Research Article

Coefficient Estimates of a New Subclass of Biunivalent Functions

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Abstract. In this paper, we try to extend and obtain some more results inspired by P. Goswami and Aljouiee [9]. Here we are introducing a new subclass of biunivalent functions by using q-derivative operator, quasi-subordination and convolution analytic bi-univalent functions. Also we find both some initial and general coefficient bounds.

Keywords. Univalent Functions, Convex and q-convex Functions, Starlike and q-starlike Functions, q-number and Generalized Confluent Hypergeometric Function

1. Introduction and Preliminary

Let $f(z)$ be analytic and univalent in $\triangle$. Then, since $f'(0) \neq 0$, the function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

In the open unit disk $\triangle$defined as $\triangle = \{z: z \in C \text{and } |z| < 1\}$, these functions are analytic and follows the normalization condition $f(0) = f'(0) - 1 = 0$

Assume subclass S of A to be univalent in $\triangle$. According to koebe one quarter theorem [1], all the functions belonging to S has their inverse in $\triangle$.Therefore if $f \in S$, then we have f^{-1}defined as

$$f^{-1}(f(z)) = z, \qquad (z \in \triangle)$$

and

$$f^{-1}(f(w)) = w, \qquad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right)$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots \tag{1.2}$$

f *is* said to be biunivalent function if its inversef^{-1} is also univalent in $\triangle$. We denote the class of biunivalent function by symbol σ.

Suppose *M* is class having functions which are of the form,

$$\phi(z) = 1 + \sum_{n=1}^{\infty} \phi_n z^n \tag{1.3}$$

and are also regular in $\triangle$.

Definition 1.1. [2] Let $P_m(\gamma)$ denote the class of analytic functions $K(z)$ in $\triangle$, satisfying the properties $K(0) = 1$, and

$$\int_0^{2\pi} \left|\frac{RK(z) - \gamma}{1 - \gamma}\right| d\theta \leq m\pi,$$

where $z = re^{i\theta}, m \geq 2 and 0 \leq \gamma < 1$.

For $m = 2, P_2(\gamma) = P(\gamma)$. When $\gamma = 0, P_m(\gamma)$ reduces to the class $P_m(0) = P_m$, defined by Pinchuk [3]. And with the help of this we get the class $P_2(0) = P$of caratheodory function of positive real parts.

Many mathematicians have worked in the field of biunivalent functions and obtained interesting results. The class σ of biunivalent functions was first investigated by Lewin [4]. He also found the bound for second coefficient. Certain subclasses of biunivalent functions similar to the subclasses of starlike, strongly starlike and convex functions are studied by Brannan and Taha [5].

In recent years, various researchers like Goyal and Goswami [8], Ali et al. [6], Aljouiee et al. [9], Srivastava et al. [7] have worked on the subclasses of bi-univalent functions and found the initial coefficient bounds.

Robertson [10], in 1970, introduced concept of quasi-subordination which is defined as follows:

Definition 1.2. If $f(z)$ and $K(z)$ be analytic function in $\triangle$, them $f(z)$ is quasi-subordinate to $K(z)$ in $\triangle$,i.e.

$$f(z) \prec_q K(z), \qquad (z \in \triangle)$$

if there exist an analytic function ψ, $(|\psi(z)| \leq 1)$, such that $\left(\frac{f(z)}{\psi(z)}\right)$is analytic in $\triangle$,and

$$\left(\frac{f(z)}{\psi(z)}\right) \prec K(z), \qquad (z \in \triangle)$$

i.e. there exist the Schwarz function $w(z)$ such that

$$f(z) = \psi(z).K(w(z))$$

And we know from [1] that $f(z)$ is subordinate to $K(z) i.e. f(z) \prec K(z)$, if there exist a Schwarz functions $w(z)$ in $\triangle$ such that $f(z) = K(w(z))$, with $w(0) = 0$and $|w(z)| < 1, (z \in \triangle)$.

Jacson [11], in 1908, introduced the concept of q-derivative, which is defined as follows:

Definition 1.3. The q-derivative of a function $f$is defined on a subset of $C$is given by

$$(D_q f)(z) = \frac{f(z)-f(zq)}{(1-q)z}, (z \neq 0) \quad (1.4)$$

and$(D_q f)(z) = f^{'}(0)$ provided $f'(0)$ exists.

