Mleand Bayes Estimator of Utilization Factor in Power Supply Queuing Model

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Abstract: In this paper,we have been obtainedMLE and Bayes Estimator of utilization factor in Power Supplyqueuing model based on requirement of power forthe number of customers at various sampled time points. We have been checked the consistency of maximum likelihood estimator of utilization factor and find the confidence interval for expected MLE of it.Bayes estimators of utilization factor havealso been obtained.We also derive the expression for Expectation of MLE of ρ, Bayes estimate of ρ under squared error loss function. Fordifferent choices of hyper-parametersthe minimum posterior risk of this Bayes estimator has also been obtained numerically by using R software packages.

Keywords: MLE, Bayes Estimator, utilization factor, Power Supply Queuing Model. Minimum Posterior Risk.

1.Introduction

The very important aspects of queuing theory are to estimate different parameters of various queuing models. One can serve this purpose by applying both classical and Bayesian framework. Several practitioners have studied Bayes estimator and maximum likelihood estimator of utilization factor (ρ), average arrival rate (λ), average service rate (μ), mean number of customer in the system (L_s), mean number of customer in the queue (L_q), mean waiting time per customer in the system (W_s), mean waiting time per customer in the queue (W_q) in M/M/1, M/E_k/1, M/G/1 and many other models. In the classical approach Clarke (1957) obtained maximum likelihood estimator of the parameters λ and μ in M/M/1 queue model withinput Poisson and negative exponential output.Muddapur (1972) derived Bayes estimator of λ and μ in M/M/1 queue using the likelihood function that Clarke used. Basawa and Prabhu (1988) have also obtained maximum likelihood estimators for performance measures based on four different sampling techniques and also proved its consistency and asymptotic normality. Acharya (1999) has also studied MLE of the arrival rate and the service rate taking GI/G/1 queuing model with the convergence rate of the distribution. Sharma and Kumar (1999) presented the MLE, UMVUE, Bayesian estimators and the testing of hypothesis of utilization factor and various other measures of performance of M/M/1 queuing model. Mukherjee and Chowdhury (2006) in their paper have shown the MLE and Bayes estimator in M/M/1 queuing model. They obtained maximum likelihood estimator of utilization factor based on number of customers present at various time pointsof sample and Bayes minimum risk. Choudhury and Borthakur (2008) obtained Bayes estimator and credibility intervals of utilization factor in M/M/1 queue. They also derived predictive distribution of system size at departure epoch for utilization factor by using natural conjugate prior and truncated Uniform prior. Chowdhury and Mukherjee (2013) have found MLEand Bayes estimator of utilization factor at ordered departure epochs in an M/M/1queuing model in equilibrium based on supported range of number within the queue. Choudhury and Basak (2018) obtained MLE of utilization factor by combining the M/M/1 queue system with the Bernoulli process. By using technique of randomised testing they also determined sample size. Most recently, Basak and Choudhury (2019) studied maximum likelihood estimator of utilization factor assuming that it is random quantity and unknown. Assuming two forms of prior information Bayes estimators of utilization factor are derived under squared error loss function. Based on principle of maximum likelihood the measure of performance of the projected Bayes estimators is compared with that of classical estimator.

From the above discussion it has been clear that almost all practitioners have obtained MLE and Bayes estimator of utilization factor of queuing models by observing mainly the number of customer arrived and the number of customer get service in the discrete set up and inter-arrival time, service time and waiting or sojourn time in the continuous set up.

But in reality, one can most easily observed the number of customers present at various time points.The main aim of this paper is to obtain maximum likelihood estimator of utilization factor of a power supply queuing model and to check its large sample properties. The exact distribution of MLE of utilization factor is also obtained. We also derived confidence interval for MLE of utilization factor. In the next section Bayes estimator of utilization factor is derived under the same set up. In the last section, Expectation of MLE of ρ, Bayes estimate of ρ under squared error loss function and the minimum posterior risk of this Bayes estimator for different choices of hyper-parameters has also been obtained numerically.

2. Model Description

In this paper, the model we deal with is the Power Supply Model; let us suppose there is an electric circuit supplying power to 'k' consumers. The requirements of consumers follow Poisson distribution with parameter λ, and the supply schedule also follows Poisson distribution with parameter μ.

