Formulas of General Solution for Linear System from Ordinary Differential Equations by Using Novel Transformation

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Abstract: In this work, we use Novel transform which has the form $N_I\big(H(t)\big) = \frac{1}{\rho}\int_0^\infty e^{-\rho t} H(t)dt$ to solve a system of linear differential equations, while homogeneous or non-homogeneous system. Moreover, general formula of the set solution of systems of first and second order are derived.

Keywords: Novel Transform, Linear System, Constant Coefficients.

1. Introduction

Linear system have great importance in applied mathematics and have a great role in other sciences such as physics, chemistry and other sciences [5]. In the last two centuries, integral transforms have been used successfully to solve many issues in mathematics. These transforms have been used on a large scale to solve differential equations [9].

It has also been extensively in physics, astronomy, and engineering, and most of these transformation are derived from the Laplace transform and Fourier transform [8], which have been used to solve ordinary and partial differential equations Elzaki, Shehu, Sumudu, Temimi,⋯ , etc. [3, 10, 4, 2].

In 2016 introduced a new integral transform which used to solve linear equations with constant coefficients, called the Novel transform[1]. It is also used to solve differential equations arising from the heat transform problem and solve other differential equations [7 ,6,11].

In this paper, some formulas of general solution for system of one and second order in dimension n, whereas homogenous or non-homogenous from using Novel transform.

In section 2, the definitions, properties and Novel transform for some fundamental functions. In section 3, we derive the general formula for a system of first order in dimension n, while homogeneous or non-homogeneous by using Novel transform. In last section, we used these formulas to solve some examples.

2. Basic Definitions and Properties of Novel Transform

The Novel transform for the function $H(t), t > 0$ is defined by the following integer:

$$\Omega(\rho) = N_I\big(H(t)\big) = \frac{1}{\rho}\int_0^\infty e^{-\rho t} H(t)dt, t > 0 \qquad (2.1)$$

where $H(t)$is a real function, $\frac{e^{-\rho t}}{\rho}$ is the kernel function, and N_I is the operator of Novel transform.

The inverse of Novel transform is given by:

$$N_I^{-1}\Omega(\rho) = H(t) \text{ for } t > 0 \qquad (2.2)$$

where N_I^{-1} returns the transformation to the original function.

To display the duality relationship Novel transform and Laplace transform, which Laplace transform

Defined by:

$$g(\rho) = L_I\big(H(t)\big) = \frac{1}{\rho}\int_0^\infty e^{-\rho t}H(t)dt\,,\ t > 0\,,\cdots \qquad (2.3)$$

Where L_I is the operator of LT.

$$\Omega(\rho) = N_I\big(H(t)\big) = \frac{1}{\rho}\int_0^\infty e^{-\rho t}H(t)dt\cdots \qquad (2.4)$$

$$\Omega(\rho) = \frac{1}{\rho}L_I\big(H(t)\big) = \frac{1}{\rho}g(\rho),\ t > 0\cdots \qquad (2.5)$$

Property: If $H_1(t)\,, H_2(t)\,, \dots, H_n(t)$ have Novel transform then:

$$N_I(\alpha_1 H_1(t) + \alpha_2 H_2(t) + \cdots + \alpha_n H_n(t)) = \alpha_1 N_I(H_1(t)) + \alpha_2 N_I(H_2(t)) + \cdots + \alpha_n N_I(H_n(t)) \ (2.6)$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are constants, the functions $y_1(t), y_2(t), \dots, and y_n(t)$are defined.

Theorem (2-1): [6] Novel transform of derivative.

If the function $H^{(n)}(t)$ is the derivative of the function $H(t)$with respect to t then its Novel transform is defined by:

$$N_I\big(H'(t)\big) = \rho N_I(H(t)) - \frac{H(0)}{\rho} \qquad (2.7)$$

$$N_I(H''(t)) = \rho^2 N_I(H(t)) - H(0) - \frac{H'(0)}{\rho} \qquad (2.8)$$

$$N_I(H'''(t)) = \rho^3 N_I(H(t)) - \rho H(0) - H'(0) - \frac{H''(0)}{\rho} \qquad (2.9)$$

where n represent the derivatives, $n \in N$

$$N_I\left(H^{(n)}(t)\right) = \rho^n N_I\big(H(t)\big) - \rho^{n-2}H(0) - \rho^{n-3}H'(0) - \cdots - H^{(n-2)}(0) - \frac{1}{\rho}H^{(n-1)} \qquad (2.10)$$

where, $H^n(t)$ is the n-order derivative of $H(t)$.

Table 1. The Novel transform for some function

ID	Function, $H(t)$	$\Omega(s) = \frac{1}{s}L(H(t))$
1	C	$\frac{c}{\rho^2}$
2	t^n	$\frac{n!}{\rho(\rho^{n+1})}$
3	e^{rt}	$\frac{1}{\rho(\rho - r)}$
4	sin rt	$\frac{r}{\rho(\rho^2 + r^2)}$
5	cosrt	$\frac{1}{(\rho^2 + r^2)}$
6	sinh rt	$\frac{r}{\rho(\rho^2 - r^2)}$
7	cosh rt	$\frac{1}{(\rho^2 - r^2)}$

3. The Formula of General Solution for System of First Order

In this section, we derive the general formula for a system of first order in dimension n, while homogeneous or non-homogeneous.

