) \le M\left((T,\varphi)f_e,(T,\varphi)f_{e_n}{}^n\right) + M\left((T,\varphi)f_{e_n}{}^n, f_e\right)$$
$$\le (\alpha,\psi)f_{e_n}{}^n(\beta,\emptyset)f_e.M\left((T,\varphi)f_e,(T,\varphi)f_{e_n}{}^n\right) + M\left((T,\varphi)f_{e_n}{}^n, f_e\right)$$
$$r.\mathrm{M}\left(f_{e_n}{}^n, f_e\right) + \mathrm{M}\left(f_{e_{n+1}}{}^{n+1}, f_{e_n}{}^n\right) + \mathrm{M}\left(f_{e_n}{}^n, f_{e_{n+1}}{}^{n+1}\right)$$
$$= r.\max\left\{\begin{matrix} M\left(f_{e_n}{}^n, f_e\right), M\left((T,\varphi)f_{e_n}{}^n, f_{e_n}{}^n\right), M\left((T,\varphi)f_e, f_e\right) \\ \frac{1}{2}\left[M\left((T,\varphi)f_{e_n}{}^n, f_e\right), M\left(f_{e_n}{}^n,(T,\varphi)f_e\right)\right] \end{matrix}\right\}$$
$$+M\left(f_{e_n}{}^n, f_e\right).$$

Letting $n \to \infty$, we get

$$M\big((T,\varphi)f_e, f_e\big) \leq r.M\big((T,\varphi)f_e,(T,\varphi)f_e\big) + M\big((T,\varphi)f_e, f_e\big)$$
$$M\big((T,\varphi)f_e, f_e\big)$$

This is a contradiction. Then$(T,\varphi)f_e = f_e$. This shows that f_e is a fixed point of (T,φ).

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