Research Article

Solutions Of Boundary-Value Problems For A System Of Differential Equations Of The Fourth Order With The Method Of Finite Differences

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Abstract: The paper considers a system of fourth-order ordinary differential equations. The system of equations is written in vector form. Computational algorithms are presented using the finite difference method with an error of O(h2). The resulting algebraic equations are solved by the matrix sweep method. Exact and approximate solutions of test problems are given. And also the errors of the considered numerical method are estimated.

Keywords: mechanics of a deformed solid, structural strength, elastic and elastic-plastic deformation, software packages, variational methods, computational algorithm, boundary conditions, uniform mesh, Approximations, approximation error, matrix form, matrix sweep, sweep coefficients, difference problem, reverse sweep, accuracy, error.

1. Introduction

The solution of the equations of mechanics of a deformed rigid body in general form can be obtained only numerically. Compared to the pre-computer sometimes, the possibilities of obtaining a representation and analysis of solutions have grown significantly. Until relatively recently, the only way to bring the calculation of the strength of a structure to a number was to use relatively elastic and elastic-plastic deformation problems [7-9]

Numerical calculation in many cases allows one to obtain a solution to the equations of mechanics of a deformed solid in fairly complex areas, without greatly simplifying the configuration. For this purpose, engineering software packages, both universal and specialized, have been created and are used, which allow “typing” structures in a relatively realistic geometry and carrying out calculations using complex material models [11].

The efficiency of one or another approximate solution method is known to be determined by many factors, among which the time spent on solving the problem and the accuracy of the results obtained are, apparently, the most important.

The analysis of widely used approximate methods leads to the conviction that variational methods are very laborious in the preparatory work, even if all integrals are calculated on a computer, and the finite difference method, although universal, is connected with a large number of algebraic equations.

2. Analysis and results

In this paper, we consider the question of constructing an approximate solution to a system of linear ordinary differential equations of the fourth order with variable coefficients and relatively general boundary conditions [6-10].

It is required to define in the area $[a,b]$ unknown function vector $U(x)=\{U_1(x),U_2(x),\dots,U_n(x)\}$ satisfying the system of differential equations

$$[K(x)U''(x)]''+a_5(x)[a_7(x)U''(x)]'+a_4(x)[a_6(x)U'(x)]'++a_3(x)U''(x)+a_2(x)U'(x)+a_1(x)U(x)=f(x), \quad (1)$$

written in matrix form under the boundary conditions

$$\left\{\alpha_i U(x)+\beta_i U'(x)+\gamma_i K(x)U''(x)+\theta_i\left[K(x)U''(x)\right]'\right\}\Big|_{x=a}=d_i; \quad (2)$$

$$\left\{\alpha_i U(x)+\beta_{i+2} U'(x)+\gamma_{i+2} K(x)U''(x)+\theta_{i+2}\left[K(x)U''(x)\right]'\right\}\Big|_{x=b}=d_{i+2}, \quad (3)$$

Where

$K(x), \alpha_j(x)\ \left(j=\overrightarrow{1,7}\right), d_\vartheta, \beta_\vartheta, \gamma_{\vartheta,}\theta_\vartheta\ \left(\vartheta=\overrightarrow{1,4}\right)-$
given square matrices in order n;

Let us present a computational algorithm for the above problems (1) - (3).

Let us introduce the notation

$$W(x)=K(x)U''(x) \qquad (4)$$

Let's rewrite the equation:

$$K(x)U''(x)-W(x)=0$$
$$W''(x)=a_5(a_7K^{-1}W)'+a_4(a_6U')+a_3K^{-1}W+a_2U'+a_1U=f \qquad (5)$$

Let's build a uniform mesh with a step h:

$$\overrightarrow{\omega_h}=\left\{x_i=a+ih,\quad i=0,1\ldots\ldots,N;\quad h=\frac{b-a}{N}\right\}.$$

According to the balance method [1,2], from the second equation (5) with an approximation error O (h^2) we have [3].

$$A_i^1W_{i+1}+A_i^2W_i+A_i^3W_{i-1}+A_i^4U_{i+1}+A_i^5U_i+A_i^6U_{i-1}=\overrightarrow{f_i}. \qquad (6)$$

