Quasi-continuity on Product Spaces

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Abstract: In this present paper, the notion of Quasi-continuity on a product space is introduced. The set of all such bounded Quasi-continuous functions defined on a closed and bounded interval is established to be a commutative Banach algebra under supremum norm.

Keywords: Quasi-continuity, Right and Left Limits, Commutative Banach Algebra, Product space.

1. Introduction

Quasi-continuity is a weaker form of continuity. In 1932, *Kempisty* introduced the notion of quasi-continuous mappings for real functions of several real variables in his classical research article [1]. There are various reasons for the interest in the study of quasi-continuous functions. There are mainly two reasons. The first one is relatively good connection between the continuity and quasi-continuity inspite of the generality of the latter. The second one is a deep connection of quasi-continuity with mathematical analysis and topology.

In this present work, we define quasi-continuity on a product space in a different approach and establish that the set of all such bounded Quasi-continuous functions on a closed and bounded interval forms a commutative Banach algebra under the supremum norm.

In what follows I and X stand for the closed unit interval $[0,1]$ and a commutative Banach algebra with identity over the field $\mathbb{R}$ of real numbers respectively.

2. Preliminaries

In this section we present a few basic definitions that are needed further study of this paper.

Definition – 1.1: Let $f : I \to X$, $q \in X$ and $x_0 \in [0,1)$. We say that $f(x_0+) = q$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|f(t) - q\| < \varepsilon$ for all $t \in (x_0, x_0 + \delta) \subset [0,1]$.

Definition – 1.2: Let $f : I \to X$, $p \in X$ and $x_0 \in (0,1]$. We say that $f(x_0-) = p$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|f(t) - p\| < \varepsilon$ for all $t \in (x_0 - \delta, x_0) \subset [0,1]$.

Definition – 1.3: A function $f : I \to X$ is said to be continuous at $x_0 \in I$ if $f(x_0+) = f(x_0-) = f(x_0)$. We say that f is continuous on I if f is continuous at every point of I .

Definition – 1.4: Let S be any non-empty set. If $c \in \mathbb{R}$, $f : S \to X$ and $g : S \to X$ then we define

a. $(f+g)(x) = f(x) + g(x)$ for all $x \in S$

b. $(cf)(x) = cf(x)$ for all $x \in S$

c. $(fg)(x) = f(x)g(x)$ for all $x \in S$

Definition – 1.5: A function $f : I \to X$ is said to be *bounded* on I if there exists a positive real number M such that $\|f(t)\| \le M$ for all $t \in I$.

Definition – 1.6: Let N and N' be normed linear spaces. An *isometric isomorphism* of N into N' is a one-to-one linear transformation T of N into N' such that $\|T(x)\| = \|x\|$ for every x in N and N is said to be *isometrically isomorphic* to if N' there exists an isometric isomorphism of N into N'.

1. Quasi-Continuity on I^2

In this section, we introduce the notion of quasi-continuity on I^2 and present a few results in this context.

Definition – 2.1: A function $f : I \to X$ is said to be *quasi-continuous* on I if

a. $f(0+)$ *and* $f(1-)$ exist.

b. $f(p+)$ *and* $f(p-)$ exist for every $p \in (0,1)$.

Definition – 2.2: Let $f : I^2 \to X$ and $(x, y) \in I^2$. We define $f_x : I \to X$ and $f_y : I \to X$ by $f_x(t) = f(x,t)$ and $f_y(t) = f(t, y)$ for all $t \in I$.

Definition – 2.3: Let $(x, y) \in I^2$. A function $f : I^2 \to X$ is said to be quasi-continuous at (x, y) if the functions $f_x : I \to X$ and $f_y : I \to X$ are quasi-continuous on I. We say that $f : I^2 \to X$ is quasi-continuous on I^2, if f is quasi-continuous at every point of I^2.

Definition – 2.4: Let $I^2 = I \times I$. A function $f : I^2 \to X$ is said to be *bounded* on I^2 if there exists a positive real number M such that $\|f(s,t)\| \le M$ for all $(s,t) \in I^2$.

Notation – 2.5:

1. The set of all quasi-continuous bounded functions from I into X is denoted by $\mathcal{Q}(I, X)$.
2. We denote the set of all quasi-continuous bounded functions from I^2 into X by the symbol $\mathcal{Q}(I^2, X)$.