If,at any time, there are xconsumers in the queue then

$$\left.\begin{matrix}\lambda_x=(k-x)\lambda \\ \mu_x=x\mu\end{matrix}\right\} 0 \le x \le k \qquad (2.1)$$

and equations leading this system are obtained as follows:

$P_0(t+\Delta t) = P_0(t)(1-k\lambda\Delta t) + P_1(t)\mu\Delta t + O(\Delta t)^2,\ for\ x=0$ (2.2)

$P_x(t+\Delta t) = P_x(t)[1-\{(k-x)\lambda + x\mu\}\Delta t] + P_{x-1}(t)(k-x+1)\lambda\Delta t + P_{x+1}(t)(x+1)\mu\Delta t, for\ 1 \le x \le k-1$ (2.3)

$P_k(t+\Delta t) = P_k(t)(1-k\mu\Delta t) + P_{k-1}(t)\lambda\Delta t, for\ x = k$ (2.4)

Thus, taking appropriate limits for steady state system, to obtain difference equations

$k\lambda P_0 = \mu P_1, for\ x = 0$ (2.5)

$[(k-x)\lambda + x\mu]P_x = (k-x-1)\lambda P_{x-1} + (x+1)\mu P_{x+1}, for\ 1 \le x \le k-1$ (2.6)

$k\mu P_k = \lambda P_{k-1}, for\ x = k$ (2.7)

From equations (2.5), (2.6) and (2.7), we get the recurrence relation is given by

$$(k-x)\lambda P_x = (x+1)\mu P_{x+1}, x = 0,1,\dots,k. for\ x = k$$

$$\therefore\ P_1 = k\left(\frac{\lambda}{\mu}\right)P_0, for\ x = 0$$

$$P_2 = \frac{k(k-1)}{2!}\left(\frac{\lambda}{\mu}\right)^2 P_0, for\ x = 1$$

$$\vdots$$

$P_x = \frac{k(k-1)(k-2)\dots(k-x+1)}{x!}\left(\frac{\lambda}{\mu}\right)^x P_0$, and

$$P_k = \left(\frac{\lambda}{\mu}\right)^k P_0$$

Since $\sum_{x=0}^{k} P_x = 1$

$$\Rightarrow P_0\left[1 + k\left(\frac{\lambda}{\mu}\right) + \frac{k(k-1)}{2!}\left(\frac{\lambda}{\mu}\right)^2 + \dots + \left(\frac{\lambda}{\mu}\right)^k\right] = 1$$

$$\Rightarrow P_0\left(1 + \frac{\lambda}{\mu}\right)^k = 1$$

$$\Rightarrow P_0 = \left(\frac{\mu}{\lambda+\mu}\right)^k$$

Thus, $P_x = \frac{k(k-1)\dots(k-x+1)}{x!}\left(\frac{\lambda}{\mu}\right)^x\left(\frac{\mu}{\lambda+\mu}\right)^k$

or, $p_x = \binom{k}{x}\left(\frac{\lambda}{\lambda+\mu}\right)^x\left(\frac{\mu}{\lambda+\mu}\right)^{k-x}, x = 0,1,2,\dots,k$ (2.8)

(Sharma, 2014)

Taking, $\rho\ (=\frac{\lambda}{\mu})$ we get,

$p_x = \binom{k}{x}\left(\frac{\rho}{1+\rho}\right)^x\left(\frac{1}{1+\rho}\right)^{k-x}, x = 0,1,2,\dots,k;$ (2.9)

which is a binomial distribution with parameters k and $\frac{\rho}{1+\rho}$.

Since for binomial distribution probability of success lies between 0 and 1, therefore here utilization factor, $\rho > 0$.

3. MLE of utilization factor(ρ)

Here, we explain maximum likelihood (ML) principle of utilization factor, ρ which is a classical estimation method. For this purpose, we at first assume that the parameter ρ is an unknown predetermined quantity and based on the requirements of consumers at various sampled time points, we have been obtained maximum likelihood estimator of ρ for power supply queuing model.

Now, likelihood function of requirements of consumers $(x_1, x_2, \dots, x_n)$ present at n different time points $t_1, t_2, \dots, t_n$is given as

$L(\rho|x) = \prod_{i=1}^{n}\binom{k}{x_i}\left(\frac{\rho}{1+\rho}\right)^{x_i}\left(\frac{1}{1+\rho}\right)^{k-x_i}$ (3.1)

and the log-likelihood function is

$$\mathrm{InL}(\rho|x) = \mathrm{In}\left[\prod_{i=1}^{n}\binom{k}{x_i}\left(\frac{\rho}{1+\rho}\right)^{x_i}\left(\frac{1}{1+\rho}\right)^{k-x_i}\right]$$

$$= \sum_{i=1}^{n}\left[\mathrm{In}\binom{k}{x_i} + x_i\mathrm{In}\left(\frac{\rho}{1+\rho}\right) + (k-x_i)\mathrm{In}\left(\frac{1}{1+\rho}\right)\right] \tag{3.2}$$

$$\therefore \frac{\partial}{\partial\rho}\mathrm{InL}(\rho|x) = \frac{\sum_{i=1}^{n}x_i}{\rho} - \frac{\sum_{i=1}^{n}x_i}{1+\rho} - \frac{nk-\sum_{i=1}^{n}x_i}{1+\rho} \tag{3.3}$$

$$Now, \quad \frac{\partial}{\partial\rho}\mathrm{InL}(\rho|x) = 0$$

$$\Rightarrow \frac{\sum_{i=1}^{n}x_i}{\rho} - \frac{\sum_{i=1}^{n}x_i}{1+\rho} - \frac{nk-\sum_{i=1}^{n}x_i}{1+\rho} = 0$$

$$\Rightarrow \hat{\rho}_{ML} = \frac{\sum_{i=1}^{n}x_i}{nk-\sum_{i=1}^{n}x_i} = \frac{y}{nk-y}, \text{ where } y = \sum_{i=1}^{n}x_i \tag{3.4}$$