If $f$is differential, then

$$\lim_{q\to 1^-} (D_q f)(z) = \lim_{q\to 1^-} \frac{f(z)-f(zq)}{(1-q)z} = \frac{df(z)}{dz},$$

From (1.4) and (1.1), we get

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} \quad (1.5)$$

Where $[n]_q = \frac{1-q^n}{1-q}, (q \neq 1)$

Definition 1.4. If $f(z)$ be a function defined by (1.1), then for any function$l(z)$ of the form,

$$l(z) = z + \sum_{n=2}^{\infty} l_n z^n$$

Convolution of $f(z)$ and $l(z)$ is defined by,

$$(f * l)(z) = z + \sum_{n=2}^{\infty} a_n l_n z^n, \ z \in \triangle \quad (1.6)$$

Sahsene Altinkaya [11] in 2018 introduced the class $T_\sigma(q,\curlywedge)$ and obtain the upper bounds for coefficient of functions of this subclass.

A function $f \in \sigma$is in $T_\sigma(q,\curlywedge)$, $(\curlywedge \geq 1)$ if satisfy the condition as follows:

$$(1-\curlywedge)\frac{f(z)}{z} + \curlywedge (D_q f)(z) \prec_q \psi(z)$$

And

$$(1-\curlywedge)\frac{F(w)}{w} + \curlywedge (D_q F)(w) \prec_q \psi(w)$$

where $F = f^{-1}$, and $\psi \in M$be univalent in $\triangle$and $\psi\ (\triangle)$ be symmetrical about the real axes with $\psi'(0) > 0$.

Definition 1.5. Let $\psi \in M$be an univalent function in $\triangle$and let $\psi(\triangle)$ be symmetrical about the real axis with $\psi'(0) > 0$. A function $f \in \sigma$, is in the class $M_\sigma^\alpha(q,\curlywedge), (\curlywedge \geq 1, \alpha \in R)$, if it satisfy the conditions given below:

$$(1-\curlywedge)\left(\frac{f(z)}{z}\right)^\alpha + \curlywedge \left((D_q f)(z)\right)^\alpha \prec_q \psi(z), (z \in \triangle) \quad (1.7)$$

and

$$(1-\curlywedge)\left(\frac{F(w)}{w}\right)^\alpha + \curlywedge \left((D_q F)(w)\right)^\alpha \prec_q \psi(w), (w \in \triangle) \quad (1.8)$$

where $F = f^{-1}$

Considering these definitions, we will define a new subclass of bi-univalent functions by q-derivative and convolution, and also obtain general and initial coefficient bounds by means of Taylor expansion formula.

1. Main Results

Lemma 2.1. [3] Suppose ξbe a function defined by $\xi(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$is convex in $\triangle$. If $\xi(z) \in P_m$, then $|c_n| \leq m$, $(m \in N)$

Definition 2.1.A function $f(z) \in \sigma$, is said to be in class $M_\sigma^\alpha(f,l;\curlywedge;t), for \curlywedge \geq 0, t \in (1/2,1], \alpha \in R$, if the following condition is satisfied:

$$(1-\curlywedge)\left(\frac{(f*l)(z)}{z}\right)^\alpha + \curlywedge \left((D_q(f*l))(z)\right)^\alpha \prec_q \psi(z), (z \in \triangle)$$

and

$$(1-\curlywedge)\left(\frac{(F*l)(w)}{w}\right)^\alpha + \curlywedge \left((D_q(F*l))(w)\right)^\alpha \prec_q \psi(w), (w \in \triangle)$$

where$F = f^{-1}$.

This is very clear from the above definition that $f \in M_\sigma^\alpha(f,l;\curlywedge;\ t)$, if there exist a function $h(|h(z)| \leq 1)$, satisfying following conditions:

$$\frac{(1-\curlywedge)\left(\frac{(f*l)(z)}{z}\right)^\alpha + \curlywedge\left((D_q(f*l))(z)\right)^\alpha}{h(z)} \prec \psi(z), \qquad (z \in \triangle) \qquad (2.1)$$

and

$$\frac{(1-\curlywedge)\left(\frac{(F*l)(w)}{w}\right)^{\alpha}+\curlywedge\left((D_q(F*l))(w)\right)^{\alpha}}{h(w)} \prec \psi(w), \qquad (w \in \triangle) \qquad (2.2)$$

where $F = f^{-1}$. Here we suppose that $\psi \in M$is of the form

$$\psi(z) = 1 + c_1 z + c_2 z^2 + \ldots, (c_n > 0, z \in \triangle)$$

and the function h analytic in $\triangle$is taken as

$$h(z) = X_0 + X_1 z + X_2 z^2 + \ldots, (|h(z)| \leq 1, z \in \triangle)$$

Now our main results are as follows:

Theorem 2.1. Let *f* be function given by (1.1) be in the class $M_{\sigma}^{\alpha}(f, l; \curlywedge; t)$, if $a_m = 0$, *for* $2 \leq m \leq n-1$, then

$$|a_n| \leq \frac{c_1 + |X_{n-1}|}{\alpha[1+([n]_q - 1)\curlywedge]|l_n|}, \qquad (n > 3).$$

Proof: We have

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

$$l(z) = z + \sum_{n=2}^{\infty} l_n z^n,$$

so

$$(f * l)(z) = z + \sum_{n=2}^{\infty} a_n l_n z^n$$

Now

$$\left[\frac{(f*l)(z)}{z}\right]^{\alpha} = \left[1 + \sum_{n=2}^{\infty} a_n l_n z^{n-1}\right]^{\alpha}$$

and

$$\left[\left(D_q(f*l)\right)(z)\right]^{\alpha} = \left[1 + \sum_{n=2}^{\infty} [n]_q a_n l_n z^{n-1}\right]^{\alpha}$$

Denoting

$$N(z) = \left(\frac{(f*l)(z)}{z}\right)^{\alpha}; Q(z) = [(D_q(f*l))(z)]^{\alpha};$$

$$V(w) = \left(\frac{(F*l)(w)}{w}\right)^{\alpha}; W(w) = [(D_q(F*l))(w)]^{\alpha}.$$

Then we have

$$(1-\curlywedge)N(z) + \curlywedge\, Q(z) \prec_q \psi(z), \qquad (2.3)$$

and

$$(1-\curlywedge)V(w) + \curlywedge\, W(w) \prec_q \psi(w), \qquad (2.4)$$

By Taylor expansion formula we obtain

$$N(z) = \left(\frac{(f*l)(z)}{z}\right)^{\alpha} = N(0) + zN'(0) + \frac{z^2}{2!}N''(0) + \ldots + \frac{z^n}{n!}N^{(n)}(0) + \ldots$$

We can calculate

$$N(0) = 1,$$
$$N'(0) = \alpha a_2 l_2$$
$$N''(0) = \alpha(\alpha-1)(a_2 l_2)^2 + 2\alpha a_3 l_3$$
$$N'''(0) = \alpha(\alpha-1)(\alpha-2)(a_2 l_2)^3 + 6\alpha(\alpha-1)a_2 a_3 l_2 l_3 + 3!\,\alpha a_4 l_4$$

…

$$N^{(n-1)}(0) = B(\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1), a_2, a_3, \ldots, a_{n-1}, l_2, l_3, \ldots l_{n-1,}) + \alpha(n-1)!\,a_n l_n,$$

where $B(\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1), a_2, a_3 \ldots, a_{n-1}, l_2, l_3, \ldots l_{n-1},)$ is the sum of the functions formed by the product of $\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1), a_2, a_3 \ldots, a_{n-1}, l_2, l_3, \ldots l_{n-1}$and atleast one of the product factor is $a_i l_i, 2 \leq i \leq n-1, so$

$$N(z) = 1 + \alpha a_2 l_2 z + \frac{z^2}{2!}[\alpha(\alpha-1)a_2^2 l_2^2 + 2\alpha a_3 l_3]$$

$$+\frac{z^3}{3!}[\alpha(\alpha-1)(\alpha-2)a_2^3 l_2^3 + 3!\,\alpha(\alpha-1)a_2 a_3 l_2 l_3 + 3!\,\alpha a_4 l_4] + \ldots$$

$$+\frac{z^{n-1}}{(n-1)!}[B(\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1), a_2, a_3, \ldots, a_{n-1}, l_2, l_3, \ldots l_{n-1}) + \alpha(n-1)!\,a_n l_n] + \ldots \qquad (2.5)$$

