Research Article

Best Approximation In Linear K-Normed Spaces

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Abstract: The article describes a new idea and established the concept the existence and uniqueness for best approximation in linear k-normed spaces, proved the mapping form k-normed space into finite dimensional subspace of k-normed space is continuous , bounded compact subset of linear k-normed is proximal and characterization of best uniform approximation in same space.

Keyword: k-normed space, best approximation, strictly convex, uniform approximation and proximal

1. Introduction

The concept of linear 2-norm spaces first investigated by Gahler [1] in 1964 and has been extensively by [2,3,4,5]. The introduce a new concept called 2-normed almost linear space and proved some of the results of best approximation in its space by Markandeya [6].
Recently, some results on best approximation theory in linear 2-norm spaces considered by [7,8] and characterization of best uniform approximation real linear 2-normed space by [9]. The theory of k-normed spaces studied by [10,11].
This paper mainly deals with existence, uniqueness , continuity of best approximation with respect to k-normed spaces , bounded compact subset of k-normed linear space is proximal and characterization of best uniform approximation in k-normed space.

2. Preliminaries

Definition 2.1:
Let $k \in \mathbb{N}$ [natural numbers] and $\mathcal{A}$ be linear space of dimensional $d \geq k$. A real valued function $\|\bullet, \dots, \bullet\|$ on $\underbrace{\mathcal{A} \times \mathcal{A} \times \dots \times \mathcal{A}}_{k-tuples} = \mathcal{A}^k$ satisfying the following conditions is called an k-normed on $\mathcal{A}$. $\forall\, r_1, ., \dots, r_k, b, c \in \mathcal{A}$

N_1 :$\|r_1, . \dots, r_k\| = 0$ iff $r_1, ., \dots, r_k$ are linearly independent.
N_2 :$\|r_1, r_2, . \dots, r_k\|$ *is* invariant under any transformation.
N_3 :$\|r_1, r_2, . \dots, \delta r_k\| = |\delta| \|r_1, r_2, . \dots, r_k\|$ for all $\delta \in \mathbb{R}$ (set of real numbers)
N_4 : $\|r_1, , . \dots, r_{k-1}, b + c\| \leq \|r_1, . \dots, r_{k-1}, b\| + \|r_1, , . \dots, r_{k-1}, c\|$.
The pair $(\mathcal{A}, \|\bullet, \dots, \bullet\|)$ is called an k-normed linear space.

Definition 2.2:
Let $\mathcal{F}$ be a subset of real k-normed space $\mathcal{A}$ and $r_1, r_2, . \dots, r_k \in \mathcal{A}$. Then $f^* \in \mathcal{F}$ is called the *best approxi.* to $r_k \in \mathcal{A}$ from $\mathcal{F}$ if

$$\|r_1, r_2, . \dots, r_k - f^*\| = \underbrace{inf}_{f \in \mathcal{F}} \|r_1, r_2, . \dots, r_k - f\|.$$

The set of all best approximation of $\mathcal{A}$ out of $\mathcal{F}$ denoted by $\Gamma_{\mathcal{F}}(f)$ and define
$\Gamma_{\mathcal{F}}(f) = \{f \in \mathcal{F};\ inf\|a_1, a_2, \dots, a_k - f\|\ \ \forall a_k \in \mathcal{A}\}$.

Definition 2.3
A linear k-normed space$(\mathcal{A}, \|\bullet, \dots, \bullet\|)$ is called strictly convex if
$\|\mathcal{R}, f\| = \|\mathcal{R}, g\| = 1, f \neq g$ and

$$\mathcal{R} = (r_1, r_2, . \dots, r_{k-1}), \quad r_1, r_2, . \dots, r_{k-1} \in \mathcal{A} \,/\, \mathcal{F}(f, g)$$

implies $\left\| \mathcal{R}, \frac{1}{2}(\mathscr{f} + \mathscr{g}) \right\| < 1$, $\mathcal{F}(\mathscr{f}, \mathscr{g})$is the subspace of $\mathcal{A}$ generated by $\mathscr{f}$ and $\mathscr{g}$.