Therefore, maximum likelihood estimator of ρ is $\hat{\rho}_{ML} = \frac{y}{nk-y}$

4. Consistency of MLE of ρ

Since $\hat{\rho}_{ML} = \frac{\sum_{i=1}^{n}x_i}{nk-\sum_{i=1}^{n}x_i}$

Now, as $x_i \sim BinomialDistribution\left(k, \frac{\rho}{1+\rho}\right)$

$$\therefore\ y = \sum_{i=1}^{n}x_i \sim BinomialDistribution\left(nk, \frac{\rho}{1+\rho}\right)$$

$\therefore E(y) = \frac{nk\rho}{1+\rho}$ and $V(y) = \frac{nk\rho}{(1+\rho)^2}$

$\hat{\rho}_{ML}$ is a one to one function of $\sum_{i=1}^{n}x_i$ and takes values $\frac{y}{nk-y}, y = 0,1,2,\ldots,k$whose probability mass function is given by

$$P\left(\frac{y}{nk-y} = v\right) = P\left(y = \frac{nkv}{1+v}\right) = \binom{k}{\frac{nkv}{1+v}}\left(\frac{\rho}{1+\rho}\right)^{\frac{nkv}{1+v}}\left(\frac{1}{1+\rho}\right)^{k-\frac{nkv}{1+v}} \tag{4.1}$$

which will give us

$$E(\hat{\rho}_{ML}) = \sum_{y=0}^{k}\frac{y}{nk-y}\binom{k}{y}\left(\frac{\rho}{1+\rho}\right)^{y}\left(\frac{1}{1+\rho}\right)^{k-y}$$

Now, when n is large,

$$E(\hat{\rho}_{ML}) \cong \frac{E(\sum_{i=1}^{n}x_i)}{nk-E(\sum_{i=1}^{n}x_i)} = \frac{\frac{nk\rho}{1+\rho}}{nk-\frac{nk\rho}{1+\rho}} = \rho$$

$$V(\hat{\rho}_{ML}) \cong [\left(\frac{\partial\hat{\rho}_{ML}}{\partial y}\right)^2 \mathrm{Vay}(y)]_{E(y)=\frac{nk\rho}{1+\rho}} = \frac{n^2k^2}{(nk-\frac{nk\rho}{1+\rho})^4}.\frac{nk\rho}{(1+\rho)^2} = \frac{\rho(1+\rho)^2}{nk} \to 0 \text{ as } k\to\infty \tag{4.2}$$

$$\therefore \hat{\rho}_{ML} \text{ is a consistent estimator of } \rho.$$

5. Confidence Interval for utilization factor(ρ)

We now consider construction of large sample confidence intervals which are invariant under the transformation of parameters. Here, our probability mass function (pmf) is given by

$$p_x = \binom{k}{x}\left(\frac{\rho}{1+\rho}\right)^{x}\left(\frac{1}{1+\rho}\right)^{k-x}, x = 0,1,2,\ldots,k$$

$$\therefore \frac{\partial}{\partial\rho}\mathrm{InL}(\rho|x) = \frac{\sum_{i=1}^{n}x_i}{\rho} - \frac{\sum_{i=1}^{n}x_i}{1+\rho} - \frac{nk-\sum_{i=1}^{n}x_i}{1+\rho}$$

$$E\left(\frac{\partial}{\partial\rho}\mathrm{InL}(\rho|x)\right) = E\left(\frac{\sum_{i=1}^{n}x_i}{\rho} - \frac{\sum_{i=1}^{n}x_i}{1+\rho} - \frac{nk-\sum_{i=1}^{n}x_i}{1+\rho}\right) = 0$$

$$\frac{\partial^2}{\partial\rho^2}\mathrm{InL}(\rho|x) = -\frac{\sum_{i=1}^{n}x_i}{\rho^2} + \frac{\sum_{i=1}^{n}x_i}{(1+\rho)^2} + \frac{nk-\sum_{i=1}^{n}x_i}{(1+\rho)^2}$$

$$V\left(\frac{\partial^2}{\partial\rho^2}\mathrm{InL}(\rho|x)\right) = -E\left(\frac{\partial^2}{\partial\rho^2}\mathrm{InL}(\rho|x)\right) = E\left(\frac{\sum_{i=1}^{n}x_i}{\rho^2} - \frac{\sum_{i=1}^{n}x_i}{(1+\rho)^2} - \frac{nk-\sum_{i=1}^{n}x_i}{(1+\rho)^2}\right) = \frac{nk}{\rho(1+\rho)^2}$$