Proposition – 2.6: Let $c \in \mathbb{R}$. If $f : I \to X$ and $g : I \to X$ are quasi-continuous on I then so are $f + g,\ fg$ and cf.

Proposition – 2.7: If $f_n \in \mathcal{Q}(I, X)$ for $n = 1, 2, 3, ...$ and if $f_n \to f$ uniformly on I then $f \in \mathcal{Q}(I, X)$.

Proposition – 2.8: $\mathcal{Q}(I, X)$ is a commutative Banach algebra with identity over the field $\mathbb{R}$ of real numbers under the supremum norm $\|f\| = \sup\{\|f(x)\| : x \in I\}$.

Remark – 2.9: It is easy to observe the following.

a. $(f + g)_x = f_x + g_x$

b. $(f + g)_y = f_y + g_y$

c. $(fg)_x = f_x g_x$

d. $(fg)_y = f_y g_y$

e. $(cf)_x = cf_x$

f. $(cf)_y = cf_y$

Proposition – 2.10: If $f \in \mathcal{Q}(I^2, X)$ and $g \in \mathcal{Q}(I^2, X)$ then

a. $f + g \in \mathcal{Q}(I^2, X)$

b. $fg \in \mathcal{Q}\left(I^2, X\right)$

c. $cf \in \mathcal{Q}\left(I^2, X\right)$ where $c \in \square$.

Proposition – 2.11: The space $\mathcal{Q}\left(I^2, X\right)$ forms a normed linear space under the supremum norm $\|f\| = \sup\left\{\|f(s,t)\| : (s,t) \in I^2\right\}$.

3. Isometric Isomorphism between $\mathcal{Q}\left(I^2, X\right)$ and $\mathcal{Q}^2$

Throughout this section we take $\mathcal{Q}^2$ to be the product space $\mathcal{Q}\left(I, X\right)\times\mathcal{Q}\left(I, X\right)$ and establish an isometric isomorphism between $\mathcal{Q}\left(I^2, X\right)$ and $\mathcal{Q}^2$.

Proposition – 3.1: There exists an isometric isomorphism between $\mathcal{Q}\left(I^2, X\right)$ and $\mathcal{Q}^2$.

Proof: Fix $\left(x, y\right) \in I^2$. Define $\varphi : \mathcal{Q}\left(I^2, X\right) \to \mathcal{Q}^2$ by $\varphi\left(f\right) = \left(f_x, f_y\right)$.

First we prove that φ is 1-1.

Suppose that $\varphi\left(f\right) = \varphi\left(g\right)$ for $f, g \in \mathcal{Q}\left(I^2, X\right)$

$\Rightarrow \left(f_x, f_y\right) = \left(g_x, g_y\right)$

$\Rightarrow f_x = g_x$ and $f_y = g_y$

$\Rightarrow f_x(t) = g_x(t)$ and $f_y(s) = g_y(s)$ for all $s, t \in I$

$\Rightarrow f(x,t) = g(x,t)$ and $f(s, y) = g(s, y)$ for all $s, t \in I$

$\Rightarrow f(x, y) = g(x, y)$

Since $\left(x, y\right)$ is an arbitrary point in I^2, we have $f(x, y) = g(x, y)$ for every $\left(x, y\right)$ in I^2 and hence $f = g$.

Thus the mapping $\varphi : \mathcal{Q}\left(I^2, X\right) \to \mathcal{Q}^2$ is 1-1.

The norm of $\left(f, g\right) \in \mathcal{Q}^2$ is defined by $\left\|\left(f, g\right)\right\| = \max\left\{\|f\|, \|g\|\right\}$.

Now we prove that the mapping $\varphi : \mathcal{Q}\left(I^2, X\right) \to \mathcal{Q}^2$ preserves norm. we have

$$\|\varphi(f)\| = \left\|\left(f_x, f_y\right)\right\|$$

$$= \max\left\{\|f_x\|, \|f_y\|\right\}$$

$$\geq \|f_x\|$$

$$= \sup\left\{\|f_x(t)\| : t \in I\right\}$$

$$\geq \|f_x(t)\| \quad \forall \; t \in I$$

$$= \|f(x,t)\| \quad \forall \; t \in I$$

In particular, $\|\varphi(f)\| \geq \|f(x, y)\|$

Since $\left(x, y\right)$ is an arbitrary point in I^2, we have $\|\varphi(f)\| \geq \|f(x, y)\|$ for all $\left(x, y\right) \in I^2$.