3.1. The Formula of General Solution of a Homogeneous System of Order One

The system of first order has the formula $H' = CH$

Where $H' = \begin{pmatrix} \frac{dH_1}{dt} \\ \frac{dH_2}{dt} \\ \vdots \\ \frac{dH_n}{dt} \end{pmatrix}, C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \ H = \begin{pmatrix} H_1 \\ H_2 \\ \vdots \\ H_1 \end{pmatrix}$ so,

$$\begin{pmatrix} H_1 \\ H_2 \\ \vdots \\ H_n \end{pmatrix}' = \begin{pmatrix} c_{11} & c_{12} & \cdots c_{1n} \\ c_{21} & c_{22} & \cdots c_{2n} \\ \vdots & \vdots & \cdots \vdots \\ c_{m1} & c_{m2} & \cdots c_{mn} \end{pmatrix} \begin{pmatrix} H_1 \\ H_2 \\ \vdots \\ H_n \end{pmatrix} \tag{3.1}$$

After taking Novel transform for both sides, yields:

$$\rho N_I(H_1) - \frac{H_1(0)}{\rho} = c_{11}N_I(H_1) + c_{12}N_I(H_2) + \cdots + c_{1n}N_I(H_n)$$
$$\rho N_I(H_2) - \frac{H_2(0)}{\rho} = c_{21}N_I(H_1) + c_{22}N_I(H_2) + \cdots + c_{2n}N_I(H_n)$$
$$\vdots$$
$$\rho N_I(H_n) - \frac{H_n(0)}{\rho} = c_{n1}N_I(H_1) + c_{n2}N_I(H_2) + \cdots + c_{mn}N_I(H_n),$$

where $H_1(0),\ H_2(0),\ .\ .\ .\ .\ , H_n(0)$ are initial conditions.

$$(\rho - c_{11})N_I(H_1) - c_{12}N_I(H_2) - \cdots - c_{1n}N_I(H_n) = \frac{H_1(0)}{\rho}$$
$$(\rho - c_{22})N_I(H_2) - c_{21}N_I(H_1) - \cdots - c_{2n}N_I(H_n) = \frac{H_2(0)}{\rho}$$
$$\vdots$$
$$(\rho - c_{mn})N_I(H_n) - c_{m1}N_I(H_1) - \cdots - c_{m2}N_I(H_n) = \frac{H_n(0)}{\rho}.$$

Moreover, simple calculation to obtain $N_I(H_1),\ \cdots\ , N_I(H_n)$,

$$\Delta = \begin{vmatrix} (\rho - c_{11}) & -c_{12} & \ldots -c_{1n} \\ -c_{21} & (\rho - c_{22}) & \ldots -c_{2n} \\ \vdots & \vdots & \ldots \ \vdots \\ -c_{m1} & -c_{m2} & \ldots\ (\rho - c_{mn}) \end{vmatrix}$$

Also,

$$N_I(H_1) = \frac{1}{\Delta} \begin{vmatrix} \frac{H_1(0)}{\rho} & -c_{12} \ldots -c_{1n} \\ \frac{H_2(0)}{\rho} & (\rho - c_{22}) \ldots -c_{2n} \\ & \vdots \vdots \cdots \vdots \\ \frac{H_n(0)}{\rho} & -c_{m2} \ldots (\rho - c_{mn}) \end{vmatrix}$$
$$\vdots$$
$$N_I(H_n) = \frac{1}{\Delta} \begin{vmatrix} (\rho - c_{11)} - c_{12} \ldots & \frac{H_1(0)}{\rho} \\ -c_{21}(\rho - c_{22}) \ldots & \frac{H_2(0)}{\rho} \\ \vdots \vdots \cdots \vdots & \\ -c_{m1} - c_{m2} \ldots & \frac{H_n(0)}{\rho} \end{vmatrix}$$

The set solution of system (3.1) yields from taking the inverse of Novel transform for $N_I(H_i)$, $i = 1,2,3, \cdots , n$.

3.2. The Formula of General Solution of Non-homogeneous System of Order One

A non-homogeneous system has the formula $H' = CH + K$

where $H' = \begin{pmatrix} \frac{dH_1}{dt} \\ \frac{dH_2}{dt} \\ \vdots \\ \frac{dH_n}{dt} \end{pmatrix}, C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, H = \begin{pmatrix} H_1 \\ H_2 \\ \vdots \\ H_n \end{pmatrix},\ K = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}$ so,

$$\begin{pmatrix} \mathrm{H}_1 \\ \mathrm{H}_2 \\ \vdots \\ \mathrm{H}_\mathrm{n} \end{pmatrix}' = \begin{pmatrix} c_{11} c_{12} \dots c_{1n} \\ c_{21} c_{22} \dots c_{2n} \\ \vdots\vdots \cdots \vdots \\ c_{m1} c_{m2} \dots c_{mn} \end{pmatrix} \begin{pmatrix} \mathrm{H}_1 \\ \mathrm{H}_2 \\ \vdots \\ \mathrm{H}_\mathrm{n} \end{pmatrix} + \begin{pmatrix} K_1 \\ K_2 \\ \vdots \\ K_n \end{pmatrix} \tag{3.2}$$

Novel transform for both side of the above system, yields:

$$\rho N_I(H_1) - \frac{H_1(0)}{\rho} = c_{11} N_I(H_1) + c_{12} N_I(H_2) + \cdots + c_{1n} N_I(H_n) + N_I(k_1)$$
$$\rho N_I(H_2) - \frac{H_2(0)}{\rho} = c_{21} N_I(H_1) + c_{22} N_I(H_2) + \cdots + c_{2n} N_I(H_n) + N_I(k_2)$$
$$\vdots$$
$$\rho N_I(H_n) - \frac{H_n(0)}{\rho} = c_{n1} N_I(H_1) + c_{n2} N_I(H_2) + \cdots + c_{mn} N_I(H_n) + N_I(k_n),$$

where $H_1(0),\ H_2(0), \cdots, H_n(0)$ are initial conditions.