By the CLT, their sample mean

$$\frac{1}{n}\frac{\partial}{\partial\rho}\mathrm{InL} \sim N(0, \frac{nk}{\rho(1+\rho)^2})$$

i.e., $z = \frac{\frac{\sum_{i=1}^{n} x_i}{\rho} - \frac{\sum_{i=1}^{n} x_i}{1+\rho} - \frac{nk-\sum_{i=1}^{n} x_i}{1+\rho}}{\sqrt{\frac{nk}{\rho(1+\rho)^2}}} \sim N(0,1)$ (5.1)

Hence, 100(1-α)% confidence interval for ρ is found out from the equation

$$P\left[-z_{\frac{\alpha}{2}} < z < z_{\frac{\alpha}{2}}\right] = 1-\alpha$$

$$\Rightarrow P\left[-z_{\frac{\alpha}{2}} < \frac{\frac{\sum_{i=1}^{n} x_i}{\rho} - \frac{\sum_{i=1}^{n} x_i}{1+\rho} - \frac{nk-\sum_{i=1}^{n} x_i}{1+\rho}}{\sqrt{\frac{nk}{\rho(1+\rho)^2}}} < z_{\frac{\alpha}{2}}\right] = 1-\alpha$$

$$\Rightarrow P\left[-z_{\frac{\alpha}{2}} < \frac{\sum_{i=1}^{n} x_i + (\sum_{i=1}^{n} x_i - nk)\rho}{\sqrt{nk\rho}} < z_{\frac{\alpha}{2}}\right] = 1-\alpha$$

$$\Rightarrow P\left[\left|\frac{\sum_{i=1}^{n} x_i + (\sum_{i=1}^{n} x_i - nk)\rho}{\sqrt{nk\rho}}\right| < z_{\frac{\alpha}{2}}\right] = 1-\alpha$$

$$\therefore \left(\frac{\sum_{i=1}^{n} x_i + (\sum_{i=1}^{n} x_i - nk)\rho}{\sqrt{nk\rho}}\right)^2 = z^2_{\frac{\alpha}{2}}$$

$$= \left(\frac{y + (y-nk)\rho}{\sqrt{nk\rho}}\right)^2 = z^2_{\frac{\alpha}{2}}$$

$$\Rightarrow (y-nk)^2\rho^2 + \left[2y(y-nk) - nkz^2_{\frac{\alpha}{2}}\right]\rho + y^2 = 0 \qquad (5.2)$$

Solving above quadratic equation in ρ, we have

$$\rho = \frac{-\left[2y(y-nk) - nkz^2_{\frac{\alpha}{2}}\right] \pm \sqrt{4\left[2y(y-nk) - nkz^2_{\frac{\alpha}{2}}\right]^2 - 4(y-nk)^2y^2}}{2(y-nk)^2}$$

∴ 100(1-α)% confidence interval for utilization factor, ρ is given by

$$\left(\frac{-\left[2y(y-nk)-nkz^2_{\frac{\alpha}{2}}\right] - \sqrt{4\left[2y(y-nk)-nkz^2_{\frac{\alpha}{2}}\right]^2 - 4(y-nk)^2y^2}}{2(y-nk)^2}, \frac{-\left[2y(y-nk)-nkz^2_{\frac{\alpha}{2}}\right] + \sqrt{4\left[2y(y-nk)-nkz^2_{\frac{\alpha}{2}}\right]^2 - 4(y-nk)^2y^2}}{2(y-nk)^2}\right) \qquad (5.3)$$

6. Bayesian Estimation of utilization factor(ρ)

Suppose we want to estimate Bayes estimator of utilization factor, ρ.For this purpose, we assume that the appropriate prior distribution is the beta distribution with parameters a and b so that ρ has the probability density function (pdf)given by

$$g(\rho|a,b) = \frac{1}{B(a,b)} \frac{\rho^{a-1}}{(1+\rho)^{a+b}}, \ \ \rho > 0$$

The joint probability mass function of $(x_1, x_2, \ldots, x_n)$ is given by

$$L(\rho|x) = \prod_{i=1}^{n} \binom{k}{x_i} \left(\frac{\rho}{1+\rho}\right)^{x_i} \left(\frac{1}{1+\rho}\right)^{k-x_i} = \left[\prod_{i=1}^{n} \binom{k}{x_i}\right] \left(\frac{\rho}{1+\rho}\right)^{\sum_{i=1}^{n} x_i} \left(\frac{1}{1+\rho}\right)^{nk-\sum_{i=1}^{n} x_i}$$

It may be seen that beta prior is the conjugate prior distribution as defined by (Raiffa and Schlaifer, 1961) of the binomial distribution, i.e., the posterior distribution of ρ corresponding to this prior distribution is also of the beta form.