Now,

$$Q(z) = [(D_q(f*l))(z)]^\alpha = \left[1 + \sum_{n=2}^{\infty}[n]_q a_n l_n z^{n-1}\right]^\alpha$$
$$= [1 + [2]_q a_2 l_2 z^1 + [3]_q a_3 l_3 z^2 + \ldots]^\alpha$$

By Taylor expansion formula,

$$Q(z) = Q(0) + zQ'(0) + \frac{z^2}{2!}Q''(0) + \ldots + \frac{z^n}{n!}Q^{(n)}(0) + \ldots$$

By calculations, we get

$$Q(0) = 1,$$
$$Q'(0) = \alpha[2]_q a_2 l_2,$$
$$Q''(0) = \alpha(\alpha-1)\left([2]_q a_2 l_2\right)^2 + 2\alpha[3]_q a_3 l_3,$$
$$Q'''(0) = \alpha(\alpha-1)(\alpha-2)\left([2]_q a_2 l_2\right)^3 + 6\alpha(\alpha-1)\left([2]_q a_2 l_2\right)\left([3]_q a_3 l_3\right) + 3!\,\alpha\left([4]_q a_4 l_4\right),$$

…

$$Q^{(n-1)}(0) = Y(\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1), a_2, a_3, \ldots, a_{n-1}, l_2, l_3, \ldots l_{n-1}) + \alpha(n-1)!\,[n]_q a_n l_n,$$

Therefore, we get

$$Q(z) = 1 + (\alpha[2]_q a_2 l_2)z + \frac{z^2}{2!}(\alpha(\alpha-1)([2]_q a_2 l_2)^2 + 2\alpha[3]_q a_3 l_3) + \ldots$$
$$+ \frac{z^{n-1}}{(n+1)!}[Y(\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1), a_2, a_3, \ldots, a_{n-1}, l_2, l_3 \ldots l_{n-1}) + \alpha(n-1)!\,[n]_q a_n l_n] + \ldots \quad (2.6)$$

where $Y(\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1), a_2, a_3, \ldots, a_{n-1}, l_2, l_3 \ldots l_{n-1})$ the sum of the functions formed by the product of $\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1), a_2, a_3, \ldots, a_{n-1}, l_2, l_3 \ldots l_{n-1}$ and at least one of the product factors is $a_i l_i, 2 \le m \le n-1,$

Using (2.5) and (2.6) in (2.3), the coefficients of $z^{n-1}, if\ a_m = 0$ for $2 \le i \le n-1$, is given by

$$[1 + ([n]_q - 1) \curlywedge]\alpha a_n l_n$$

Similarly, we can find the coefficient of w^{n-1} in (2.4), i.e.

$$[1 + ([n]_q - 1) \curlywedge]\alpha b_n l_n$$

Where

$$F(w) = w + \sum_{n=2}^{\infty} b_n w^n, \qquad F = f^{-1}$$

From definition (2.2), it is clear that there exist two Schwarz functions $\phi(z) = \sum_{n=1}^{\infty} d_n z^n$ and $\varphi(w) = \sum_{n=1}^{\infty} s_n w^n, |d_n| \le 1, |s_n| \le 1$, such that

$$(1-\curlywedge)\left[\frac{(f*l)(z)}{z}\right]^\alpha + \curlywedge\left[(D_q(f*l))(z)\right]^\alpha = h(z)\psi(\phi(z)), \quad (2.7)$$

and

$$(1-\curlywedge)\left[\frac{(F*l)(w)}{w}\right]^\alpha + \curlywedge\left[(D_q(F*l))(w)\right]^\alpha = h(w)\psi(\phi(w)), \quad (2.8)$$

Thus from definition (2.2), and (2.7)

$$[1 + ([n]_q - 1) \curlywedge]\alpha a_n l_n = X_{n-1} + \sum_{t=1}^{\infty}\sum_{k=1}^{\infty} c_k \triangle_n^k (d_1, d_2, \ldots d_n).X_{n-(t+1)}, (X_0 = 1) \quad (2.9)$$

