Research Article

Solve the Laplace, Poisson and Telegraph Equations using the Shehu Transform

Athraa Neamah Albukhuttar[1], Banin Shakir Jubear[2], Marwah Tahseen Neamah[3]
[1,2,3]Faculty of education for girls, University of Kufa, Najaf 54002, Iraq.

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Abstract: In this paper, we present new applications for the Shehu transform to solve some important partial differential equations. Through obtained formulas for general solution of Laplace, Poisson and Telegraph equations under initial and boundary condition.

Keywords: Shehu Transform, General Formula of Solution, Initial Condition.

1. Introduction

Integral transform has played an important role in solving differential equations and integral equation, through transform these equations to algebraic equation. In addition, there are many integral transforms that have been used in many of the solution to the problems under initial conditions, which are difficult to resolve the classic ways like Laplace, Elzaki, Temimi and Novel …etc. [6,3,7,8].

Laplace transform is introduced by Pierre –Simon Laplace that became one of the famed transform in mathematics, engineering and physics[6,11].

=====o=f= Laplace transform, that applied to differential equations and the initial value problems with variable coefficients also [2,5]. Moreover, it applied to solve systems of ordinary differential equations its defined by:

$$E\,[(t)] = T\,(v) = v\int_0^{\infty} f(t)\ e^{-\frac{t}{v}}\ \mathrm{dt}\ ,\ \mathrm{t} \geq 0 \qquad (1.1)$$

In 2016 introduced a new transform called the Novel transform to solve many differential equations defined for the function is as follows:

$$\psi(s) = N_1(\mathrm{y(t)}) = \frac{1}{s}\int_0^{\infty} e^{-st}\mathrm{y(t)}\ \ \mathrm{dt}\ ,\ \mathrm{t} > 0 \qquad (1.2)$$

where y(t) is a real function, $t > 0$, $\frac{e^{-st}}{s}$ is the kernel function [8,10].

We benefited in this paper from Laplace - type integral transform for solving both ordinary and partial differential equations named Shehu transform [13,11].Moreover, it applied to solve transport and heat equation[4].

In this paper, we applied Shehu transform to solve Laplace, Poisson and Telegraph equations which are homogeneous or non- homogeneous, through the derivation of general formulas for solutions of these equations.

In section 2, the definitions, properties and of Shehu transform for some functions are showed In section 3, we obtained the general formulas to the solution of Laplace, Poisson and Telegraph equations. In last section, we utilize from these formulas to solve some examples

2. Basic Definitions and Properties of Shehu Transform

The Shehu transform is defined by:

$$\mathbb{s}[\pounds(\mathrm{t})] = \mathbb{s}[\pounds(\mu,\varepsilon)] = \int_0^{\infty} \exp\left(\frac{-\mu \mathrm{t}}{\varepsilon}\right)\pounds(\mathrm{t})\ \mathrm{dt}$$

$$= \lim_{\alpha\to\infty}\int_0^{\alpha} \exp\left(\frac{-\mu \mathrm{t}}{\varepsilon}\right)\pounds(\mathrm{t})\mathrm{dt} = \Omega(\mu,\varepsilon)\ ;\quad \mu > 0\ , \varepsilon > 0 \qquad (2.1)$$

Where $\pounds(\mathrm{t})$is a real function, $\exp\left(\frac{-\mu \mathrm{t}}{\varepsilon}\right)$ is the kernel function, and $\mathbb{s}$ is the operator of Shehu transform [11].

The inverse of Shehu transform is given by:

$$\mathbb{s}^{-1}[\Omega(\mu,\varepsilon)] = \Omega(t),\ \text{for}\ t \geq 0 \quad (2.2)$$
$$\Omega(t) = \mathbb{s}^{-1}[\Omega(\mu,\varepsilon)] = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{\varepsilon}\exp\left(\frac{\mu t}{\varepsilon}\right)\Omega(\mu,\varepsilon)\,dt \quad (2.3)$$

Property: If $£_1(x), £_2(x), \ldots, £_n(x)$ have Shehu transform, then

$$\mathbb{s}(ß_1 £_1(x) + ß_2 £_2(x) + \cdots + ß_n £_n(x)) = ß_1 \mathbb{s}(£_1(x)) + ß_2\, \mathbb{s}(£_2(x)) + \ldots + \mathbb{s}(£_n(x)) \quad (2.4)$$

where $ß_1, ß_2, \ldots, ß_n$ are constants and $£_1(x)$, $£_2(x), \ldots, £_n(x)$ are defined function.

Theorem (2.1) [5]: Shehu transform of derivative.

If the function $£^{(n)}(t)$ is the derivative of the function $£(t)$ with respect to t then its Shehu transform is defined by:

$$\mathbb{s}[£'(t)] = \frac{\mu}{\varepsilon}\,\mathbb{s}[£(\mu,\varepsilon)] - £(x,0) \quad (2.5)$$
$$\mathbb{s}[£''(t)] = \frac{\mu^2}{\varepsilon^2}\mathbb{s}[£(\mu,\varepsilon)] - \frac{\mu}{\varepsilon}£(x,0) - £'(x,0) \quad (2.6)$$
$$\mathbb{s}[£'''(t)] = \frac{\mu^3}{\varepsilon^3}\mathbb{s}[£(\mu,\varepsilon)] - \frac{\mu^2}{\varepsilon^2}£(x,0) - \frac{\mu}{\varepsilon}£'(x,0) - £''(x,0) \quad (2.7)$$
$$\mathbb{s}\left[£^{(n)}(t)\right] = \frac{\mu^n}{\varepsilon^n}\mathbb{s}[£(\mu,\varepsilon)] - \sum_{k=0}^{n-1}\left(\frac{\mu}{\varepsilon}\right)^{n-(k+1)} £^k(x,0) \quad (2.8)$$

Table 1: The Shehu transform for some functions:

S. No.	$£(t)$	$\mathbb{s}[£(t)]$
1	1	$\frac{\varepsilon}{\mu}$
2	T	$\frac{\varepsilon^2}{\mu^2}$
3	$\exp(\propto (t))$	$\frac{\varepsilon}{\mu - \propto \varepsilon}$
4	$\sin(\propto t)$	$\frac{\propto \varepsilon^2}{\mu^2 + \propto^2 \varepsilon^2}$
5	$\cos(\propto t)$	$\frac{\varepsilon\mu}{\mu^2 + \propto^2 \varepsilon^2}$
6	$t\exp(\propto t)$	$\frac{\varepsilon^2}{(\mu - \propto \varepsilon)^2}$
7	$\sin h(\propto t)$	$\frac{\propto \varepsilon^2}{\mu^2 - \propto^2 \varepsilon^2}$
8	$\cosh(\propto t)$	$\frac{\varepsilon\,\mu}{\mu^2 - \propto^2 \varepsilon^2}$

3. General Formulas of Laplace, Poisson and Telegraph Equations

Formula(1)

Consider Laplace equations $\Delta £ = \sum_{i=1}^{n} £_{x_i x_i} = 0$, $x \in R^n$

If n=2, then Laplace equations in two dimension has the form:

$$£_{tt}(x,t) + £_{xx}(x,t) = 0 \quad (3.1)$$

with the initial and boundary conditions $£(x,0) = \lambda_1(x)$, $£_t(x,0) = \lambda_2(x)$ and $£(0,t) = £(1,t) = 0$.

By Applying the Shehu transform to both sides:

$$\frac{\mu^2}{\varepsilon^2}\,\mathbb{s}[£(x,\mu,\varepsilon)] - \frac{\mu}{\varepsilon}£(x,0) - £_t(x,0) + \frac{d^2}{dx^2}\mathbb{s}[£(x,\mu,\varepsilon)] = 0$$

After substitute the initial condition we get:

$$\frac{\mu^2}{\varepsilon^2}\ \mathbb{s}[£(x,\mu,\varepsilon)]\ -\frac{\mu}{\varepsilon}\ \lambda_1(x)-\lambda_2\ (x)+\frac{d^2}{dx^2}\ \mathbb{s}[£(x,\mu,\varepsilon)]\ =0$$

$$\frac{d^2}{dx^2}\ \mathbb{s}[£(x,\mu,\varepsilon)]+\frac{\mu^2}{\varepsilon^2}\mathbb{s}[£(x,\mu,\varepsilon)]\ =\frac{\mu}{\varepsilon}\ \lambda_1(x)+\lambda_2\ (x) \qquad (3.2)$$

The above equation represent nonhomogeneous ODE of order two with dependent variable, $\mathbb{s}[£(x,\mu,\varepsilon)]$,which has the complementary solution:

$$\mathbb{s}_c[\,£(x,\mu,\varepsilon)]=ß_1\cos\left(\frac{\mu}{\varepsilon}\right)x+ß_2\ \sin\left(\frac{\mu}{\varepsilon}\right)x$$

and particular solution can be obtained by variation of parameters

$$\mathbb{s}_p\ [\,£(x,\mu,\varepsilon)]=ß_1(x)\cos\left(\frac{\mu}{\varepsilon}\right)x+ß_2(x)\ \sin\left(\frac{\mu}{\varepsilon}\right)x$$

Since

$$ß_1'(x)\cos\left(\frac{\mu}{\varepsilon}\right)x+\ ß_2'(x)\sin\left(\frac{\mu}{\varepsilon}\right)x=0 \qquad (3.3)$$

$$-\frac{\mu}{\varepsilon}\ ß_1'(x)\sin\left(\frac{\mu}{\varepsilon}\right)x+\ \frac{\mu}{\varepsilon}ß_2'(x)\cos\left(\frac{\mu}{\varepsilon}\right)x=\frac{\mu}{\varepsilon}\lambda_1(x)+\lambda_2(x) \qquad (3.4)$$

where

$$\Delta=\begin{vmatrix}\cos\left(\frac{\mu}{\varepsilon}\right)x & \sin\left(\frac{\mu}{\varepsilon}\right)x\\ -\frac{\mu}{\varepsilon}\sin\left(\frac{\mu}{\varepsilon}\right)x & \frac{\mu}{\varepsilon}\cos\left(\frac{\mu}{\varepsilon}\right)x\end{vmatrix}=\frac{\mu}{\varepsilon}$$

$$ß_1'(x)=\frac{1}{\Delta}\begin{vmatrix}0 & \sin\left(\frac{\mu}{\varepsilon}\right)x\\ \frac{\mu}{\varepsilon}\lambda_1(x)+\ \lambda_2(x) & \frac{\mu}{\varepsilon}\cos\left(\frac{\mu}{\varepsilon}\right)x\end{vmatrix}=\frac{-\left(\frac{\mu}{\varepsilon}\ \lambda_1(x)+\lambda_2\ (x)\ \right)\sin\left(\frac{\mu}{\varepsilon}\right)x}{\frac{\mu}{\varepsilon}}\Longrightarrow$$

So

$$ß_1(x)=-\ \frac{\varepsilon}{\mu}\int\left(\ \frac{\mu}{\varepsilon}\lambda_1(x)+\lambda_2\ (x)\ \right)\sin\left(\frac{\mu}{\varepsilon}\right)x\ dx$$