$$(\rho - c_{11}) N_I(H_1) - c_{12} N_I(H_2) - \cdots - c_{1n} N_I(H_n) = \frac{H_1(0)}{\rho} + N_I(k_1)$$
$$(\rho - c_{22}) N_I(H_2) - c_{21} N_I(H_1) - \cdots - c_{2n} N_I(H_n) = \frac{H_2(0)}{\rho} + N_I(k_2)$$
$$\vdots$$
$$(\rho - c_{mn}) N_I(H_n) - c_{m1} N_I(H_1) - \cdots - c_{m2} N_I(H_n) = \frac{H_n(0)}{\rho} + N_I(k_n).$$

Similarly, with the formula (3.1), we have

$$\Delta = \begin{vmatrix} (\rho - c_{11}) - c_{12} \dots - c_{1n} \\ -c_{21} (\rho - c_{22}) \dots - c_{2n} \\ \vdots\vdots \cdots \vdots \\ -c_{m1} - c_{m2} \dots\ (\rho - c_{mn}) \end{vmatrix}$$

$$N_I(H_1) = \frac{1}{\Delta} \begin{vmatrix} \frac{\mathrm{H}_1(0)}{\rho} + \mathrm{N_I}(\mathrm{K}_1) - \mathrm{c}_{12} \cdots - \mathrm{c}_{1n} \\ \frac{H_2(0)}{\rho} + N_I(K_2)(\rho - c_{22}) \cdots - c_{2n} \\ \vdots\vdots \cdots \vdots \\ \frac{H_n(0)}{\rho} + N_I(K_n) - c_{m2} \dots (\rho - c_{mn}) \end{vmatrix}$$
$$\vdots$$
$$N_I(H_n) = \frac{1}{\Delta} \begin{vmatrix} (\rho - c_{11}) - c_{12} \cdots \frac{H_1(0)}{\rho} + N_I(K_1) \\ -c_{21} (\rho - c_{22}) \cdots \frac{H_2(0)}{\rho} + N_I(K_2) \\ \vdots\vdots \cdots \vdots \\ -c_{m1} - c_{m2} \cdots \frac{H_n(0)}{\rho} + N_I(K_n) \end{vmatrix}$$

After taking inverse of Novel transform to $N_I(H_1), \cdots, N_I(H_n)$, obtaining the solution of the system (3.2).

3.3. The Formula of General Solution of Homogeneous System of Second Order

System of second order with constants has the formula $H'' = CH' + RH$

where $H'' = \begin{pmatrix} \frac{d^2H_1}{dt^2} \\ \frac{d^2H_2}{dt^2} \\ \vdots \\ \frac{d^2H_n}{dt^2} \end{pmatrix}$, $C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$, $H' = \begin{pmatrix} \frac{dH_1}{dt} \\ \frac{dH_2}{dt} \\ \vdots \\ \frac{dH_n}{dt} \end{pmatrix}$, $R = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}$, $H = \begin{pmatrix} H_1 \\ H_2 \\ \vdots \\ H_n \end{pmatrix}$ so,

$$\begin{pmatrix} \mathrm{H}_1 \\ \mathrm{H}_2 \\ \vdots \\ \mathrm{H}_n \end{pmatrix}'' = \begin{pmatrix} c_{11}c_{12}\cdots c_{1n} \\ c_{21}c_{22}\cdots c_{2n} \\ \vdots\vdots\cdots\vdots \\ c_{m1}c_{m2}\cdots c_{mn} \end{pmatrix}\begin{pmatrix} \mathrm{H}_1 \\ \mathrm{H}_2 \\ \vdots \\ \mathrm{H}_n \end{pmatrix}' + \begin{pmatrix} r_{11}r_{12}\cdots r_{1n} \\ r_{21}r_{22}\cdots r_{2n} \\ \vdots\vdots\cdots\vdots \\ r_{m1}r_{m2}\cdots r_{mn} \end{pmatrix}\begin{pmatrix} \mathrm{H}_1 \\ \mathrm{H}_2 \\ \vdots \\ \mathrm{H}_n \end{pmatrix} \quad (3.3)$$

By taking Novel transform to (3.3),

$$\rho^2 N_I(H_1) - H_1(0) - \frac{H_1'(0)}{\rho} = c_{11}\rho N_I(H_1) - c_{11}\frac{H_1(0)}{\rho} + c_{12}\rho N_I(H_2) - c_{12}\frac{H_2(0)}{\rho} + \cdots + c_{1n}\rho N_I(H_n) - c_{1n}\frac{H_n(0)}{\rho} + r_{11}N_I(H_1) + r_{12}N_I(H_2) + \cdots + r_{1n}N_I(H_n)$$

$$\rho^2 N_I(H_2) - H_2(0) - \frac{H_2'(0)}{\rho} = c_{21}\rho N_I(H_1) - c_{21}\frac{H_1(0)}{\rho} + c_{22}\rho N_I(H_2) - c_{22}\frac{H_2(0)}{\rho} + \cdots + c_{2n}\rho N_I(H_n) - c_{2n}\frac{H_n(0)}{\rho} + r_{21}N_I(H_1) + r_{22}N_I(H_2) + \cdots + r_{2n}N_I(H_n)$$

$$\vdots$$

$$\rho^2 N_I(H_n) - H_n(0) - \frac{H_n'(0)}{\rho} = c_{m1}\rho N_I(H_1) - c_{m1}\frac{H_1(0)}{\rho} + c_{m2}\rho N_I(H_2) - c_{m2}\frac{H_2(0)}{\rho} + \cdots + c_{mn}\rho N_I(H_n) - c_{mn}\frac{H_n(0)}{\rho} + r_{m1}N_I(H_1) + r_{m2}N_I(H_2) + \cdots + r_{mn}N_I(H_n),$$