Similarly by definition (2.2) and (2.8), we get

$$[1 + ([n]_q - 1) \curlywedge]\alpha b_n l_n = X_{n-1} + \sum_{t=1}^{\infty}\sum_{k=1}^{\infty} c_k \triangle_n^k (s_1, s_{2\ldots,} s_n).X_{n-(t+1)},$$

For $a_m = 0, \quad (2 \le m \le n-1)$, we have $b_n = -a_n$ and so

$$\alpha[1 + ([n]_q - 1) \curlywedge]a_n l_n = \alpha a_n l_n + \alpha \curlywedge ([n]_q - 1) = c_1 d_{n-1} + X_{n-1} \quad (2.10)$$

and

$$\alpha[1 + ([n]_q - 1) \curlywedge]b_n l_n = c_1 s_{n-1} + X_{n-1} \quad (2.11)$$

Now taking the absolute value of the above equations, we get

$$|a_n| = \frac{|c_1 d_{n-1} + X_{n-1}|}{|\alpha[1 + ([n]_q - 1)\curlywedge]||l_n|} \le \frac{c_1 + |X_{n-1}|}{\alpha[1 + ([n]_q - 1)\curlywedge]|l_n|}, \qquad (n > 3).(2.12)$$

This completes proof.

Theorem 2.2. Let the function $f \in M_\sigma^\alpha(f, l; \curlywedge; t)$, be given by (1.1). If $a_k = 0$ for $2 \le k \le n-1$, then we have

$$|a_n| \le \frac{m(1-\curlywedge)}{\alpha|l_n|}, \qquad (n \ge 3)$$

Proof: We have from (2.5)

$$\left(\frac{(f*l)(z)}{z}\right)^\alpha = 1 + \alpha a_2 l_2 z + \frac{z^2}{2!}(\alpha(\alpha-1)(a_2 l_2)^2 + 2\alpha a_3 l_3) + \ldots + \frac{z^{n-1}}{(n-1)!}$$
$$[B(\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1), a_2, a_3, \ldots, a_{n-1}, l_2, l_3 \ldots l_{n-1} + \alpha(n-1)!\,a_n l_n] + \ldots, (2.13)$$

Similarly, for $F = f^{-1} = w + \sum_{n=2}^{\infty} b_n w^n,$

$$\left(\frac{(F*l)(w)}{w}\right)^{\alpha} = 1+\alpha b_2 l_2 w + \frac{w^2}{2!}(\alpha(\alpha-1)(b_2l_2)^2 + 2\alpha b_3 l_3) + \ldots + \frac{w^{n-1}}{(n-1)!}$$
$$[A(\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1), b_2, b_3, \ldots, b_{n-1}, l_2, l_3 \ldots l_{n-1}) + \alpha(n-1)!\, b_n l_n], \quad (2.14)$$

By definition and Lemma (2.1), there exist two functions

$$u(z) = 1+\sum_{n=1}^{\infty} u_n z^n \in P_m, \quad (2.15)$$
$$v(w) = 1+\sum_{n=1}^{\infty} v_n w^n \in P_m, \quad (2.16)$$
$$|u_n| \le m, |v_n| \le m,$$

such that

$$\left(\frac{(f*l)(z)}{z}\right)^{\alpha} = \lambda + (1-\lambda)u(z)$$
$$= 1 + (1-\lambda)u_1 z + (1-\lambda)u_2 z^2 + \ldots, \quad (2.17)$$
$$\left(\frac{(F*l)(w)}{w}\right)^{\alpha} = \lambda + (1-\lambda)v(w)$$
$$= 1 + (1-\lambda)v_1 w + (1-\lambda)v_2 w^2 + \ldots, \quad (2.18)$$

Now comparing the coefficients of (2.13) and (2.17),

$$\frac{1}{(n-1)!}[B(\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1), a_2, a_3, \ldots, a_{n-1}, l_2, l_3, \ldots l_{nn-1} + \alpha(n-1)!\, a_n l_n)] = (1-\lambda)u_{n-1}$$
(2.19)

also comparing the coefficients of (2.14) and (2.18)

$$\frac{1}{(n-1)!}[A(\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1), b_2, b_3, \ldots, b_{n-1}, l_2, l_3, \ldots l_{n-1} + \alpha(n-1)!\, b_n l_n)] = (1-\lambda)v_{n-1}$$
(2.20)

If $a_k, l_k = 0$ for $2 \le k \le n-1$, then

$$\frac{\alpha(n-1)!\, a_n l_n}{(n-1)!} = (1-\lambda)u_{n-1}$$

or

$$a_n = \frac{1}{\alpha l_n}(1-\lambda)u_{n-1}$$

Similarly

$$b_n = \frac{1}{\alpha l_n}(1-\lambda)v_{n-1}$$

Taking absolute value, we get

$$|a_n| \le \frac{(1-\lambda)|u_{n-1}|}{\alpha|l_n|} \le \frac{(1-\lambda)m}{\alpha|l_n|} \quad (2.21)$$

Here we get the desired result.