In similar way:

$$ß_2'(x)=\frac{\left(\frac{\mu}{\varepsilon}\ \lambda_1(x)+\ \lambda_2\ (x)\right)\cos\left(\frac{\mu}{\varepsilon}\right)x}{\frac{\mu}{\varepsilon}}$$

$$\therefore ß_2(x)=\ \frac{\varepsilon}{\mu}\int\left(\frac{\mu}{\varepsilon}\ \lambda_1(x)+\lambda_2\ (x)\right)\cos\left(\frac{\mu}{\varepsilon}\right)x\ \ dx$$

so the general solution of equation (2) is:

$$\mathbb{s}[£(x,\mu,\varepsilon)]\ =\left[C_1\cos\ \left(\frac{\mu}{\varepsilon}\right)xC_2\ \sin\left(\frac{\mu}{\varepsilon}\right)x\right]+$$
$$\left[\left(-\ \frac{\varepsilon}{\mu}\int\left(\frac{\mu}{\varepsilon}\ \lambda_1(x)+\ \lambda_2\ (x)\right)\sin\left(\frac{\mu}{\varepsilon}\right)x\ dx\right)\cos\left(\frac{\mu}{\varepsilon}\right)x+\left(\frac{\varepsilon}{\mu}\int\left(\frac{\mu}{\varepsilon}\ \lambda_1(x)+\lambda_2\ (x)\ \right)\cos\left(\frac{\mu}{\varepsilon}\right)x\ dx\right)\sin\left(\frac{\mu}{\varepsilon}\right)x\right]$$

From utilizing to the boundary condition yields:

$$C_1=C_2=0$$

$$\mathbb{s}\ [£(x,\mu,\varepsilon)]=\left[\left(-\ \frac{\varepsilon}{\mu}\int\left(\frac{\mu}{\varepsilon}\ \lambda_1(x)+\ \lambda_2\ (x)\right)\sin\left(\frac{\mu}{\varepsilon}\right)x\ dx\right)\cos\left(\frac{\mu}{\varepsilon}\right)x\right.$$
$$\left.+\left(\frac{\varepsilon}{\mu}\int\left(\frac{\mu}{\varepsilon}\ \lambda_1(x)+\lambda_2\ (x)\ \right)\cos\left(\frac{\mu}{\varepsilon}\right)x\ dx\right)\sin\left(\frac{\mu}{\varepsilon}\right)x\right]$$

After Taking the inverse of both sides we get the general solution of equation (3.1):

$$£(x,t)=\mathbb{s}^{-1}\left[\left(-\frac{\varepsilon}{\mu}\int\left(\frac{\mu}{\varepsilon}\lambda_1(x)+\lambda_2(x)\right)\sin\left(\frac{\mu}{\varepsilon}\right)xdx\right)\cos\left(\frac{\mu}{\varepsilon}\right)x+\left(\frac{\varepsilon}{\mu}\int\left(\frac{\mu}{\varepsilon}\lambda_1(x)+\lambda_2(x)\right)\cos\left(\frac{\mu}{\varepsilon}\right)x\,dx\right)\sin\left(\frac{\mu}{\varepsilon}\right)x\right] \quad (3.5)$$

Formula(2):

Consider Poisson equations $\Delta £(t.x)$= f(x, t) (3.6)

with the initial and boundary conditions $£(x,0)=\lambda_1(x)$, $£_t(x,0)=\lambda_2(x)$ and $£(0,t)=£(1,t)=0$

Applying the Shehu transform to both sides:

$$\frac{\mu^2}{\varepsilon^2}\mathbb{s}[£(x,\mu,\varepsilon)] - \frac{\mu}{\varepsilon}£(x,0) - £_t(x,0)+\frac{d^2}{dx^2}\mathbb{s}[£(x,\mu,\varepsilon)] =\mathbb{s}[f(x,t)]$$

Also,

$$\frac{d^2}{dx^2}\mathbb{s}[£(x,\mu,\varepsilon)] + \frac{\mu^2}{\varepsilon^2}\mathbb{s}[£(x,\mu,\varepsilon)] = \mathbb{s}[f(x,t)]+\frac{\mu}{\varepsilon}\lambda_1(x)+\lambda_2(x)$$

The above equation is ordinary equation of order two with $\mathbb{s}[£(x,\mu,\varepsilon)]$ dependent variable and it has the solution:

$$\mathbb{s}[£(x,\mu,\varepsilon)] = \left[C_1\cos\left(\frac{\mu}{\varepsilon}\right)x+C_2\sin\left(\frac{\mu}{\varepsilon}\right)x\right]+\left[\left(-\frac{\varepsilon}{\mu}\int\left(\left(\mathbb{s}[f(x,t)]+\frac{\mu}{\varepsilon}\lambda_1(x)+\lambda_2(x)\right)\sin\left(\frac{\mu}{\varepsilon}\right)x\right)dx\right)\cos\left(\frac{\mu}{\varepsilon}\right)x + \left(\frac{\varepsilon}{\mu}\int\left(\mathbb{s}[f(x,t)]+\frac{\mu}{\varepsilon}\lambda_1(x)+\lambda_2(x)\right)\cos\left(\frac{\mu}{\varepsilon}\right)xdx\right)\sin\left(\frac{\mu}{\varepsilon}\right)x\right]$$

by substitute boundary conditions $£(0,t)=0$ *and* $£(1,t)=0$,we get $C_1=C_2=0$, then:

$$\mathbb{s}[£(x,\mu,\varepsilon)] = \left[\left(-\frac{\varepsilon}{\mu}\int\left(\mathbb{s}[f(x,t)]+\frac{\mu}{\varepsilon}\lambda_1(x)+\lambda_2(x)\right)\sin\left(\frac{\mu}{\varepsilon}\right)x\,dx\right)\cos\left(\frac{\mu}{\varepsilon}\right)x + \left(\frac{\varepsilon}{\mu}\int\left(\mathbb{s}[f(x,t)]+\frac{\mu}{\varepsilon}\lambda_1(x)+\lambda_2(x)\right)\cos\left(\frac{\mu}{\varepsilon}\right)x\,dx\right)\sin\left(\frac{\mu}{\varepsilon}\right)x\right]$$

after taking the inverse of both sides , we get the solution of equation (4)

$$£(x,t)=\mathbb{s}^{-1}\left[\left(-\frac{\varepsilon}{\mu}\int\left(\mathbb{s}[f(x,t)]+\frac{\mu}{\varepsilon}\lambda_1(x)+\lambda_2(x)\right)\sin\left(\frac{\mu}{\varepsilon}\right)x\,dx\right)\cos\left(\frac{\mu}{\varepsilon}\right)x+\left(\frac{\varepsilon}{\mu}\int\left(\mathbb{s}[f(x,t)]+\frac{\mu}{\varepsilon}\lambda_1(x)+\lambda_2(x)\right)\cos\left(\frac{\mu}{\varepsilon}\right)x\,dx\right)\sin\left(\frac{\mu}{\varepsilon}\right)x\right] \quad (3.7)$$