Here

$$A_i^1=E+\frac{h}{2}a_5(x_i)a_7\left(x_{i+\frac{1}{2}}\right)K^{-1}\left(x_{i+\frac{1}{2}}\right);$$
$$A_i^2=-2E+\frac{h}{2}a_5(x_i)\left[a_7\left(x_{i+\frac{1}{2}}\right)K^{-1}\left(x_{i+\frac{1}{2}}\right)-a_7\left(x_{i-\frac{1}{2}}\right)K^{-1}\left(x_{i-\frac{1}{2}}\right)\right]+$$
$$+h\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}a_5(x)K^{-1}(x)dx;$$
$$A_i^3=E-\frac{h}{2}a_5(x_i)a_7\left(x_{i-\frac{1}{2}}\right)K^{-1}\left(x_{i-\frac{1}{2}}\right);\qquad A_i^4=a_4(x_i)a_6\left(x_{i+\frac{1}{2}}\right)++\frac{h}{2}a_2(x_i);$$
$$A_i^5=-a_4(x_i)\left[a_6\left(x_{i+\frac{1}{2}}\right)+a_6\left(x_{i-\frac{1}{2}}\right)\right]+h\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}a_1(x)dx;$$
$$A_i^6=a_4(x_i)a_6\left(x_{i-\frac{1}{2}}\right)-\frac{h}{2}a_2(x_i);\qquad \overrightarrow{f_i}=h\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}f(x)dx;$$

E- unit matrix.

I performed a similar procedure with the first equation in (5) and denoted $\begin{pmatrix}U_i\\W_i\end{pmatrix}=\vartheta_i,$ (7)

we represent the first equation (5) and equation (6) in the form [1-5].

$$A_i\vartheta_{i-1}-C_i\vartheta_i+B_i\vartheta_{i+1}=-F_i,\qquad i=1,2,\ldots,N-1, \quad (8)$$

Where

$$A_i=\begin{pmatrix}K(x_i) & 0\\ A_i^6 & A_i^3\end{pmatrix};\qquad C_i=\begin{pmatrix}2x(x_i) & h^2E\\ -A_i^5 & -A_i^2\end{pmatrix};$$
$$B_i=\begin{pmatrix}K(x_i) & 0\\ A_i^4 & A_i^1\end{pmatrix};\qquad F_i=\begin{pmatrix}0\\ \overrightarrow{f_i}\end{pmatrix};$$

Here, to find N + 1 unknown vectors, we have N + 1 matrix equations, and the missing equations are obtained on the boundary conditions (2) and (3) taking into account equation (4), using the three-point approximation for the values of the derivatives U'(x) and W'(x) with accuracy O (h^2):

$$\left.\begin{matrix}A_0\vartheta_0-C_0\vartheta_1+B_0\vartheta_2=-F_0\\ A_N\vartheta_{N-2}-C_N\vartheta_{N-1}+B_N\vartheta_N=-F_N\end{matrix}\right\} \qquad (9)$$

где

$$F_0=-2h\begin{pmatrix}d_1\\d_2\end{pmatrix};\quad B_0=-\begin{pmatrix}\beta_1 & \theta_1\\ \beta_2 & \theta_2\end{pmatrix};\quad C_0=4B_0;$$
$$A_0=2h\begin{pmatrix}\alpha_1 & \gamma_1\\ \alpha_2 & \gamma_2\end{pmatrix}++3B_0;\ A_N=\begin{pmatrix}\beta_3 & \theta_3\\ \beta_4 & \theta_4\end{pmatrix};\quad C_N=4A_N;$$
$$B_N=2h\begin{pmatrix}\alpha_3 & \gamma_3\\ \alpha_4 & \gamma_4\end{pmatrix}+3A_N;\quad F_N=-2h\begin{pmatrix}d_3\\d_4\end{pmatrix};$$

So, we have completely formulated the difference problem (8) - (9), the solution of which, based on the matrix sweep method [1,5], is sought in the form

$$\vartheta_i=X_{i+1}\vartheta_{i+1}+Z_{i+1},\qquad i=1,2\ldots,N-1; \qquad (10)$$

Where

$X_i = \{X_i^{P,S}\} \qquad p,s = 1,2 \ldots, 2n; \qquad Z_i = \{Z_{i1}, Z_{i2}, \ldots Z_{i2n}\}$

the matrix and vector sweep coefficients, respectively, determined from the relations

$X_{i+1} = (C_i - A_i X_i)^{-1} B_i; \qquad Z_{i+1} = (C_i - A_i X_i)^{-1}(F + A_i Z_i);$ (11)

Formulas for calculating the values of X_2 and Z_2, which make it possible to start counting for the sweep coefficients according to formulas (11), we obtain as follows: we multiply on the left by equation (8) for i = 1 matrix $A_0 A_1^{(-1)}$ and, subtracting the found relation from the first equation (9), we reduce to the equality

$\vartheta_1 = (C_0 - A_0 A_1^{-1} C_1)^{-1}[(B_0 - A_0 A_1^{-1} B_1)\vartheta_2 + F_0 - A_0 A_1^{-1} F_1].$ (12)