Where $H_1(0), H_2(0), \cdots, H_n(0)$ and $H_1'(0), H_2'(0), \cdots, H_n'(0)$ are initial conditions.

$$(\rho^2 - c_{11}\rho - r_{11})N_I(H_1) - (c_{12}\rho + r_{12})N_I(H_2) - \cdots - (c_{1n}\rho + r_{1n})N_I(H_n) = H_1(0) + \frac{H_1'(0)}{\rho} - c_{11}\frac{H_1(0)}{\rho} - c_{12}\frac{H_2(0)}{\rho} - \cdots - c_{1n}\frac{H_n(0)}{\rho}$$

$$(\rho^2 - c_{22}\rho - r_{22})N_I(H_2) - (c_{21}\rho + r_{21})N_I(H_1) - \cdots - (c_{2n}\rho + r_{2n})N_I(H_n) = H_2(0) + \frac{H_2'(0)}{\rho} - c_{21}\frac{H_1(0)}{\rho} - c_{22}\frac{H_2(0)}{\rho} - \cdots - c_{2n}\frac{H_n(0)}{\rho}$$

$$\vdots$$

$$(\rho^2 - c_{mn}\rho - r_{mn})N_I(H_n) - (c_{m1}\rho + r_{m1})N_I(H_1) - \cdots - (c_{m2}\rho + r_{m2})N_I(H_2) = H_n(0) + \frac{H_n'(0)}{\rho} - c_{m1}\frac{H_1(0)}{\rho} - c_{m2}\frac{H_2(0)}{\rho} - \cdots - c_{mn}\frac{H_n(0)}{\rho}.$$

Through simple steps can be find the formula of $N_I(H_1), \cdots, N_I(H_n)$:

$$\Delta = \begin{vmatrix} (\rho^2 - c_{11}\rho - r_{11}) & -(c_{12}\rho + r_{12})\cdots - (c_{1n}\rho + r_{1n}) \\ -(c_{21}\rho + r_{21}) & (\rho^2 - c_{22}\rho - r_{22})\cdots -(c_{2n}\rho + r_{2n}) \\ & \vdots\vdots\cdots\vdots \\ -(c_{m1}\rho + r_{m1}) & -(c_{m2}\rho + r_{m2}) \quad \cdots \quad (\rho^2 - c_{m1}\rho - c_{mn}) \end{vmatrix}$$

Also,

$$N_I(H_1) = \frac{1}{\Delta}\begin{vmatrix} \varphi_1 - (c_{12}\rho + r_{12}) \quad \cdots - (c_{1n}\rho + r_{1n}) \\ \varphi_2 (\rho^2 - c_{22}\rho - r_{22})\cdots -(c_{2n}\rho + r_{2n}) \\ \vdots\vdots\cdots\vdots \\ \varphi_m - (c_{m2}\rho + r_{m2}) \quad \cdots \quad (\rho^2 - c_{m1}\rho - c_{mn}) \end{vmatrix}$$

$$\vdots$$

$$N_I(H_n)=\frac{1}{\Delta}\begin{vmatrix} (\rho^2-c_{11}\rho-r_{11}) & -(c_{12}\rho+r_{12}) & \cdots & \varphi_1 \\ -(c_{21}\rho+r_{21}) & (\rho^2-c_{22}\rho-r_{22}) & \cdots & \varphi_2 \\ \vdots & \vdots & \cdots & \vdots \\ -(c_{m1}\rho+r_{m1}) & -(c_{m2}\rho+r_{m2}) & \cdots & \varphi_m \end{vmatrix}$$

where,

$$\varphi_1=H_1(0)+\frac{H_1'(0)}{\rho}-c_{11}\frac{H_1(0)}{\rho}-c_{12}\frac{H_2(0)}{\rho}-\cdots-c_{1n}\frac{H_n(0)}{\rho}$$
$$\varphi_2=H_2(0)+\frac{H_2'(0)}{\rho}-c_{21}\frac{H_1(0)}{\rho}-c_{22}\frac{H_2(0)}{\rho}-\cdots-c_{2n}\frac{H_n(0)}{\rho}$$
$$\vdots$$
$$\varphi_n=H_n(0)+\frac{H_n'(0)}{\rho}-c_{m1}\frac{H_1(0)}{\rho}-c_{m2}\frac{H_2(0)}{\rho}-\cdots-c_{mn}\frac{H_n(0)}{\rho}.$$

After taking the inverse of Novel transform for $(N_I(H_i))i=1,2,3,\dots,n$, we obtained the set solution of system (3.3).