Formula(3):

The Telegraph equation has the form:

$$£_{tt}(x,t)+2Ð£_t(x,t)+ß^2£(x,t)=£_{xx}+f(x,t) \quad (3.8)$$

If $f(x,t)$ =0 then equation homogeneous Telegraph equation,

under the conditions $£(x,0)=\lambda_1(x)$, $£_t(x,0)=\lambda_2(x)$ *and* $£(0,t)=£(1,t)=0$

By taking the Shehu transform to both sides, yields:

$$\frac{\mu^2}{\varepsilon^2}\mathbb{s}[£(x,\mu,\varepsilon)] - \frac{\mu}{\varepsilon}£(x,0) - £_t(x,0) -2Ð\frac{\mu}{\varepsilon}\mathbb{s}[£(x,\mu,\varepsilon)] -2Ð£(x,0) + ß^2\mathbb{s}[£(x,\mu,\varepsilon)] = \frac{d^2}{dx^2}\mathbb{s}[£(x,\mu,\varepsilon)] +\mathbb{s}[f(x,t)]$$

After substituting initial conditions:

$$\frac{μ^2}{\varepsilon^2}\, 𝕤[£(x,μ,\varepsilon)] - \frac{μ}{\varepsilon}\lambda_1(x) - \lambda_2(x) - 2Đ\frac{μ}{\varepsilon}𝕤[£(x,μ,\varepsilon)] - 2Đ\,\lambda_1(x) + ß^2 𝕤[£(x,μ,\varepsilon)] = \frac{d^2}{dx^2}𝕤[£(x,μ,\varepsilon)] + 𝕤[f(x,t)],$$

$$\frac{d^2}{dx^2}𝕤[£(x,μ,\varepsilon)] - \left(\frac{μ^2}{\varepsilon^2} + 2Đ\frac{μ}{\varepsilon} + ß^2\right)𝕤[£(x,μ,\varepsilon)] = -\frac{μ}{\varepsilon}\lambda_1(x) - 2Đ\,\lambda_1(x) - \lambda_2(x) - 𝕤[f(x,t)] \qquad (3.9)$$

The general solution of equation (3.9), after using variation of parameters to obtain the particular solution, has the form

$$𝕤[£(x,μ,\varepsilon)] = c_1 e^{\sqrt{\left(\frac{μ^2}{\varepsilon^2}+2Đ\frac{μ}{\varepsilon}+ß^2\right)}x} + c_2 e^{-\sqrt{\left(\frac{μ^2}{\varepsilon^2}+2Đ\frac{μ}{\varepsilon}+ß^2\right)}x} + \left[\frac{1}{2\sqrt{\left(\frac{μ^2}{\varepsilon^2}+2Đ\frac{μ}{\varepsilon}+ß^2\right)}}\left[\int\left(-\frac{μ}{\varepsilon}\lambda_1(x) - 2Đ\,\lambda_1(x) - \lambda_2(x) - 𝕤[f(μ,t)]\right)e^{-\sqrt{\left(\frac{μ^2}{\varepsilon^2}+2Đ\frac{μ}{\varepsilon}+ß^2\right)}x}\, d(x)\right]e^{\sqrt{\left(\frac{μ^2}{\varepsilon^2}+2Đ\frac{μ}{\varepsilon}+ß^2\right)}x} - \frac{1}{2\sqrt{\left(\frac{μ^2}{\varepsilon^2}+2Đ\frac{μ}{\varepsilon}+ß^2\right)}}\left[\int\left(-\frac{μ}{\varepsilon}\lambda_1(x) - 2Đ\,\lambda_1(x) - \lambda_2(x) - 𝕤[f(μ,t)]\right)e^{\left(\frac{μ^2}{\varepsilon^2}+2Đ\frac{μ}{\varepsilon}+ß^2\right)x}\, d(x)\right]e^{-\sqrt{\left(\frac{μ^2}{\varepsilon^2}+2Đ\frac{μ}{\varepsilon}+ß^2\right)}x}\right] \qquad (3.10)$$

From utilizing the boundary conditions $C_1 = C_2 = 0$ and the equation (3.10) become:

$$𝕤[£(x,μ,\varepsilon)] = \frac{1}{2\sqrt{\left(\frac{μ^2}{\varepsilon^2}+2Đ\frac{μ}{\varepsilon}+ß^2\right)}}\left[\int -\frac{μ}{\varepsilon}\lambda_1(x) - 2Đ\,\lambda_1(x) - \lambda_2(x)\, 𝕤[f(μ,t)]\, e^{-\sqrt{\left(\frac{μ^2}{\varepsilon^2}+2Đ\frac{μ}{\varepsilon}+ß^2\right)}x}\, d(x)\right]e^{\sqrt{\left(\frac{μ^2}{\varepsilon^2}+2Đ\frac{μ}{\varepsilon}+ß^2\right)}x} - \frac{1}{2\sqrt{\left(\frac{μ^2}{\varepsilon^2}+2Đ\frac{μ}{\varepsilon}+ß^2\right)}}\left[\int -\frac{μ}{\varepsilon}\lambda_1(x) - 2Đ\,\lambda_1(x) - \lambda_2(x) - 𝕤[f(μ,t)]\, e^{\left(\frac{μ^2}{\varepsilon^2}+2Đ\frac{μ}{\varepsilon}+ß^2\right)x}\, d(x)\right]e^{-\sqrt{\left(\frac{μ^2}{\varepsilon^2}+2Đ\frac{μ}{\varepsilon}+ß^2\right)}x} \qquad (3.11)$$