Definition 2.4

Let $\mathcal{A}$ be k-normed space. The set $\mathcal{F}$ is said to be proximal if $\Gamma_{\mathcal{F}}(\mathscr{f}) \neq \emptyset$ for every $a_i \in \mathcal{A}, 1 \leq i \leq k$ where$\Gamma_{\mathcal{F}}(\mathscr{f})$ is the set of all best approximation of $\mathscr{f}$ to a_i ,$1 \leq i \leq k$.

Definition 2.5

Let $f \in C([\dot{r}_0, \dot{r}_1] \times [\dot{r}_1, \dot{r}_2] \times \dots \times [\dot{r}_{k-1}, \dot{r}_k])$ and $\|\mathscr{f}\|_\infty = sup\{|\mathscr{f}(\mathrm{u}_1, \mathrm{u}_2, \dots, \mathrm{u}_k|: \mathrm{u}_1 \in [\dot{r}_0, \dot{r}_1], \mathrm{u}_2 \in [\dot{r}_1, \dot{r}_2], \dots, \mathrm{u}_k \in [\dot{r}_{k-1}, \dot{r}_k]\}$.

The set of extreme points of function

$\mathscr{f} \in C([\dot{r}_1, \dot{r}_2] \times [\dot{r}_1, \dot{r}_2] \times \dots \times [\dot{r}_{k-1}, \dot{r}_k])$ is define by

$\mathcal{E}(\mathscr{f}) = \{\|\mathscr{f}\|_\infty = |\mathscr{f}(x_1, x_2, \dots, x_k|: x_1 \in [a_0, a_1], x_2 \in [a_2, a_3], \dots, x_k \in [a_{k-1}, a_k]\}$.

Best approximation with respect to this norm is called best uniform approximation.

3. Main Results

In this part, we prove existenss, uniqueness of best approximation in k-normed considered , the mapping from k-normed space $\mathcal{A}$ into finite dimensional subspace $\mathcal{F}$ is continuous ,bounded &closed subspace of k-normed homomorphism space is proximal & characterization of best uniform approximation in k-normed space.

Theorem 3.1

Let $\mathcal{F} = \{\mathscr{f}_1, \mathscr{f}_2, \dots, \mathscr{f}_n\} \subset \mathcal{A}$. Then for all $a_i \in \mathcal{A},\ 1 \leq i \leq k$, there is best approximation$\mathscr{f} \in \mathcal{F}$.

Proof : Let $\dot{r}_i \in \mathcal{A},\ 1 \leq i \leq k$ and $\mathcal{R} = \dot{r}_1, , \dots \dot{r}_{k-1},$. Then by using definition of infimum, there is a sequence $\{\mathscr{f}_k\} \in \mathcal{F}$ such that

$\|\mathcal{R}, \dot{r}_k - \mathscr{f}_k\| \to \underbrace{inf}_{\mathscr{f} \in \mathcal{F}} \|\mathcal{R}, \dot{r}_k - \mathscr{f}\|$.

This implies that there exist absolute constant $c > 0$, such that for k.

$$\|\mathcal{R}, \mathscr{f}_k\| - \|\mathcal{R}, \dot{r}_k\| \leq \underbrace{inf}_{\mathscr{f} \in \mathcal{F}} \|\mathcal{R}, \dot{r}_k - \mathscr{f}\| + 1c \leq \|\mathcal{R}, \dot{r}_k\| .$$

For k hence

$$\|\mathcal{R}, \dot{r}_k - \mathscr{f}_k\| \leq 2\|\mathcal{R}, \dot{r}_k\| + c.$$

Thus $\{\mathscr{f}_k\}$ is bounded sequences. Then there exists subsequence $\{\mathscr{f}_{k_l}\}$ of $\{\mathscr{f}_k\}$ convergent to $\mathscr{f}^* \in \mathcal{F}$