3.4. The Formula of General Solution of Non-homogeneous System of Order Two

A non-homogeneous system has the formula $H''=CH'+RH+K$

where $H''=\begin{pmatrix}\frac{d^2H_1}{dt^2}\\ \frac{d^2H_2}{dt^2}\\ \vdots\\ \frac{d^2H_n}{dt^2}\end{pmatrix}$, $C=\begin{pmatrix}c_1\\c_2\\ \vdots\\ c_n\end{pmatrix}$, $H'=\begin{pmatrix}\frac{dH_1}{dt}\\ \frac{dH_2}{dt}\\ \vdots\\ \frac{dH_n}{dt}\end{pmatrix}$, $R=\begin{pmatrix}r_1\\r_2\\ \vdots\\ r_n\end{pmatrix}$, $H=\begin{pmatrix}H_1\\H_2\\ \vdots\\ H_n\end{pmatrix}$, $K=\begin{pmatrix}k_1\\k_2\\ \vdots\\ k_n\end{pmatrix}$ so,

$$\begin{pmatrix}H_1\\H_2\\ \vdots\\ H_n\end{pmatrix}''=\begin{pmatrix}c_{11}c_{12}\cdots c_{1n}\\ c_{21}c_{22}\cdots c_{2n}\\ \vdots\vdots\cdots\vdots\\ c_{m1}c_{m2}\cdots c_{mn}\end{pmatrix}\begin{pmatrix}H_1\\H_2\\ \vdots\\ H_n\end{pmatrix}'+\begin{pmatrix}r_{11}r_{12}\cdots r_{1n}\\ r_{21}r_{22}\cdots r_{2n}\\ \vdots\vdots\cdots\vdots\\ r_{m1}r_{m2}\cdots r_{mn}\end{pmatrix}\begin{pmatrix}H_1\\H_2\\ \vdots\\ H_n\end{pmatrix}+\begin{pmatrix}k_1\\k_2\\ \vdots\\ k_n\end{pmatrix} \quad (3.4)$$

Novel transform for both side of the above system, yields:

$$\rho^2N_I(H_1)-H_1(0)-\frac{H_1'(0)}{\rho}$$
$$=c_{11}\rho N_I(H_1)-c_{11}\frac{H_1(0)}{\rho}+c_{12}\rho N_I(H_2)-c_{12}\frac{H_2(0)}{\rho}+\cdots+c_{1n}\rho N_I(H_n)-c_{1n}\frac{H_n(0)}{\rho}$$
$$+r_{11}N_I(H_1)+r_{12}N_I(H_2)+\cdots+r_{1n}N_I(H_n)+N_I(k_1)$$
$$\rho^2N_I(H_2)-H_2(0)-\frac{H_2'(0)}{\rho}$$
$$=c_{21}\rho N_I(H_1)-c_{21}\frac{H_1(0)}{\rho}+c_{22}\rho N_I(H_2)-c_{22}\frac{H_2(0)}{\rho}+\cdots+c_{2n}\rho N_I(H_n)-c_{2n}\frac{H_n(0)}{\rho}$$
$$+r_{21}N_I(H_1)+r_{22}N_I(H_2)+\cdots+r_{2n}N_I(H_n)+N_I(k_2)$$
$$\vdots$$
$$\rho^2N_I(H_n)-H_n(0)-\frac{H_n'(0)}{\rho}=c_{m1}\rho N_I(H_1)-c_{m1}\frac{H_1(0)}{\rho}+c_{m2}\rho N_I(H_2)-c_{m2}\frac{H_2(0)}{\rho}+\cdots+c_{mn}\rho N_I(H_n)-$$
$$c_{mn}\frac{H_n(0)}{\rho}+r_{m1}N_I(H_1)+r_{m2}N_I(H_2)+\cdots+r_{mn}N_I(H_n)+N_I(k_n),$$

where $H_1(0), H_2(0),\cdots, H_n(0)$ $and H_1'(0), H_2'(0),\cdots, H_n'(0)$ are initial conditions.

$$(\rho^2-c_{11}\rho-r_{11})N_I(H_1)-(c_{12}\rho+r_{12})N_I(H_2)-\cdots-(c_{1n}\rho+r_{1n})N_I(H_n)$$
$$=H_1(0)+\frac{H_1'(0)}{\rho}-c_{11}\frac{H_1(0)}{\rho}-c_{12}\frac{H_2(0)}{\rho}-\cdots-c_{1n}\frac{H_n(0)}{\rho}+N_I(k_1)$$

$$(\rho^2 - c_{22}\rho - r_{22})N_I(H_2) - (c_{21}\rho + r_{21})N_I(H_1) - \cdots - (c_{2n}\rho + r_{2n})N_I(H_n)$$
$$= H_2(0) + \frac{H_2'(0)}{\rho} - c_{21}\frac{H_1(0)}{\rho} - c_{22}\frac{H_2(0)}{\rho} - \cdots - c_{2n}\frac{H_n(0)}{\rho} + N_I(k_2)$$
$$\vdots$$
$$(\rho^2 - c_{mn}\rho - r_{mn})N_I(H_n) - (c_{m1}\rho + r_{m1})N_I(H_1) - \cdots - (c_{m2}\rho + r_{m2})N_I(H_2)$$
$$= H_n(0) + \frac{H_n'(0)}{\rho} - c_{m1}\frac{H_1(0)}{\rho} - c_{m2}\frac{H_2(0)}{\rho} - \cdots - c_{mn}\frac{H_n(0)}{\rho} + N_I(k_n)$$