Comparing relation (12) with formula (10) for i = 1, we have

$X_2 = (C_0 - A_0 A_1^{-1} C_1)^{-1}(B_0 - A_0 A_1^{-1} B_1);$

$Z_2 = (C_0 - A_0 A_1^{-1} C_1)^{-1}(F_0 - A_0 A_1^{-1} F_1).$

Xi and Zi for all i, then solving the equations

$\vartheta_{N-1} = X_n \vartheta_N + Z_N;$

$A_{N-1}\vartheta_{N-2} - C_{N-1}\vartheta_{N-1} + B_{N-1}\vartheta_N = -F_N$

together with the second equation in (8), we obtain

$\vartheta_N = [B_N - A_N A_{N-1}^{-1} B_{N-1} - (C_N - A_N A_{N-1}^{-1} C_{N-1}) X_N]^{-1} *$

$* [(C_N - A_N A_{N-1}^{-1} C_{N-1}) Z_N - F_N - A_N A_{N-1}^{-1} F_{N-1}].$

Next, using the backward sweep (10), we calculate $\vartheta_{N-1}, \vartheta_{N-2}, \ldots \vartheta_1$. After that, we find ϑ_0 by the formula

$$\vartheta_0 = A_1^{-1}(C_1\vartheta_1 - B_1\vartheta_2 - F_1).$$

Based on the above algorithm, a computer program in the Python environment has been developed.

In this paper, we have considered the implementation algorithm for the tasks. Here are some methodological problems, the solution of which is realized by computerization. Practical results were obtained on the basis of object - oriented programming.

Consider the equations

$[(1+x)U'']'' + (2+x^3)[(2+x)U'']' + (3+x)[(4+x)U']' + (2+x^3)U'' +$

$+(5+x)U' - (1-x)U = 49x^5 + 8x^4 + 145x^3 + 91x^2 - 18x - 16$

with boundary conditions

$U(0) = U'(0) = U(1) = U'(1) = 0.$

The exact solution to this problem is as follows.

$U = x^2(1-x)^2.$

For this task, a condition can be set by direct computation to ensure that the matrix sweep method is applicable.

Table 1 shows the exact and approximate values

$U(x), \quad U'(x), \quad KU''(x), \quad [KU''(x)]'$

Table 1. Comparison of results

x	**Value**	$U(x)$	$U'(x)$	$KU''(x)$	$[KU''(x)]'$
0	Accuracy	0	0	2	-10
	Approx.	0,000000000	0,000000000	1,999975821	-10,000012714
0.25	Accuracy	0,03515625	0,1875	-0,3125	-8
	Approx.	0,035156193	0,187501317	-0,312501726	-8,000017324
0.5	Accuracy	0,0625	0	-1,5	-1
	Approx.	0,062499768	0,000001473	-1,499974161	-0,999993519
0.75	Accuracy	0,0351625	0,1875	-0,432501765	1,25
	Approx.	0,035156194	0,187501324	0,4325	1,249976434
1	Accuracy	0	0	4	26
	Approx.	0,0000010151	0,000000421	4,000001147	25,99945677

Consider the following equation

$[(1+x)U''(x)]''+xU''(x)-2U(x)=6[6(2+2x)+x^2(1-2x^2)]$

Under boundary conditions

$U(0)=U'(0)=0;\quad U''(1)-9U(1)=0;\quad U''(1)=\frac{30}{7}U'(1)=0.$

The exact solution to the problem will be as follows:

$U(x)=x^3(1+x).$

Table 2. gives exact and approximate values for

$U(x),\quad U'(x),\ KU''(x),\quad [KU''(x)]'$

Table 2: Comparison of results

x	Value	$U(x)$	$U'(x)$	$KU''(x)$	$[KU''(x)]'$
0	Accuracy	0			
	Approx.	0,000000000	0,000000000	0,000000000	6,000033271
0.25	Accuracy	0,0195314	0,25	2,8125	17
	Approx.	0,019530753	0,249994613	2,81254201	17,00033706
0.5	Accuracy	0,1875	1,25	9	33
	Approx.	0,1874994997	1,249995918	9,00002783	33,00028527
0.75	Accuracy	0,73828053	3,375	12,803750	51
	Approx.	0,738281791	3,374998643	12,80371953	51,00017631
1	Accuracy	2	7	36	78
	Approx.	2,000001120	6,999945675	36,00005231	77,999766129

3. Conclusion/Recommendations

It can be seen from the above tabular data that the accuracy of determining the numerical results agrees well with the error of the approximation method. The integration steps were taken into account with an accuracy of h = 0.001. Numerous other computer calculations have shown that the above computational algorithms stably determine the calculated values within a fairly wide range of changes in the input parameters of the problems under consideration.

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