Hence, Posterior distribution of ρ is given by

$$g_x(\rho) = \frac{g(\rho|a,b)\mathrm{L}(\rho|\mathrm{x})}{\int g(\rho|a,b)\mathrm{L}(\rho|\mathrm{x})d\rho}$$

$$= \frac{\frac{\rho^{a-1}}{B(a,b)(1+\rho)^{a+b}}\left[\prod_{i=1}^{n}\binom{k}{x_i}\right]\left(\frac{\rho}{1+\rho}\right)^{y}\left(\frac{1}{1+\rho}\right)^{nk-y}}{\int_0^{\propto}\frac{\rho^{a-1}}{B(a,b)(1+\rho)^{a+b}}\left[\prod_{i=1}^{n}\binom{k}{x_i}\right]\left(\frac{\rho}{1+\rho}\right)^{y}\left(\frac{1}{1+\rho}\right)^{nk-y}d\rho}$$

$$= \frac{\rho^{a+y-1}}{B(a+y,\ b+nk-y)(1+\rho)^{a+b+nk}} \quad (6.1)$$

Thus, under squared error loss function the Bayes estimate of ρ is given by

$$E_{\mathrm{x}}(\rho) = \int_0^{\propto} \frac{\rho^{a+y}}{B(a+y,\ \mathrm{b}+\mathrm{nk}-y)(1+\rho)^{a+b+nk}} d\rho$$

$$= \frac{a+y}{\mathrm{b}+\mathrm{nk}-y-1} \quad (6.2)$$

The minimum posterior risk in association with this Bayes estimator (Martz, 1982) is

$$V_x(\hat{\rho}^B) = \int_0^{\propto} \left(\rho - E_{\mathrm{x}}(\rho)\right)^2 \frac{\rho^{a+y}}{B(a+y,\ \mathrm{b}+\mathrm{nk}-y)(1+\rho)^{a+b+nk}} d\rho$$

$$= \frac{(a+y)(a+b+nk-1)}{(\mathrm{b}+\mathrm{nk}-y-1)^2(\mathrm{b}+\mathrm{nk}-y-2)} \quad (6.3)$$

7. Simulation

To simulate, $\hat{\rho}_{\mathrm{ML}} = \frac{\sum_{i=1}^{n} x_i}{nk-\sum_{i=1}^{n} x_i} = \frac{n\bar{x}}{nk-n\bar{x}} = \frac{\bar{x}}{k-\bar{x}}$ generate $n = 10, 100, 1000, 10000$ random values from a $Bin(k, \frac{\rho}{1+\rho})$ population where $\rho = 0.2, 0.5, 0.8$. Pay particular attention to the fact that $k = 10, 100, 1000, 10000$ and $n = 10, 100, 1000, 10000$

To generate a random sample of size n=1000, k=10 and ρ=0.2 we use the following R programming.

```
set.seed (23)
> x̄ <-mean(rbinom(1000, 10, 0.16666667))
> ρ <-x̄/(10 - x̄)
> ρ
[1] 0.1933174
```

Table1. Expected ML estimates of utilization factor for different sample sizes n=10, 100, 1000, 10000 along with rmse when the electric circuit supplying current to k=10, 100, 1000, 10000 customers.

k=10	ρ=0.2	rmse	ρ=0.5	rmse	ρ=0.8	rmse
n=10	0.1627907	0.043051	0.3888889	0.079422	0.5384615	0.18516
n=100	0.1750881	0.011402	0.4749263	0.029968	0.754386	0.049416
n=1000	0.1933174	0.003703	0.4819206	0.008936	0.7793594	0.010747
n=10000	0.1987677	0.001251	0.4935404	0.004075	0.793336	0.004479
k=100	ρ=0.2	rmse	ρ=0.5	rmse	ρ=0.8	rmse
n=10	0.1723329	0.013035	0.4662757	0.018826	0.7241379	0.050691
n=100	0.193175	0.006121	0.4797277	0.009986	0.7736786	0.01573
n=1000	0.1972607	0.001619	0.4951035	0.003683	0.7923395	0.002989
n=10000	0.1990321	0.000436	0.4984641	0.001539	0.7961802	0.00196
k=1000	ρ=0.2	rmse	ρ=0.5	rmse	ρ=0.8	rmse

n=10	0.1936023	0.006558	0.484781	0.008539	0.7927573	0.009892
n=100	0.197404	0.002012	0.4911574	0.003936	0.7898693	0.00554
n=1000	0.1992378	0.000518	0.4987223	0.000595	0.7980146	0.001053
n=10000	0.1995976	0.000253	0.4998085	0.000296	0.7994832	0.00039
k=10000	ρ=0.2	rmse	ρ=0.5	rmse	ρ=0.8	rmse
n=10	0.1979491	0.001334032	0.4950812	0.005086	0.7950099	0.003882
n=100	0.1994636	0.000420275	0.4982979	0.001461	0.7952902	0.001991
n=1000	0.1998641	0.0000784619	0.4997205	0.000191	0.7996063	0.000418
n=10000	0.1999361	0.0000519571	0.499808	0.000129	0.7999107	0.000157

It is seen that the expected ML estimate of ρ approaches the true value of ρ as n (number of time points at which the queue has been observed) increases and also as the supplying of electricity to the number of customer increases. We also found that as sample size increases then the root mean square error decreases. Further, we observed that as the electric circuit supplying power to customer increases then the expected value of ρ closer to the true value of ρ as sample size increases.