$\Rightarrow \left\|\varphi\left(f\right)\right\| \geq \|f\| \quad \to \quad (1)$

Since $\|f\| = \sup\left\{\|f(s,t)\| : (s,t) \in I^2\right\}$, it follows that $\|f\| \geq \left\|f\left(s,t\right)\right\| \quad \forall \; \left(s,t\right) \in I^2$

$\Rightarrow \quad \|f\| \geq \|f(x,t)\|$ and $\|f\| \geq \|f(s,y)\|$ for every s and t in I

$\Rightarrow \quad \|f\| \geq \|f_x(t)\|$ and $\|f\| \geq \|f_y(s)\|$ f or every s and t in I

$\Rightarrow \quad \|f\| \geq \|f_x\|$ and $\|f\| \geq \|f_y\|$

$\Rightarrow \quad \|f\| \geq \max\{\|f_x\|, \|f_y\|\}$

$\Rightarrow \quad \|f\| \geq \|\varphi(f)\| \qquad \rightarrow \qquad (2)$

From (1) and (2) $\|\varphi(f)\| = \|f\|$.

Hence φ preserves norm.

Now it remains to show that φ is linear. If $f, g \in \mathcal{Q}(I^2, X)$ and $c \in \square$, then

$$\varphi(f+g) = ((f+g)_x, (f+g)_y)$$
$$= (f_x + g_x, f_y + g_y)$$
$$= (f_x, f_y) + (g_x, g_y)$$
$$= \varphi(f) + \varphi(g)$$

Similarly, $\varphi(cf) = ((cf)_x, (cf)_y)$

$$= (cf_x, cf_y)$$
$$= c(f_x, f_y)$$
$$= c\varphi(f)$$

Hence φ is an isometric isomorphism of $\mathcal{Q}(I^2, X)$ into $\mathcal{Q}^2$.

Proposition – 3.2: If $f_n \in \mathcal{Q}(I^2, X)$ for $n = 1,2,3...$ and $f_n \rightarrow f$ uniformly on I^2 then $f \in \mathcal{Q}(I^2, X)$.

Proof: Let $\varepsilon > 0$ be given and take $\delta = \varepsilon$.

For $f, g \in \mathcal{Q}(I^2, X)$, suppose that $\|f - g\| < \delta$.

Since $\varphi : \mathcal{Q}(I^2, X) \rightarrow \mathcal{Q}^2$ is an isometric isomorphism, we have $\|\varphi(f-g)\| < \delta$

$\Rightarrow \quad \|\varphi(f) - \varphi(g)\| < \varepsilon$

Hence $\varphi : \mathcal{Q}(I^2, X) \rightarrow \mathcal{Q}^2$ is uniformly continuous on $\mathcal{Q}(I^2, X)$.

$\Rightarrow \quad \varphi : \mathcal{Q}(I^2, X) \rightarrow \mathcal{Q}^2$ is continuous on $\mathcal{Q}(I^2, X)$.

Let $(x, y) \in I^2$. Suppose that $f_n \in \mathcal{Q}(I^2, X)$ for $n = 1,2,3...$ and $f_n \rightarrow f$ uniformly on I^2.

$\Rightarrow \quad \varphi(f_n) \rightarrow \varphi(f)$ in $\mathcal{Q}^2$

$\Rightarrow \quad ((f_n)_x, (f_n)_y) \rightarrow (f_x, f_y)$ in $\mathcal{Q}^2$

$\Rightarrow \quad (f_n)_x \rightarrow f_x$ and $(f_n)_y \rightarrow f_y$ uniformly on I in $\mathcal{Q}(I, X)$

$\Rightarrow \quad f_x \in \mathcal{Q}(I, X)$ and $f_y \in \mathcal{Q}(I, X)$.

$\Rightarrow \quad f$ is quasi-continuous at $(x, y) \in I^2$.

Since $(x, y) \in I^2$ is arbitrary, $f \in \mathcal{Q}(I^2, X)$.

Proposition – 3.3: $\mathcal{Q}\left(I^{2}, X\right)$ is a commutative Banach algebra with identity over the field $\mathbb{R}$ of real numbers under the supremum norm.

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