The general solution of equation (3.8) result from taking the inverse of Shehu transform to both sides of equation (3.11):

$$£(x,t) = 𝕤^{-1}\left[\frac{1}{2\sqrt{\left(\frac{μ^2}{\varepsilon^2}+2Đ\frac{μ}{\varepsilon}+ß^2\right)}}\left[\int\left(-\frac{μ}{\varepsilon}\lambda_1(x) - 2Đ\,\lambda_1(x) - \lambda_2(x) - 𝕤[f(μ,t)]\right)e^{-\sqrt{\left(\frac{μ^2}{\varepsilon^2}+2Đ\frac{μ}{\varepsilon}+ß^2\right)}x}\, d(x)\right]e^{\sqrt{\left(\frac{μ^2}{\varepsilon^2}+2Đ\frac{μ}{\varepsilon}+ß^2\right)}x} - \frac{1}{2\sqrt{\left(\frac{μ^2}{\varepsilon^2}+2Đ\frac{μ}{\varepsilon}+ß^2\right)}}\left[\int\left(-\frac{μ}{\varepsilon}\lambda_1(x) - 2Đ\,\lambda_1(x)\,\lambda_2(x) - 𝕤[f(μ,t)]\right)e^{\left(\frac{μ^2}{\varepsilon^2}+2Đ\frac{μ}{\varepsilon}+ß^2\right)x}\, d(x)\right]e^{-\sqrt{\left(\frac{μ^2}{\varepsilon^2}+2Đ\frac{μ}{\varepsilon}+ß^2\right)}x}\right] \qquad (3.12)$$

Formula(4):

Moreover, if $f(x,t) = 0\ t$ then the equation (3.8) has the form

$$£_{tt} + 2Đ£_t(x,t) + ß^2£(x,t) = £_{xx} \qquad (3.13)$$

which represent homogeneous Telegraph equation with the same condition

$$£(x,0) = \lambda_1(x),\ £_t(x,0) = \lambda_2(x)\ and\ £(0,t) = £(1,t) = 0$$

So, from equation (3.12) the general solution of (3.13), has the form

$$£(x,t)=\mathbb{s}^{-1}\left[\frac{1}{2\sqrt{\left(\frac{\mu^2}{\varepsilon^2}+2Đ\frac{\mu}{\varepsilon}+ß^2\right)}}\left[\int\left(-\frac{\mu}{\varepsilon}\,\lambda_1(x)-2Đ\,\lambda_1(x)-\lambda_2(x)\right)e^{-\sqrt{\left(\frac{\mu^2}{\varepsilon^2}+2Đ\frac{\mu}{\varepsilon}+ß^2\right)}\,x}\,d(x)\right]e^{\sqrt{\left(\frac{\mu^2}{\varepsilon^2}+2Đ\frac{\mu}{\varepsilon}+ß^2\right)}\,x}-\frac{1}{2\sqrt{\left(\frac{\mu^2}{\varepsilon^2}+2Đ\frac{\mu}{\varepsilon}+ß^2\right)}}\left[\int\left(-\frac{\mu}{\varepsilon}\,\lambda_1(x)-2Đ\,\lambda_1(x)-\lambda_2(x)\right)e^{\left(\frac{\mu^2}{\varepsilon^2}+2Đ\frac{\mu}{\varepsilon}+ß^2\right)\,x}\,d(x)\right]e^{-\sqrt{\left(\frac{\mu^2}{\varepsilon^2}+2Đ\frac{\mu}{\varepsilon}+ß^2\right)}\,x}\right] \qquad (3.14)$$

4. Applications

In this section, some examples are solved by using the formulas that obtained in the previous section.

Example (1):

To solve Laplace equation

$$£_{tt}(x,t)+£_{xx}(x,t)=0 \qquad (4.1)$$

with the conditions $£(x,0)=0$, $£_t(x,0)=x$ and $£(0,t)=£(1,t)=0$

Sol:

Using the for formula (1) with the equation (3.5)

$$£(x,t)=\mathbb{s}^{-1}\left[\left(-\frac{\varepsilon}{\mu}\int(x)\sin\left(\frac{\mu}{\varepsilon}\right)x\,dx\right)\cos\left(\frac{\mu}{\varepsilon}\right)x+\left(\frac{\varepsilon}{\mu}\int(x)\cos\left(\frac{\mu}{\varepsilon}\right)x\,dx\right)\sin\left(\frac{\mu}{\varepsilon}\right)x\right]$$

$$£(x,t)=\mathbb{s}^{-1}\left[\frac{\varepsilon^2}{\mu^2}\cos\left(\frac{\mu}{\varepsilon}\right)^2x-\frac{\varepsilon^3}{\mu^3}\sin\left(\frac{\mu}{\varepsilon}\right)x\cos\left(\frac{\mu}{\varepsilon}\right)x+\frac{\varepsilon^2}{\mu^2}\sin\left(\frac{\mu}{\varepsilon}\right)^2+\frac{\varepsilon^3}{\mu^3}\sin\left(\frac{\mu}{\varepsilon}\right)x\cos\left(\frac{\mu}{\varepsilon}\right)x\right]$$

$$£(x,t)=\mathbb{s}^{-1}\left[\frac{\varepsilon^2}{\mu^2}x\right]$$

$$£(x,t)=t\,x$$

which represent the solution of equation (4.1) as shown in figure(1).