$\lim_{l \to \infty} \left\|\mathcal{R}, \dot{r}_k - \mathscr{f}_{k_l}\right\| = \|\mathcal{R}, \dot{r}_k, \mathscr{f}^*\|$

$\mathcal{R}$=$\dot{r}_1, ,. \dots \dot{r}_{k-1} \in \mathcal{A}$

implies $\mathscr{f}^*$ is best approximation to $\dot{r}_i \in \mathcal{A}$, $1 \leq i \leq k$.

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Theorem 3.2

$\mathcal{L}$et $\mathcal{A}$ be the strictly convex lined k-normed $s$pace and $\mathcal{F} = \{\mathscr{f}_1, \mathscr{f}_2, \dots, \mathscr{f}_n\} \subset \mathcal{A}$. Then every $\dot{r}_i \in \mathcal{A},\ 1 \leq i \leq k$, there is a only one best approximation from $\mathcal{F}$.

Proof:

Let $\dot{r}_i \in \mathcal{A},\ 1 \leq i \leq k$, we have $\mathcal{F}$ finite dimensional. So, from theorem 3.1 there is element $\mathscr{f}^* \in \mathcal{F}$ such that$\mathscr{f}^*$ is best approximation $\dot{r}_i \in \mathcal{A}$, $1 \leq i \leq k$.

For that first, we prove that $\mathcal{F}$ is convex. Let $\mathscr{f}_1, \mathscr{f}_2 \in \mathcal{F}$, $\mathcal{R}$=$\dot{r}_1, ,. \dots \dot{r}_{k-1} \in \mathcal{A}$ and $0 \leq \lambda \leq 1$.Then

$$\begin{aligned} \|\mathcal{R}, \dot{r}_k - (\lambda\mathscr{f}_1 + (1-\lambda)\mathscr{f}_2\| &= \|\mathcal{R}, \lambda(\dot{r}_k - \mathscr{f}_1) + (1-\lambda)(\dot{r}_k - \mathscr{f}_2)\| \\ &\leq \|\mathcal{R}, \lambda(\dot{r}_k - \mathscr{f}_1)\| + \|\mathcal{R}, (1-\lambda)(\dot{r}_k - \mathscr{f}_2)\| \\ &= \lambda\|\mathcal{R}, \dot{r}_k - \mathscr{f}_1)\| + (1-\lambda)\|\mathcal{R}, \dot{r}_k - \mathscr{f}_2\| \\ &= \lambda \underbrace{inf}_{\mathscr{f} \in \mathcal{F}} \|\mathcal{R}, \dot{r}_k - \mathscr{f})\| + (1-\lambda) \underbrace{inf}_{\mathscr{f} \in \mathcal{F}} \|\mathcal{R}, \dot{r}_k - \mathscr{f}\| \\ &= \underbrace{inf}_{\mathscr{f} \in \mathcal{F}} \|\mathcal{R}, \dot{r}_k - \mathscr{f})\| \end{aligned}$$

implies $\lambda\mathscr{f}_1 + (1-\lambda)\mathscr{f}_2 \in \mathcal{F}$

so, $\mathcal{F}$ is convex space.

We shall suppose that$\mathscr{f}^{**} \in \mathcal{F}$, implies $\frac{1}{2}(\mathscr{f}^* + \mathscr{f}^{**}) \in \mathcal{F}$.

$$\left\|\mathcal{R}, \frac{1}{2}((\dot{r}_k - \mathscr{f}^*) + (\dot{r}_k - \mathscr{f}^{**}))\right\| + \left\|\mathcal{R}, \dot{r}_k - \frac{1}{2}(\mathscr{f}^* + \mathscr{f}^{**})\right\| = \underbrace{inf}_{\mathscr{f} \in \mathcal{F}} \|\mathcal{R}, \dot{r}_k - \mathscr{f})\|$$

implies $\dot{r}_k - f^* = \dot{r}_k - f^{**}$, we obtain $f^* = f^{**}$.
Thus $\mathcal{F}$ contain only one best approximation to $\dot{r}_i \in \mathcal{A},\ 1 \le i \le k$.