Through simple steps can be find the formula of $N_I(H_1), \cdots, N_I(H_n)$:

$$\Delta = \begin{vmatrix} (\rho^2 - c_{11}\rho - r_{11}) & -(c_{12}\rho + r_{12}) & \ldots & -(c_{1n}\rho + r_{1n}) \\ -(c_{21}\rho + r_{21}) & (\rho^2 - c_{22}\rho - r_{22}) & \ldots & -(c_{2n}\rho + r_{2n}) \\ \vdots & \vdots & \cdots & \vdots \\ -(c_{m1}\rho + r_{m1}) & -(c_{m2}\rho + r_{m2}) & \ldots & (\rho^2 - c_{m1}\rho - c_{mn}) \end{vmatrix}$$

$$N_I(H_1) = \frac{1}{\Delta}\begin{vmatrix} \gamma_1 & -(c_{12}\rho + r_{12}) & \ldots & -(c_{1n}\rho + r_{1n}) \\ \gamma_2 & (\rho^2 - c_{22}\rho - r_{22}) & \ldots & -(c_{2n}\rho + r_{2n}) \\ \vdots & \vdots & \cdots & \vdots \\ \gamma_m & -(c_{m2}\rho + r_{m2}) & \ldots & (\rho^2 - c_{m1}\rho - c_{mn}) \end{vmatrix}$$
$$\vdots$$
$$N_I(H_n) = \frac{1}{\Delta}\begin{vmatrix} (\rho^2 - c_{11}\rho - r_{11}) & -(c_{12}\rho + r_{12}) & \ldots & \gamma_1 \\ -(c_{21}\rho + r_{21}) & (\rho^2 - c_{22}\rho - r_{22}) & \ldots & \gamma_2 \\ \vdots & \vdots & \cdots & \vdots \\ -(c_{m1}\rho + r_{m1}) & -(c_{m2}\rho + r_{m2}) & \ldots & \gamma_m \end{vmatrix}$$

where,

$$\gamma_1 = H_1(0) + \frac{H_1'(0)}{\rho} - c_{11}\frac{H_1(0)}{\rho} - c_{12}\frac{H_2(0)}{\rho} - \cdots - c_{1n}\frac{H_n(0)}{\rho} + N_I(k_1)$$
$$\gamma_2 = H_2(0) + \frac{H_2'(0)}{\rho} - c_{21}\frac{H_1(0)}{\rho} - c_{22}\frac{H_2(0)}{\rho} - \cdots - c_{2n}\frac{H_n(0)}{\rho} + N_I(k_2)$$
$$\vdots$$
$$\gamma_m = H_n(0) + \frac{H_n'(0)}{\rho} - c_{m1}\frac{H_1(0)}{\rho} - c_{m2}\frac{H_2(0)}{\rho} - \cdots - c_{mn}\frac{H_n(0)}{\rho} + N_I(k_n)\ .$$

After taking the inverse of Novel transform for $(N_I(H_i)), i = 1,2,3, \ldots .n$, we obtained the set solution of system (3.4).

4. Applications

In this section, using the formulas found in the previous section, we apply them to a number of systems.

Example (1): To solvethe system of order one in dimension two

$$H' = AH \text{ where } A=\begin{pmatrix} 5 & -4 \\ 3 & -2 \end{pmatrix},\ H(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \qquad (4.1)$$

Solution: By using Novel transform and apply formal (1), yields:

$$N_I(H_1) = \frac{1}{(\rho - 2)(\rho - 1)}\begin{vmatrix} \frac{3}{\rho} & 4 \\ \frac{2}{\rho} & (\rho + 2) \end{vmatrix}$$

After simple calculation using partition fraction:

$$N_I(H_1) = \frac{4}{\rho(\rho - 2)} - \frac{1}{\rho(\rho - 1)}$$

Now, taking inverse of Novel transform to both sides of the above equation, we obtain:

$$H_1(t) = 4e^{2t} - e^t$$

In similar way, $N_I(H_2)$ can be obtained by:

$$N_I(H_2) = \frac{1}{(\rho-2)(\rho-1)}\begin{vmatrix}(\rho-5) & \frac{3}{\rho} \\ -3 & \frac{2}{\rho}\end{vmatrix}$$

$$N_I(H_2) = \frac{3}{\rho(\rho-2)} - \frac{1}{\rho(\rho-1)}$$

Also, by the inverse of Novel transform for the above equation,

$$H_2(t) = 3e^{2t} - e^t,$$

where $H_1(t) and H_2(t)$represent the set solution of the system (4.1). Figure (1)