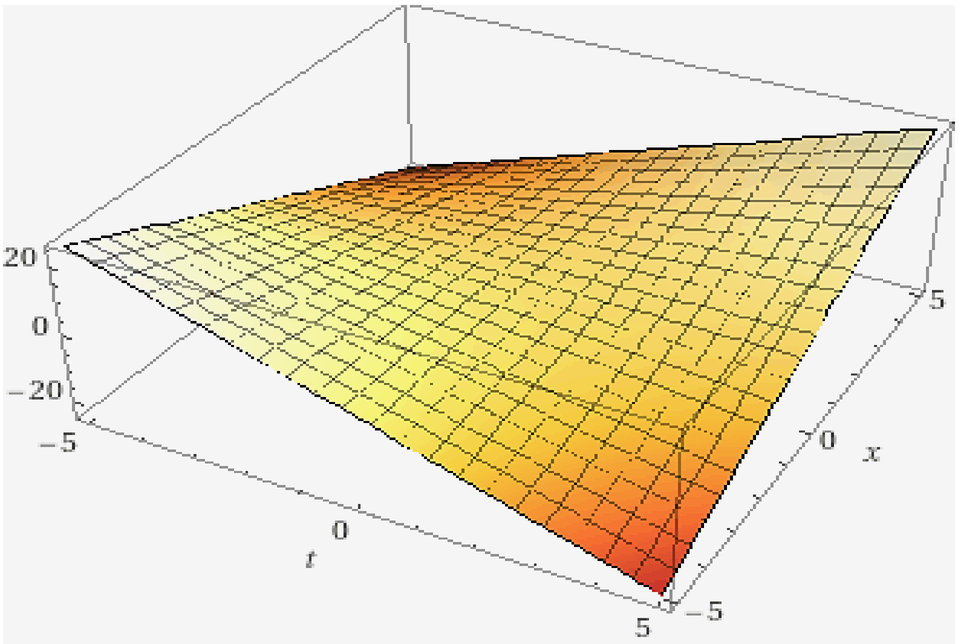

Figure 1.

Example (2):

To solve Poisson equation

$$\pounds_{tt}(x,t)+\pounds_{xx}(x,t)= xe^{t} \qquad (4.2)$$

with the conditions $\pounds(x,0) = 0$, $\pounds_t(x,0)$=x and $\pounds(0,t)$=$\pounds(1,t)$=0

Sol:

Using equation (3.7) and after substitute the conditions

$$\pounds(x,t) = \mathbb{s}^{-1}\left[\left(-\frac{\varepsilon}{\mu}\int\left(\mathrm{x}\left(\frac{\varepsilon}{\mu-\varepsilon}\right)+x\right)\sin\left(\frac{\mu}{\varepsilon}\right)xdx\right)\cos\left(\frac{\mu}{\varepsilon}\right)x\right.$$
$$\left.+\left(\frac{\varepsilon}{\mu}\int\left(\mathrm{x}\left(\frac{\varepsilon}{\mu-\varepsilon}\right)+x\right)\cos\left(\frac{\mu}{\varepsilon}\right)x\,dx\right)\sin\left(\frac{\mu}{\varepsilon}\right)x\right]$$

$$\pounds(x,t)=\mathbb{s}^{-1}\left[\left(\frac{\varepsilon^2}{\mu^2-\mu\varepsilon}x\cos\left(\frac{\mu}{\varepsilon}\right)x-\frac{\varepsilon^3}{\mu^3-\varepsilon^2\mu}\sin\left(\frac{\mu}{\varepsilon}\right)x\right)\cos\left(\frac{\mu}{\varepsilon}\right)x+\left(\frac{\varepsilon^2}{\mu^2-\mu\varepsilon}x\sin\left(\frac{\mu}{\varepsilon}\right)x+\frac{\varepsilon^3}{\mu^3-\varepsilon^2\mu}\cos\left(\frac{\mu}{\varepsilon}\right)x\right)\sin\left(\frac{\mu}{\varepsilon}\right)x\right]$$

Simplification of the above equation yields

$$\pounds(x,t)=\mathbb{s}^{-1}\left[\frac{\varepsilon^2}{\mu^2-\mu\varepsilon}x\cos\left(\frac{\mu}{\varepsilon}\right)^2x-\frac{\varepsilon^3}{\mu^3-\varepsilon^2\mu}\sin\left(\frac{\mu}{\varepsilon}\right)x\cos\left(\frac{\mu}{\varepsilon}\right)x+\frac{\varepsilon^2}{\mu^2-\mu\varepsilon}x\sin\left(\frac{\mu}{\varepsilon}\right)^2x+\frac{\varepsilon^3}{\mu^3-\varepsilon^2\mu}\sin\left(\frac{\mu}{\varepsilon}\right)x\cos\left(\frac{\mu}{\varepsilon}\right)x\right]$$
$$\pounds(x,t)=\mathbb{s}^{-1}\left[\frac{\varepsilon^2}{\mu^2-\mu\varepsilon}x\right]$$
$$\pounds(x,t)=\mathbb{s}^{-1}\left[\frac{\varepsilon}{\mu-\varepsilon}x-\frac{\varepsilon}{\mu}x\right]$$
$$\pounds(x,t)=e^{t}x-x\ ,$$

which represent the solution of equation (4.2) as shown in figure (2).