Table 2. Expected ML estimates of utilization factor for ρ= 0.2 for different sample sizes n=10, 100, 1000, 10000 along with 95% confidence interval when the electric circuit supplying current to k=10, 100, 1000, 10000 customers.

ρ=0.2		k=10			k=100	
n=	Lower confidence limit	Expected ML estimate of ρ	Upper confidence limit	Lower confidence limit	Expected ML estimate of ρ	Upper confidence limit
10	0.096612	0.1627907	0.4140257	0.1581677	0.1723329	0.2528961
100	0.1581677	0.1750881	0.2528961	0.1856676	0.193175	0.2154388
1000	0.1856676	0.1933174	0.2154388	0.195351	0.1972607	0.2047596
10000	0.195351	0.1987677	0.2047596	0.198518	0.1990321	0.2014931
ρ=0.2		k=1000			k=10000	
n=	Lower confidence limit	Expected ML estimate of ρ	Upper confidence limit	Lower confidence limit	Expected ML estimate of ρ	Upper confidence limit
10	0.1856676	0.1936023	0.2154388	0.195351	0.1979491	0.2047596
100	0.195351	0.197404	0.2047596	0.198518	0.1994636	0.2014931
1000	0.198518	0.1992378	0.2014931	0.1995302	0.1998641	0.200471
10000	0.1995302	0.1995976	0.200471	0.1998513	0.1999361	0.2001488

From the above table, it is seen that as sample size increases and also the supplying of the electricity to the number of customer increases then the difference between the expected ML estimate of ρ and the lower confidence interval and the difference between expected ML estimate of ρ and the upper confidence interval is decreases. That is, size of the confidence interval decreases as the electric circuit supplying power to customer increases along with the increase of the sample size.

Table 3.Show the values of Bayes estimate of ρ under squared error loss function for different choices of hyper-parameters a and b for given different values of k, n and ρ=0.2

k=10	b a	1	4	7	10
n=10	1	0.1744186	0.1685393	0.1630435	0.1578947
	4	0.2093023	0.2022472	0.1956522	0.1894737
	7	0.244186	0.2359551	0.2282609	0.2210526
	10	0.2790698	0.2696629	0.2608696	0.2526316
k=10	b a	1	4	7	10
n=100	1	0.1762632	0.175644	0.1750292	0.1744186
	4	0.1797885	0.1791569	0.1785298	0.177907
	7	0.1833137	0.1826698	0.1820303	0.1813953
	10	0.186839	0.1861827	0.1855309	0.1848837
k=10	b a	1	4	7	10

n=1000	1	0.1934368	0.1933675	0.1932984	0.1932292
	4	0.1937947	0.1937254	0.1936561	0.1935868
	7	0.1941527	0.1940833	0.1940138	0.1939445
	10	0.1945107	0.1944411	0.1943716	0.1943021
k=10	b a	1	4	7	10
n=10000	1	0.1987797	0.1987725	0.1987654	0.1987582
	4	0.1988156	0.1988085	0.1988013	0.1987942
	7	0.1988516	0.1988444	0.1988373	0.1988301
	10	0.1988875	0.1988804	0.1988732	0.1988661
k=100	b a	1	4	7	10
n=10	1	0.1735053	0.1728972	0.1722934	0.1716937
	4	0.1770223	0.1764019	0.1757858	0.175174
	7	0.1805393	0.1799065	0.1792782	0.1786543
	10	0.1840563	0.1834112	0.1827707	0.1821346
k=100	b a	1	4	7	10
n=100	1	0.1932944	0.1932252	0.1931561	0.193087
	4	0.1936523	0.193583	0.1935138	0.1934446
	7	0.1940103	0.1939408	0.1938715	0.1938021
	10	0.1943682	0.1942987	0.1942292	0.1941597
k=100	b a	1	4	7	10
n=1000	1	0.1972726	0.1972656	0.1972585	0.1972514
	4	0.1973086	0.1973015	0.1972944	0.1972873
	7	0.1973445	0.1973374	0.1973303	0.1973232
	10	0.1973804	0.1973733	0.1973662	0.1973591
k=100	b a	1	4	7	10
n=10000	1	0.1990333	0.1990326	0.1990319	0.1990312
	4	0.1990369	0.1990362	0.1990355	0.1990348
	7	0.1990405	0.1990398	0.1990391	0.1990384
	10	0.1990441	0.1990434	0.1990427	0.199042
k=1000	b a	1	4	7	10
n=10	1	0.1937217	0.1936523	0.193583	0.1935138
	4	0.1940797	0.1940103	0.1939408	0.1938715
	7	0.1944378	0.1943682	0.1942987	0.1942292
	10	0.1947959	0.1947262	0.1946565	0.1945869
k=1000	b a	1	4	7	10
n=100	1	0.197416	0.1974089	0.1974018	0.1973947
	4	0.1974519	0.1974448	0.1974377	0.1974306
	7	0.1974878	0.1974808	0.1974737	0.1974666
	10	0.1975238	0.1975167	0.1975096	0.1975025
k=1000	b a	1	4	7	10
n=1000	1	0.199239	0.1992382	0.1992375	0.1992368
	4	0.1992426	0.1992418	0.1992411	0.1992404
	7	0.1992462	0.1992454	0.1992447	0.199244
	10	0.1992498	0.199249	0.1992483	0.1992476
k=1000	b a	1	4	7	10
n=10000	1	0.1995977	0.1995976	0.1995975	0.1995975
	4	0.199598	0.199598	0.1995979	0.1995978
	7	0.1995984	0.1995983	0.1995983	0.1995982
	10	0.1995988	0.1995987	0.1995986	0.1995985
k=10000	b	1	4	7	10