If we relax the condition $a_k = 0$ for $2 \le k \le n-1$, then we have the following consequence:

Corollary 2.1. If $a_k \ne 0$ for $2 \le k \le n-1$, then we have,

$$|a_2| \le \begin{cases} \sqrt{\dfrac{2m(1-\lambda)}{\alpha[(\alpha-1)|l_2|^2 + 2|l_3|]}}, & 0 \le \lambda \le 1 - \dfrac{2\alpha|l_2|^2}{m[(\alpha-1)|l_2|^2 + 2|l_3|]} \\ \dfrac{m(1-\lambda)}{\alpha|l_2|}, & 1 - \dfrac{2\alpha|l_2|^2}{m[(\alpha-1)|l_2|^2 + 2|l_3|]} \le \lambda < 1 \end{cases}$$

and

$$|a_3| \le \begin{cases} \sqrt{\dfrac{2m(1-\lambda)}{\alpha[(\alpha-1)|l_2|^2 + 2|l_3|]}}, & 0 \le \lambda \le 1 - \dfrac{2\alpha|l_2|^2}{m[(\alpha-1)|l_2|^2 + 2|l_3|]} \\ \dfrac{m^2(1-\lambda)^2}{\alpha^2|l_2|^2} + \dfrac{m(1-\lambda)}{\alpha|l_3|}, & 1 > \lambda \ge 1 - \dfrac{2\alpha|l_2|^2}{m[(\alpha-1)|l_2|^2 + 2|l_3|]}, \end{cases}$$

Proof: If $a_k \ne 0$ for $2 \le k \le n-1$, then from (2.21), we have

$$|a_2| \le \frac{(1-\lambda)m}{\alpha|l_2|} \quad (2.22)$$

Again, on comparing the coefficients of z^2 in (2.13) and (2.17), we get

$$\frac{1}{2!}(\alpha(\alpha-1)(a_2l_2)^2 + 2\alpha a_3 l_3) = (1-\lambda)c_2 \quad (2.23)$$

Using (2.14) and (2.18), comparing the coefficients of w^2, we get

$$\frac{1}{2!}(\alpha(\alpha-1)(a_2l_2)^2 + 2\alpha(2a_2^2 - a_3)l_3) = (1-\lambda)d_2 \quad (2.24)$$

Now adding (2.23) and (2.24), we get

$$a_2^2 = \frac{(1-\lambda)(c_2+d_2)}{\alpha((\alpha-1)l_2^2 + 2l_3)}$$

taking absolute value, we get the result.

Again subtracting (2.24) from (2.23), we get

$$a_3 = \frac{(1-\lambda)(c_2-d_2)+2\alpha a_2^2 l_3}{2l_3\alpha} \tag{2.25}$$

Taking absolute value, and using the value of $|a_2|^2$, we get the required result.
Remark 2.1 For $\alpha = 1$, we get,

$$|a_2| \le \begin{cases} \frac{\sqrt{m(1-\lambda)}}{|l_3|}, 0 \le \lambda \le 1 - \frac{|l_2|^2}{m|l_3|} \\ \frac{m(1-\lambda)}{|l_2|}, 1 - \frac{|l_2|^2}{m|l_3|} \le \lambda < 1 \end{cases}$$

and

$$|a_3| \le \begin{cases} \frac{2m(1-\lambda)}{|l_3|}, 0 \le \lambda \le 1 - \frac{|l_2|^2}{m|l_3|} \\ \frac{m^2(1-\lambda)^2}{|l_2|^2} + \frac{m(1-\lambda)}{|l_3|}, 1 - \frac{|l_2|^2}{m|l_3|} \le \lambda < 1 \end{cases}$$

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