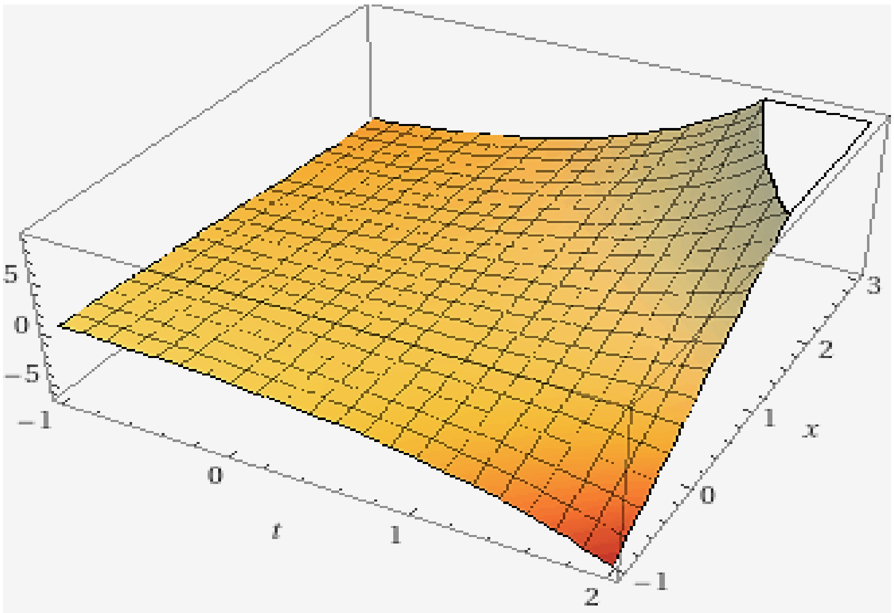

Figure 2.

Example (3)

To obtain the solution of Telegraph equation:

$$\pounds_{tt}(x,t)+\pounds_t(x,t)+\pounds(x,t)=\pounds_{xx}(x,t)+2e^{-2t}\sinh(x) \qquad (4.3)$$

Under the conditions $\pounds(x,0) = \sinh(x)$, $\pounds_{\,t}(x,0)$= -2 $\sinh(x)$ *and* $\pounds(0,t)$=$\pounds(1,t)$=0

Sol:

From formula (3) with the equation (3.12)

$$£(x,t)=\mathbb{s}^{-1}\left[\frac{1}{2\sqrt{\left(\frac{\mu^2}{\varepsilon^2}+\frac{\mu}{\varepsilon}+1\right)}}\left(\int\left(\sinh(x)-\frac{\mu}{\varepsilon}\sinh(x)-2\sinh(x)\frac{\varepsilon}{\mu+2\varepsilon}\right)e^{-\sqrt{\left(\frac{\mu^2}{\varepsilon^2}+\frac{\mu}{\varepsilon}+1\right)}\,x}\,dx\right)e^{\sqrt{\left(\frac{\mu^2}{\varepsilon^2}+\frac{\mu}{\varepsilon}+1\right)}\,x}-\frac{1}{2\sqrt{\left(\frac{\mu^2}{\varepsilon^2}+\frac{\mu}{\varepsilon}+1\right)}}\left(\int\sinh(x)-\frac{\mu}{\varepsilon}\sinh(x)-2\sinh(x)\frac{\varepsilon}{\mu+2\varepsilon}e^{\sqrt{\left(\frac{\mu^2}{\varepsilon^2}+\frac{\mu}{\varepsilon}+1\right)}\,x}\,dx\right)e^{-\sqrt{\left(\frac{\mu^2}{\varepsilon^2}+\frac{\mu}{\varepsilon}+1\right)}\,x}\right]$$

Using Euler's formula and some simple calculations:

$$£(x,t)=\mathbb{s}^{-1}\left[\frac{1-\frac{\mu}{\varepsilon}-\frac{2\varepsilon}{\mu+2\varepsilon}}{4\sqrt{\left(\frac{\mu^2}{\varepsilon^2}+\frac{\mu^2}{\varepsilon^2}+1\right)}}\left(e^{x}\left(\frac{2\sqrt{\left(\frac{\mu^2}{\varepsilon^2}+\frac{\mu^2}{\varepsilon^2}+1\right)}}{1-\left(\frac{\mu^2}{\varepsilon^2}+\frac{\mu^2}{\varepsilon^2}+1\right)}\right)-e^{-x}\left(\frac{2\sqrt{\left(\frac{\mu^2}{\varepsilon^2}+\frac{\mu^2}{\varepsilon^2}+1\right)}}{1-\left(\frac{\mu^2}{\varepsilon^2}+\frac{\mu^2}{\varepsilon^2}+1\right)}\right)\right)\right]$$

$$£(x,t)=\mathbb{s}^{-1}\left[\frac{1-\frac{\mu}{\varepsilon}-\frac{2\varepsilon}{\mu+2\varepsilon}}{1-\left(\frac{\mu^2}{\varepsilon^2}+\frac{\mu^2}{\varepsilon^2}+1\right)}\sinh(x)\right]$$

$$£(x,t)=\mathbb{s}^{-1}\left[\sinh(x)\frac{\varepsilon}{\mu+2\varepsilon}\right]$$

Lastly, the solution of equation (4.3) is

$$£(x,t)=\sinh(x)\,e^{-2t}$$

as shown in figure (3).

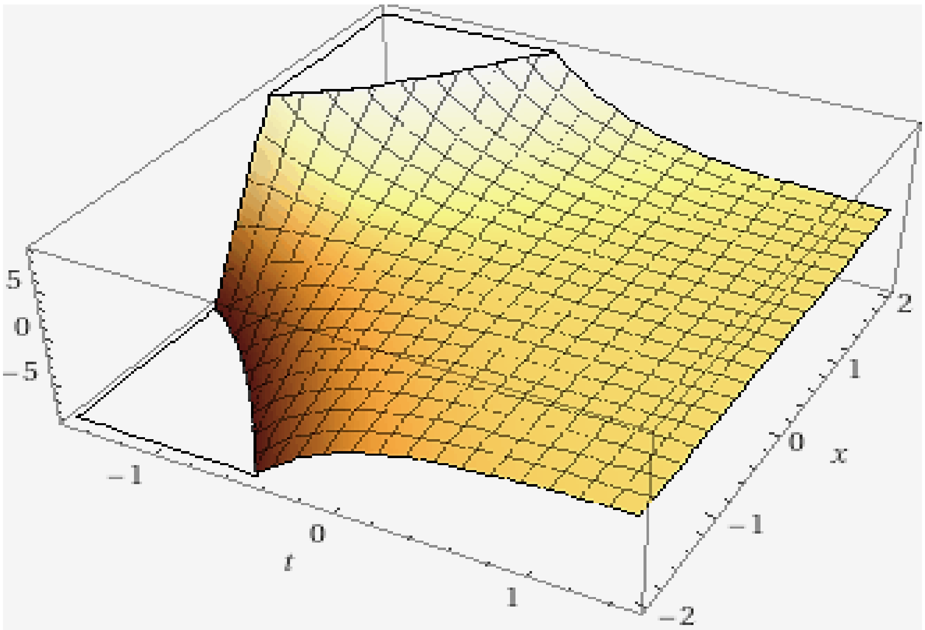

Figure 3.