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Theorem 3.3
Let $\mathcal{F} = \{f_1, f_2, \dots, f_n\} \subset \mathcal{A}$ with the property that every function with domain $\mathcal{A} \times \mathcal{A} \times \dots \times \mathcal{A}$ has only one best approximation from $\mathcal{F}$. Then for all á, à $\in \mathcal{A}$,

$$\left| \underbrace{inf}_{f\in\mathcal{F}} \|\dot{r}_1, \dot{r}_2, \dots \dot{r}_{k-1}, á - f\| - \underbrace{inf}_{f\in\mathcal{F}} \|\dot{r}_1, \dot{r}_2, \dots \dot{r}_{k-1}, à - f\| \right|$$

$\le \|\dot{r}_1, \dot{r}_2, \dots \dot{r}_{k-1}, á - à\|, \dot{r}_1, \dot{r}_2, \dots \dot{r}_{k-1} \in \mathcal{A}/\mathcal{F}$ and
$Q_{\mathcal{F}}: \mathcal{A} \times \mathcal{A} \times \dots \times \mathcal{A} \to \mathcal{F}$ is continuous.

Proof :
Suppose that $Q_{\mathcal{F}}$ discontinuous. Then there exist an element $â \in \mathcal{A}$ and sequence $\{â_k\}$ in $\mathcal{A}$ such that $Q_{\mathcal{F}}(â_k) \nrightarrow Q_{\mathcal{F}}(â)$.
Since $\mathcal{F}$ is finite dimensional, there exist subsequence $\{â_{k_l}\}$ of $\{â_k\}$ such that
$Q_{\mathcal{F}}: \{â_{k_l}\} \to f \in \mathcal{F}$,$f \ne Q_{\mathcal{F}}(â)$ and we shall show that the mapping
$f \to \underbrace{inf}_{f\in\mathcal{F}} \|a_1, a_2, \dots, a_{k-1}, â - f\|$ is continuous, $â. \in \mathcal{A}$.
Let á, à $\in \mathcal{A}$ and $\mathcal{R} = \dot{r}_1, \dots \dot{r}_{k-1} \in \mathcal{A}$. Then there exist $f_1 \in \mathcal{F}$ such that
$\|\mathcal{R}, à - f_1\| = inf\|\mathcal{R}, à - f\|$ and

$$\underbrace{inf}_{f\in\mathcal{F}} \|\mathcal{R}, á - f\| \le \|\mathcal{R}, á - f_1\| \le \|\mathcal{R}, á - à\| + \|\mathcal{R}, à - f_1\|$$

$$= \|\mathcal{R}, á - à\| + \underbrace{inf}_{f\in\mathcal{F}} \|\dot{r}_1, \dot{r}_2, \dots \dot{r}_{k-1}, à - f_1\|$$

implies

$$\underbrace{inf}_{f\in\mathcal{F}} \|\mathcal{R}, á - f\| - \underbrace{inf}_{f\in\mathcal{F}} \|\mathcal{R}, à - f\| \le \|\mathcal{R}, á - à\|, \qquad ,\mathcal{R} = \dot{r}_1, \dots \dot{r}_{k-1} \in \mathcal{A}/\mathcal{F}$$

.
This proves that

$$\left| \underbrace{inf}_{f\in\mathcal{F}} \|\mathcal{R}, á - f\| - \underbrace{inf}_{f\in\mathcal{F}} \|\mathcal{R}, à - f\| \le \|\mathcal{R}, á - à\|. \right|$$