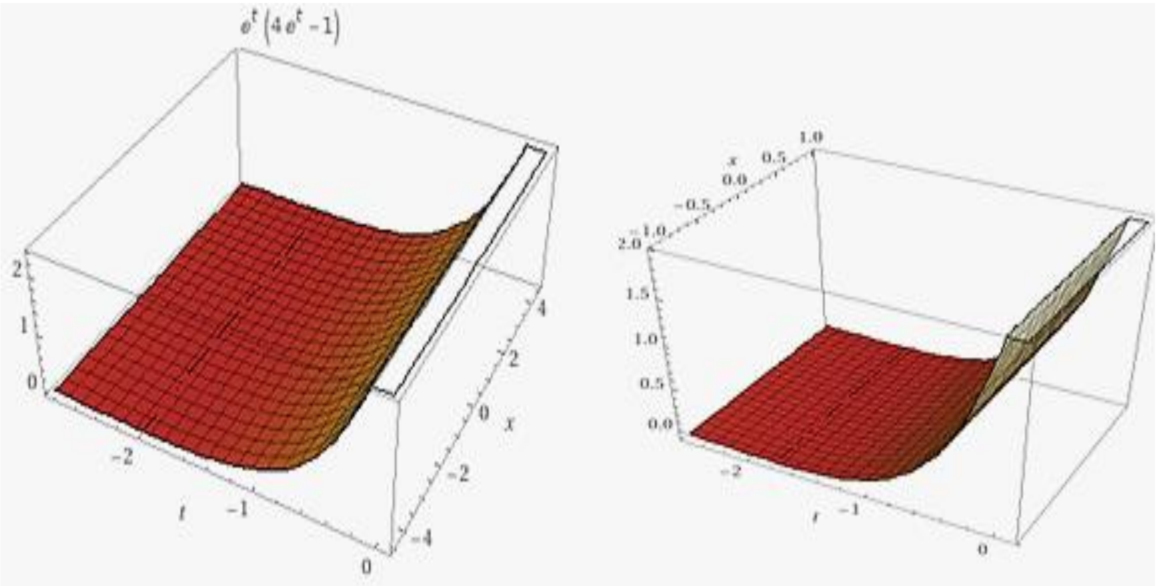

Figure 1.

Example (2): To find the general solution of the system $H' = AH + K$

where $A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, K = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}, H(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ (4.2)

Solution: Using formal (3.2), yields:

$$N_I(H_1) = \frac{1}{\rho^2-1}\begin{vmatrix}\frac{2}{\rho} + \frac{1}{\rho(\rho^2+1)} & 1 \\ \frac{0}{\rho} + \frac{1}{\rho^2+1} & (\rho-0)\end{vmatrix} = \frac{1}{\rho^2-1}\left(\frac{2\rho^2+2}{\rho^2+1}\right)$$

Simple fiction and taking inverse Novel transform to both sides of the above equation,

$$H_1(t) = 2\cosh(t)$$

In similar way, $N_I(H_2)$ can be obtained by:

$$N_I(H_2) = \frac{1}{\rho^2-1}\begin{vmatrix}(\rho-0) & \frac{2}{\rho}+\frac{1}{\rho(\rho^2+1)} \\ 1 & \frac{0}{\rho}+\frac{1}{\rho^2+1}\end{vmatrix} = \frac{1}{\rho^2-1}\left(\frac{\rho^2-2\rho^2+3}{\rho(\rho^2+1)}\right)$$
$$= \frac{-2}{\rho(\rho^2+1)} + \frac{1}{\rho(\rho^2-1)}$$

Also, by the inverse of Novel transform for the above equation:

$$H_2 = -2sin(t) + sinh(t),$$

where $H_1(t) and H_2(t)$represent the set solution of the system (4.2). Figure (2).

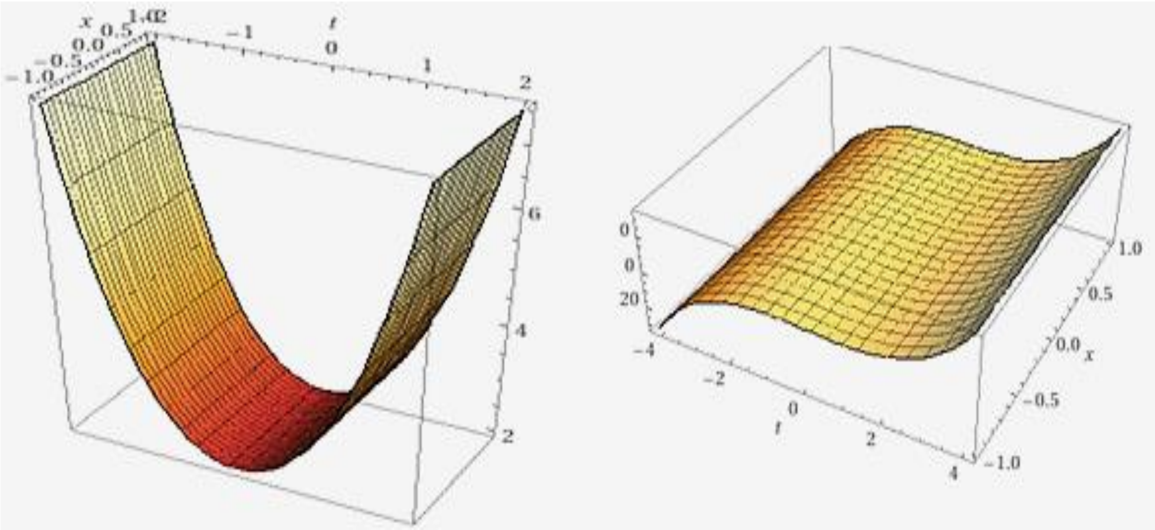

Figure 2.

Example (3): To solve the system of order two in dimension two

$\mathrm{H}'' = \mathrm{AH}$ where $\mathrm{A} = \begin{pmatrix}-3 & -2 \\ 4 & 3\end{pmatrix}$, $\mathrm{H}(0) = \begin{pmatrix}1 \\ 0\end{pmatrix}$, $\mathrm{H}'(0) = \begin{pmatrix}0 \\ 1\end{pmatrix} \cdots$ (4.3)

Solution: Using formal (3.3), yield:

$$\mathrm{N_I(H_1)} = \frac{1}{(\rho^2+1)(\rho^2-1)}\begin{vmatrix}1 & 2 \\ \frac{1}{\rho} & (\rho^2-3)\end{vmatrix}$$
$$= \frac{2\rho}{\rho(\rho^2+1)} - \frac{4}{\rho(\rho^2+1)} - \frac{\rho}{\rho(\rho^2-1)} + \frac{2}{\rho(\rho^2-1)}$$

Now, taking inverse Novel transform to both sides of the above equation:

$$\mathrm{H_1(t)} = 2\cos(\mathrm{t}) - 4\sin(\mathrm{t}) - \cosh(\mathrm{t}) + 2\sinh(\mathrm{t})$$