	a				
n=10	1	0.1979611	0.197954	0.1979469	0.1979397
	4	0.197997	0.1979899	0.1979828	0.1979757
	7	0.198033	0.1980259	0.1980187	0.1980116
	10	0.1980689	0.1980618	0.1980547	0.1980476
k=10000	b	1	4	7	10
	a				
n=100	1	0.1994648	0.1994641	0.1994634	0.1994626
	4	0.1994684	0.1994677	0.199467	0.1994662
	7	0.199472	0.1994713	0.1994706	0.1994698
	10	0.1994756	0.1994749	0.1994742	0.1994734
k=10000	b	1	4	7	10
	a				
n=1000	1	0.1998642	0.1998642	0.1998641	0.199864
	4	0.1998646	0.1998645	0.1998645	0.1998644
	7	0.199865	0.1998649	0.1998648	0.1998648
	10	0.1998653	0.1998653	0.1998652	0.1998651
k=10000	b	1	4	7	10
	a				
n=10000	1	0.1999361	0.1999361	0.1999361	0.1999361
	4	0.1999361	0.1999361	0.1999361	0.1999361
	7	0.1999362	0.1999362	0.1999362	0.1999361
	10	0.1999362	0.1999362	0.1999362	0.1999362

It is clear from above table that the values of Bayes estimate of ρ under squared error loss function almost equal for different choices of hyper-parameters a and b if sample size increases and the electric circuit supplying power to the customer also increases.

Table 4. Show the values of minimum posterior risk in association with Bayes estimator for different choices of hyper-parameters a& b for given different values of k, n and ρ=0.2.

k=10	b	1	4	7	10
	a				
n=10	1	0.002409888	0.002238009	0.002083809	0.001944952
	4	0.002977762	0.002763081	0.002570681	0.002397595
	7	0.003574269	0.003313976	0.003080922	0.002871456
	10	0.004199408	0.003890693	0.003614533	0.003366535
k=10	b	1	4	7	10
	a				
n=100	1	0.00024392	0.000242081	0.000240262	0.000238464
	4	0.000249544	0.00024766	0.000245798	0.000243956
	7	0.000255197	0.000253269	0.000251361	0.000249476
	10	0.00026088	0.000258906	0.000256954	0.000255024
k=10	b	1	4	7	10
	a				
n=1000	1	0.00002755156	0.00002753025	0.00002750896	0.00002748769
	4	0.00002761083	0.00002758947	0.00002756813	0.00002754682
	7	0.00002767013	0.00002764872	0.00002762733	0.00002760597
	10	0.00002772946	0.00002770800	0.00002768657	0.00002766516
k=10	b	1	4	7	10
	a				
n=10000	1	0.00000285661	0.00000285639	0.00000285617	0.00000285595
	4	0.00000285722	0.00000285699	0.00000285677	0.00000285655
	7	0.00000285782	0.00000285760	0.00000285737	0.00000285715
	10	0.00000285842	0.00000285820	0.00000285798	0.00000285775
k=100	b	1	4	7	10
	a				
n=10	1	0.000238978	0.000237182	0.000235406	0.00023365
	4	0.000244553	0.000242713	0.000240893	0.000239094
	7	0.000250157	0.000248272	0.000246409	0.000244566