By continuity it surveys that

$$\|\mathcal{R}, â_{k_l} - Q_{\mathcal{F}}(â_{k_l})\| = inf\|\mathcal{R}, â_{k_l} - f\|$$

implies

$$\underbrace{inf}_{f\in\mathcal{F}} \lim_{l\to\infty} \|\mathcal{R}, â_{k_l} - f\| = \|\mathcal{R}, â - f\| \text{ and}$$

$$\lim_{l\to\infty} \|\mathcal{R}, â_{k_l} - Q_{\mathcal{F}}(â_{k_l})\| \to \|\mathcal{R}, â - Q_{\mathcal{F}}(â)\|.$$

We obtain f and $Q_{\mathcal{F}}(â)$ are two distinct best approximation of â, this contradiction with the hypothesis where$\mathcal{A}$ has a unique best approximation from $\mathcal{F}$.
So, $Q_{\mathcal{F}}: \mathcal{A} \times \mathcal{A} \times \dots \times \mathcal{A} \to \mathcal{F}$ is continuous.

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Theorem 3.4
If $\mathcal{A}$ is k-normed linear spaces & $\mathcal{F}$ compact subset of $\mathcal{A}$. Then $\mathcal{F}$ is proximal in $\mathcal{A}$.

Proof :
Let $\mathcal{F}$ be closed and bounded subset of $\mathcal{A}$ and $\{f_n\}_{n=1}^{\infty}$ sequence in $\mathcal{F}$ and
$\mathcal{R} = \dot{r}_1, \dots, \dot{r}_{k-1} \in \mathcal{A}$ such that
$\|\mathcal{R}, \dot{r}_k - f\| = \lim_{n\to\infty} \|\mathcal{R}, \dot{r}_k - f_n\|$.
Since $\mathcal{F}$ is bounded for some $\epsilon > 0$, there exist positive integer $\mathbb{N}$ such that

$$\|\mathcal{R} - f_n\| \le \|\mathcal{R}, \dot{r}_k - f_1\| + \epsilon \ \le \ \mathrm{K} + \epsilon \quad \forall\, n \ge \mathbb{N}$$

Where $\mathrm{K} = \max\{\mathrm{K}_1, \mathrm{K}_2\}$

$K_1 = \|\mathcal{R}, \dot{r}_k - f_1\| + \epsilon$ and
$K_2 = max\|\mathcal{R}, \dot{r}_k - f_n\| a_k \; for \; n \leq \mathbb{N}$.
Now, $\|\mathcal{R}, f_n\| \leq \|\mathcal{R}, \dot{r}_k - f_n\| + \|\mathcal{R}, \dot{r}_k\| \leq K + \|\mathcal{R}, \dot{r}_k\|$
this, implies that $\{f_n\}_{n=1}^{\infty}$ is bounded.
So, $\{f_n\}_{n=1}^{\infty}$ approximate to f in $\mathcal{F}$.
Hence, we have
$inf\|\mathcal{R}, \dot{r}_k - f_1\| \leq \lim_{n\to\infty} \|\mathcal{R}, \dot{r}_k - f_n\| = \|\mathcal{R}, \dot{r}_k - f_1\|$
but $\|\mathcal{R}, \dot{r}_k - f\| \geq \|\mathcal{R}, \dot{r}_k - f_1\|$ implies
$inf\|\mathcal{R}, \dot{r}_k - f\| \geq \|\mathcal{R}, \dot{r}_k - f_1\|$.
We obtain $inf\|\mathcal{R}, \dot{r}_k - f\| = \|\mathcal{R}, \dot{r}_k - f_1\|$.
Thus, f_1 is best approximation to $\dot{r}_k$ from $\mathcal{F}$.Hence $\mathcal{F}$ is proximal in $\mathcal{A}$.