Example (4):

To solve the homogeneous Telegraph equation

$$£_{tt}(x,t)+£_{t}(x,t)+£(x,t)=£_{xx} \qquad (4.4)$$

Under the conditions $£(x,0)=e^{x}$, $£_{t}(x,0)$= - e^{x} and $£(0,t)=£(1,t)$ =0

By using equation (3.14) and after substituting the conditions, it yields:

$$£(x,t)=\mathbb{s}^{-1}\left[\frac{1}{2\sqrt{\left(\frac{\mu^2}{\varepsilon^2}+\frac{\mu}{\varepsilon}+1\right)}}\left(\int\left(-\frac{\mu}{\varepsilon}e^{x}\right)e^{-\sqrt{\left(\frac{\mu^2}{\varepsilon^2}+\frac{\mu}{\varepsilon}+1\right)}x}\,d(x)\right)e^{\sqrt{\left(\frac{\mu^2}{\varepsilon^2}+\frac{\mu}{\varepsilon}+1\right)}x}-\frac{1}{2\sqrt{\left(\frac{\mu^2}{\varepsilon^2}+\frac{\mu}{\varepsilon}+1\right)}}\left(\int\left(-\frac{\mu}{\varepsilon}e^{x}\right)e^{\sqrt{\left(\frac{\mu^2}{\varepsilon^2}+\frac{\mu}{\varepsilon}+1\right)}x}\,d(x)\right)e^{-\sqrt{\left(\frac{\mu^2}{\varepsilon^2}+\frac{\mu}{\varepsilon}+1\right)}x}\right]$$

$$£(x,t)=\mathbb{s}^{-1}\left[\left(\frac{-\mu e^{\left(1+\sqrt{\left(\frac{\mu^2}{\varepsilon^2}+\frac{\mu}{\varepsilon}+1\right)}\right)x}}{2\varepsilon\sqrt{\left(\frac{\mu^2}{\varepsilon^2}+\frac{\mu}{\varepsilon}+1\right)}\left(1+\sqrt{\left(\frac{\mu^2}{\varepsilon^2}+\frac{\mu}{\varepsilon}+1\right)}\right)}\right)e^{\sqrt{\left(\frac{\mu^2}{\varepsilon^2}+\frac{\mu}{\varepsilon}+1\right)}x}+\left(\frac{\mu e^{\left(1+\sqrt{\left(\frac{\mu^2}{\varepsilon^2}+\frac{\mu}{\varepsilon}+1\right)}\right)x}}{2\varepsilon\sqrt{\left(\frac{\mu^2}{\varepsilon^2}+\frac{\mu}{\varepsilon}+1\right)}\left(1+\sqrt{\left(\frac{\mu^2}{\varepsilon^2}+\frac{\mu}{\varepsilon}+1\right)}\right)}\right)e^{-\sqrt{\left(\frac{\mu^2}{\varepsilon^2}+\frac{\mu}{\varepsilon}+1\right)}x}\right]$$

After simple calculation

$$£(x,t)=\mathbb{s}^{-1}\left[\frac{-\mu e^{x}}{\varepsilon\left(1-\left(\frac{\mu^2}{\varepsilon^2}+\frac{\mu}{\varepsilon}+1\right)\right)}\right]=\mathbb{s}^{-1}\left[e^{x}\frac{\varepsilon}{\mu+\varepsilon}\right]$$

$$£(x,t)=e^{x-t}$$

which represent the solution of equation (4.4) as shown in figure (4).

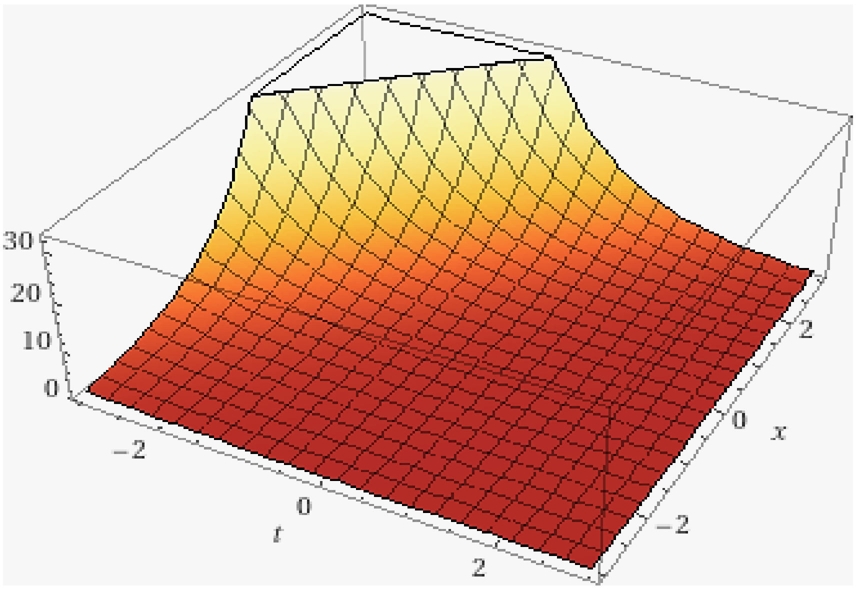

Figure 4.

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