	10	0.00025579	0.000253861	0.000251953	0.000250067
k=100	b a	1	4	7	10
n=100	1	0.00002752471	0.00002750342	0.00002748215	0.00002746091
	4	0.00002758395	0.00002756261	0.00002754130	0.00002752001
	7	0.00002764323	0.00002762184	0.00002760048	0.00002757914
	10	0.00002770253	0.00002768110	0.00002765969	0.00002763830
k=100	b a	1	4	7	10
n=1000	1	0.00000282783	0.00000282761	0.00000282739	0.00000282717
	4	0.00000282843	0.00000282821	0.00000282799	0.00000282777
	7	0.00000282903	0.00000282881	0.00000282859	0.00000282837
	10	0.00000282963	0.00000282941	0.00000282919	0.00000282897
k=100	b a	1	4	7	10
n=10000	1	0.00000028615	0.00000028614	0.00000028614	0.00000028614
	4	0.00000028615	0.00000028615	0.00000028615	0.00000028615
	7	0.00000028616	0.00000028616	0.00000028615	0.00000028615
	10	0.00000028616	0.00000028616	0.00000028616	0.00000028616
k=1000	b a	1	4	7	10
n=10	1	0.00002760532	0.00002758395	0.00002756261	0.00002754130
	4	0.00002766464	0.00002764323	0.00002762184	0.00002760048
	7	0.00002772399	0.00002770253	0.00002768110	0.00002765969
	10	0.00002778338	0.00002776187	0.00002774038	0.00002771893
k=1000	b a	1	4	7	10
n=100	1	0.00000283057	0.00000283035	0.00000283013	0.00000282991
	4	0.00000283117	0.00000283095	0.00000283073	0.00000283051
	7	0.00000283177	0.00000283155	0.00000283133	0.00000283111
	10	0.00000283237	0.00000283215	0.00000283193	0.00000283171
k=1000	b a	1	4	7	10
n=1000	1	0.00000028654	0.00000028654	0.00000028654	0.00000028653
	4	0.00000028655	0.00000028654	0.00000028654	0.00000028654
	7	0.00000028655	0.00000028655	0.00000028655	0.00000028655
	10	0.00000028656	0.00000028656	0.00000028655	0.00000028655
k=1000	b a	1	4	7	10
n=10000	1	0.00000002872	0.00000002872	0.00000002872	0.00000002872
	4	0.00000002872	0.00000002872	0.00000002872	0.00000002872
	7	0.00000002872	0.00000002872	0.00000002872	0.00000002872
	10	0.00000002872	0.00000002872	0.00000002872	0.00000002872
k=10000	b a	1	4	7	10
n=10	1	0.00000284097	0.00000284075	0.00000284052	0.00000284030
	4	0.00000284157	0.00000284135	0.00000284113	0.00000284090
	7	0.00000284217	0.00000284195	0.00000284173	0.00000284151
	10	0.00000284277	0.00000284255	0.00000284233	0.00000284211
k=10000	b a	1	4	7	10
n=100	1	0.00000028697	0.00000028697	0.00000028697	0.00000028697
	4	0.00000028698	0.00000028698	0.00000028697	0.00000028697
	7	0.00000028699	0.00000028698	0.00000028698	0.00000028698
	10	0.00000028699	0.00000028699	0.00000028699	0.00000028698
k=10000	b a	1	4	7	10
n=1000	1	0.00000002877	0.00000002877	0.00000002877	0.00000002877
	4	0.00000002877	0.00000002877	0.00000002877	0.00000002877

	7	0.00000002877	0.00000002877	0.00000002877	0.00000002877
	10	0.00000002877	0.00000002877	0.00000002877	0.00000002877
k=10000	b a	1	4	7	10
n=10000	1	0.00000000288	0.00000000288	0.00000000288	0.00000000288
	4	0.00000000288	0.00000000288	0.00000000288	0.00000000288
	7	0.00000000288	0.00000000288	0.00000000288	0.00000000288
	10	0.00000000288	0.00000000288	0.00000000288	0.00000000288

It is to be noted that as electric circuit supplying power to the customer increases and the sample size increases then the values of minimum posterior risk associated with Bayes estimator for different choices of hyper-parameters a and b decreases. It is found that for large sample hyper parameter does not have any impact on the values of minimum posterior risk associated with Bayes estimator.

8. Conclusions

In this paper we have presented an estimator of utilization factor of power supply queuing model and find that the maximum likelihood estimator of the utilization factoris consistent. We also developed a method to find the confidence interval for the utilization factor. From numerical analysis it is found that estimated utilization factor lies between the confidence interval. We also observed that if electric circuit supplying power to the customer increases and the demand of the customer in any time point increases then estimated value of utilization factor approaches the true value of the utilization factor.It is also clear thatcircuit supplying power to the customer increases and the demand of the customer in any time point also increases then the values of Bayes estimate of utilization factor under squared error loss function almost equal for different choices of hyper-parameters a and b and the values of minimum posterior risk in association with Bayes estimator decreases.

Conflict of Interest

The authors declare that there is no conflict of interest.

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