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Theorem 3.5
Let $\mathcal{F}$ be a subspace of $C([\dot{r}_0, \dot{r}_1] \times [\dot{r}_1, \dot{r}_2] \times \ldots \times [\dot{r}_{k-1}, \dot{r}_k])$ and
$\alpha \in C([\dot{r}_0, \dot{r}_1] \times [\dot{r}_1, \dot{r}_2] \times \ldots \times [\dot{r}_{k-1}, \dot{r}_k])$. Then the follow. Statement are equivalents . :

i. For every function $f \in \mathcal{F}$, the points $\mathcal{U} = \dot{u}_1, \dot{u}_2, \ldots, \dot{u}_k \in \mathcal{E}(\alpha - f_\circ)$,
$\big(\alpha(\mathcal{U}) - f_\circ(\mathcal{U})\big)\big(f(\mathcal{U})\big) \leq 0, f_\circ \in \mathcal{F}$.

ii. The function $f_\circ$ is best uniform approximation of α from $\mathcal{F}$.

Proof :
Suppose that (i) holds and we need to prove that $f_\circ$ is best uniform approximation to α from $\mathcal{F}$.
From (i) there exist points $\mathcal{U} = \dot{u}_1, \dot{u}_2, \ldots, \dot{u}_k \in \mathcal{E}(\alpha - f_\circ)$ such that
$(\alpha(\mathcal{U}) - f_\circ(\mathcal{U}))(f(\mathcal{U})) \leq 0$. So, we have

$$\begin{aligned}\|\alpha - f_\circ\|_\infty &\leq |\alpha(\mathcal{U}) - f_\circ(\mathcal{U})| + |f(\mathcal{U}) - f_\circ(\mathcal{U})| \\ &= |\alpha(\mathcal{U}) - f_\circ(\mathcal{U}) + f_\circ(\mathcal{U}) - f(\mathcal{U})| \\ &= |\alpha(\mathcal{U}) + f(\mathcal{U})| \leq \|\alpha - f\|_\infty \;.\end{aligned}$$

Which shows $f_\circ$ is best uniform approximation to α from $\mathcal{F}$.
Conversely suppose that (ii) holds and to prove (i).
Assume (i) be unsuccessful. Then there exist function $\mathfrak{f}_1 \in \mathcal{F}$ such that
$\mathcal{U} \in \mathcal{E}(\alpha - \mathfrak{f}_\circ)$,
$(\alpha(\mathcal{U}) - \mathfrak{f}_\circ(\mathcal{U}))(\mathfrak{f}_1(\mathcal{U})) > 0$. Since $\mathcal{E}(\alpha - \mathfrak{f}_\circ)$ is closed & bounded , $\exists\, C_1 > 0$ & $C_2 > 0 \ni$ for each $\mathcal{U} \in \mathcal{E}(\alpha - \mathfrak{f}_\circ)$

$$(\alpha(\mathcal{U}) - \mathfrak{f}_\circ(\mathcal{U}))(\mathfrak{f}_1(\mathcal{U})) > C_1 \;. \tag{1}$$

Further, there exists an open ball $\mathcal{N}$ of $\mathcal{E}(\alpha - \mathfrak{f}_\circ)$ such that for all
$\mathcal{U} \in \mathcal{N}$ and

$$C_1(\alpha(\mathcal{U}) - f_\circ(\mathcal{U}))(f_1(\mathcal{U})) > C_2 \;. \tag{2}$$

$$|\alpha(\mathcal{U}) - \mathfrak{f}_\circ(\mathcal{U})| \geq C_2\|\alpha - \mathfrak{f}_\circ\|_\infty \;. \tag{3}$$

Since $[\dot{r}_0, \dot{r}_1]/\mathbb{N}$ is compact, there exist a positive real number C_3 such that for all
$\mathcal{U} \in [a_0, a_1]/\mathbb{N}$,

$$|\alpha(\mathcal{U}) - \mathfrak{f}_\circ(\mathcal{U})| < \|\alpha - \mathfrak{f}_\circ\|_\infty - C_3 \;. \tag{4}$$