In similar way, $\mathrm{N_I(H_2)}$ can be obtained by:

$$\mathrm{N_I(H_2)} = \frac{1}{(\rho^2+1)(\rho^2-1)}\begin{vmatrix}(\rho^2-3) & 1 \\ -4 & \frac{1}{\rho}\end{vmatrix} = \frac{1}{(\rho^2+1)(\rho^2-1)}\left(\frac{\rho^2+3}{\rho}+4\right)$$
$$= \frac{2}{\rho(\rho-1)} - \frac{2\rho}{\rho(\rho^2+1)} - \frac{1}{\rho(\rho^2+1)}$$

taking inverse Novel transform to both sides of the above equation

$$\mathrm{H_2(t)} = 2\mathrm{e^t} - 2\cos(\mathrm{t}) - \sin(\mathrm{t})$$

Where $\mathrm{H_1(t)}$ and $\mathrm{H_2(t)}$represent the set solution of the system (4.3). Figure (3)

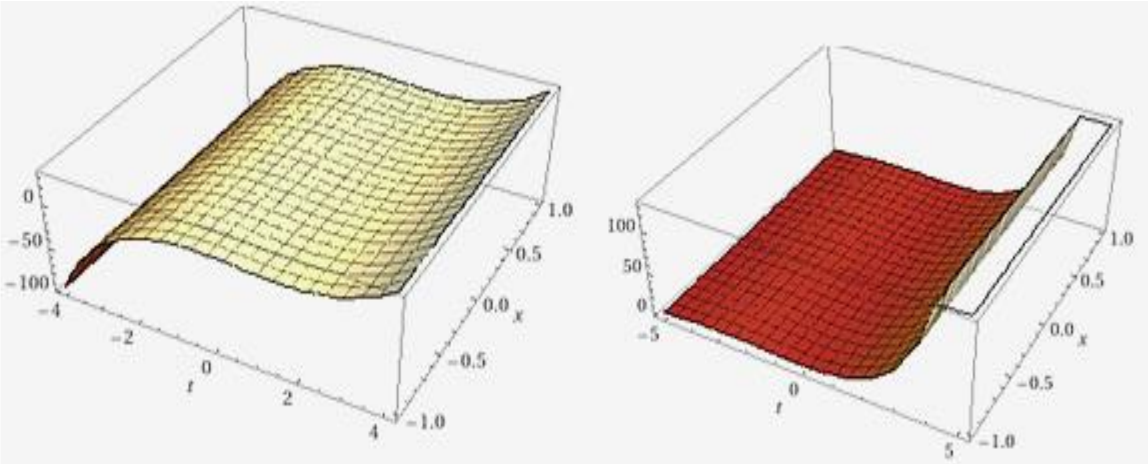

Figure 3.

Example (4): To find the general solution of the system $H'' = RH + K$,

where $R = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$, $K = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $H(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $H'(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (4.4)

Solution: By using Novel transform and apply formula (2.4), yields:

$$N_I(H_1) = \frac{1}{\rho^4}\begin{vmatrix} \frac{2}{\rho^2} & 0 \\ 1+\frac{1}{\rho^2} & (\rho^2+1) \end{vmatrix} = \frac{1}{\rho^4}\left(\frac{2\rho^2+1}{\rho^2}\right)$$

Taking inverse of Novel transform of $N_I(H_1)$:

$$H_1(t) = \frac{t^2}{2} + \frac{1}{4!}t^4$$

In similar way, $N_I(H_2)$ can be obtained by:

$$N_I(H_2) = \frac{1}{\rho^4}\begin{vmatrix} (\rho^2-1) & \frac{2}{\rho^2} \\ -1 & 1+\frac{1}{\rho^2} \end{vmatrix} = \frac{1}{\rho^4}\left(\frac{(\rho^2-1)}{\rho^2} + (\rho^2-1) + \frac{2}{\rho^2}\right)$$

Taking inverse of Novel transform to both sides of the above equation obtain:

$$H_2(t) = 1 + \frac{1}{4!}t^4,$$

where $H_1(t)$ and $H_2(t)$represent the set solution of the system (4.4).

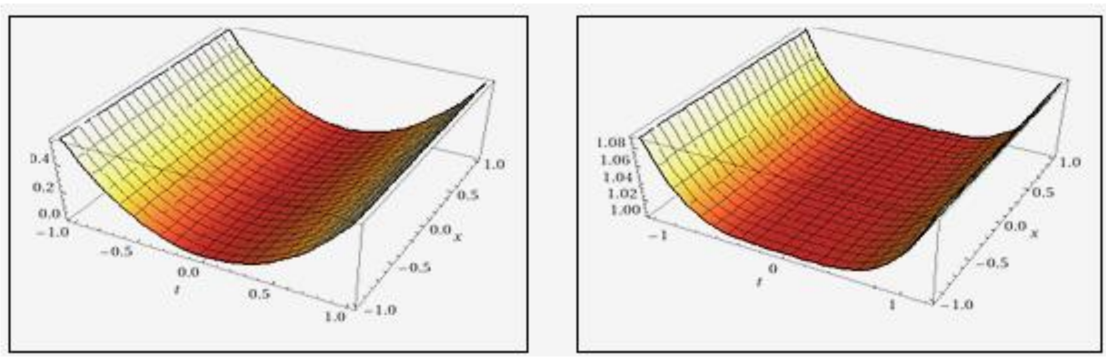

Figure 4.

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