Now we shall assume that $\|\mathfrak{f}_1\|_\infty \leq \min\{C_3, \|\alpha - \mathfrak{f}_\circ\|\}$. (5)
Let $f_2 = \mathfrak{f}_\circ + \mathfrak{f}_1$. Then by (4) and (5) for all $\dot{u}_1, \dot{u}_2, \ldots, \dot{u}_k \in [a_0, a_1]/\mathcal{N}$,

$$\begin{aligned}|\alpha(\mathcal{U}) - f_2(\mathcal{U})| &= |\alpha(\mathcal{U}) - \mathfrak{f}_\circ(\mathcal{U}) - \mathfrak{f}_1(\mathcal{U})| \\ &\leq |\alpha(\mathcal{U}) - \mathfrak{f}_\circ(\mathcal{U})| + |\mathfrak{f}_1(\mathcal{U})| \\ &\leq \|\alpha - \mathfrak{f}_\circ\|_\infty - C_3 + \|\mathfrak{f}_1\|_\infty \leq \|\alpha - \mathfrak{f}_\circ\|_\infty.\end{aligned}$$

For all $\mathcal{U} \in \mathbb{N}$ from (3),(4) and (5), we obtain

$$\begin{aligned}|\alpha(\mathcal{U}) - f_2(\mathcal{U})| &= |\alpha(\mathcal{U}) - \mathfrak{f}_\circ(\mathcal{U}) - \mathfrak{f}_1(\mathcal{U})| \leq |\alpha(\mathcal{U}) - \mathfrak{f}_\circ(\mathcal{U})| + |\mathfrak{f}_1(\mathcal{U})| \\ &\leq \|\alpha - \mathfrak{f}_\circ\|_\infty - C_3 + \|f_1\|_\infty \leq \|\alpha - \mathfrak{f}_\circ\|_\infty.\end{aligned}$$

Implies, $\|\alpha - f_2\|_\infty \leq \|\alpha - \mathfrak{f}_\circ\|_\infty$.
Hence $\mathfrak{f}_\circ$ is not best uniform approximation to α which is a contradiction.
So, (i) holds.

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Conclusion

In this paper, we conclude that when linear k-normed space is bounded compact then best approximation of the functions existence and a unique and the map from k-normed space into finite dimensional subspace of it is continuous, bounded and closed.

References

1. Gahler, S. (1964) , Linear 2-norm space, Math. Nachr., 28,. 1-43.
2. Freese, R. and Ghler, S. (1982) , Remarks on semi 2-normed spaces, Math. Nachr., 105,. 151-161.
3. Cho, Y.J.. , Diminmi, C., ect. (1992) Isosceles orthogonal triples in linear 2-normed space, Math. Nachr., 157,. 225-234.
4. Freese, R., Cho .,Y.J. and Kim, S.S. (1992), Strictly 2-convex linear 2-normed spaces, Journal Korean Math. Soc. 29,.391-400.
5. Kim ,S.S., Cho., Y.J. and White, A. (1992), Linear operators on linear 2-normed spaces , Glasink Math. 27(1992),. 63-70.
6. Makandeya, T. and Bharathi, D. (2013), Best approximation in 2-normed almost linear space, International journal of engineering research and technology, 12,.3569-3573.
7. Elumalia, S. and Ravi, R. (1992), Approximation in linear 2-normed space, Indian journal Math. 34, 53-59.
8. Kim, S.S. and Cho. ,Y.J. (1996), Strictly convexity in linear k-normed spaces, Demonstration Math. 29No.4, 739-744.
9. Vijayaragavan,R. (2013), Best approximation in real 2-normed spaces,IOSR Journal of Mathematics 6, 16-24.
10. Malceski,R. (1997), Strong convex n-normed space, Math. Bilten , No.21, 81-102.
11. Gunawan, H. and Mashadi ,M. (2001), On k-normed spaces, International journalliumath.,Math. Sci. 27, No. 10, 631-639.