Research Article

Series solution with Frobenius method

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Abstract: T In this article we explained the structure of Frobenius method to solve a homogeneous linear differential equations of order two . In any homogeneous linear differential equations of order two we have three cases of two roots (as a ,b) of the indicial equation :
Case1 : $a - b = c/d$ such that $c, d \in Z$ where Z is integer number and $d \neq 0, d \neq 1$. Case 2 : $a - b = 0$. Case 3 : $a - b = c$, such that $c \in Z$. And we explained how to find the general solution of each case with many examples .

Keywords: Frobenius method, homogeneous linear differential equations

1. Introduction

A linear , second order and homogenous (for short homo) ODE can have two independent solutions . Let us consider a method of obtaining one of the solutions . The method which is a series expansion will always work , provided the point of expansion $x = x_0$ is no worse than a regular singular point . Fortunately in the problems in physics this condition is almost always satisfied .

We write the linear , second order and homo ODE in the form :

$$y'' + p(x)y' + q(x) = 0 . \quad (1)$$

This equation is homo since each term contain $y(x)$ or a derivative . It is linear because each y , y' and y'' appears as the first power , and has no products .

Equation 1 (for short Eq. 1) can have two linearly independent solutions . Let us find (at least) one solution of Eq. 1 using a generalized power series . By using the first solution we can develop the second independent solution . We will also later prove that a third independent solution does not exist .

Let us write the most general solution of Eq. 1 as :

$$y(x) = c_1 y_1(x) + c_2 y_2(x) . \quad (2)$$

Where c_1 and c_2 any arbitrary constant .

In some cases we can have a source term as will in the ODE , leading to non-homo , linear , second order ODE ,

$$y'' + p(x)y' + q(x)y = r(x). \quad (3)$$

The function $r(x)$ represents a source or driving force .

Calling this solution y_p , we may add to it any solution of the corresponding homo Eq. 1 . Therefore the most general solution of Eq. 3 is :

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x) . \quad (4)$$

We have to fix the two arbitrary constant c_1 and c_2 and that will be done by applying boundary conditions .

2. The structure of Frobenius series in homogenous linear equation of order two :

At the moment let us $r(x) = 0$ and the our DF is homo . We will try to develop the solution of our linear , second order and homo DF , Eq. 1 , by substituting in a power series with undetermined coefficients .

This generalized power series has a parameter , which is the power of the lowest non vanishing term of series .

As a test bed let us apply this method to an important DF , the linear oscillator Eq.

$y'' + w^2 y = 0$. (5)

Its two independent solution are known

$$y = c_1 y_1(x) + c_2 y_2(x)$$

$= c_1 \sin \sin wx + c_2 \cos \cos wx$. (6)

Let us try the following power series solution

$$y(x) = x^k (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)$$

$= \sum_{l=0}^{\infty} a_l x^{k+l}$, $a_0 \neq 0$. (7)

With the exponent k and all the coefficient a_l still undetermined .

Note that k could be either positive or negative and it may be a fraction (it may even be complex , but we shall not consider this case) . a_0 is not zero since $a_0 x^k$ is to be the first term of the series .

The series Eq. 7 is called a generalized power series or Frobenius series .

By differentiating with respected to x we get :

$$y' = \sum_{l=0}^{\infty} a_l (k+l) x^{k+l-1}$$

$$y'' = \sum_{l=0}^{\infty} a_l (k+l)(k+l-1) x^{k+l-2}$$

Let us substitute the series form of $y(x)$ and $y''(x)$ into Eq. 5 . We get :

$\sum_{l=0}^{\infty} a_l (k+l)(k+l-1) x^{k+l-2} + w^2 \sum_{l=0}^{\infty} a_l x^{k+l} = 0$. (8)

The uniqueness of power series tells us that , the coefficient of each power of x on the L. H. S. of Eq. 8 must vanish individually . We have

$(a_0 k(k-1) x^{k-2} + a_1 k(k+1) x^{k-1} + a_2 (k+1)(k+2) x^k + a_3 (k+2)(k+3) x^{k+1} + \cdots + a_l (k+l)(k+l-1) x^{k+l-2} + \cdots) + (w^2 a_0 x^k + w^2 a_1 x^{k+1} + w^2 a_3 x^{k+3} + \cdots + w^2 a_l x^{k+l} + \cdots) = 0$. (9)

Combining the coefficients of x , the series is expressed as :

$$a_0 k(k-1) x^{k-2} + a_1 k(k+1) x^{k-1} + [a_2 (k+1)(k+2) + w^2 a_0] x^k$$

$+[a_3 (k+2)(k+3) + w^2 a_1] x^{k+1} + \cdots + [a_l (k+l)(k+l-1) + w^2 a_{l-2}] x^{k+l-2} + \cdots = 0$ (10)

The lowest power of x appearing in Eq. 10 is x^{k-2} for $l = 0$. The requirement that the coefficient vanish yields .

$a_0 k(k-1) = 0$. (11)

We had chosen a_0 as the coefficient of the lowest non-vanishing term of series , Eq. 7 hence by definition , $a_0 \neq 0$. Therefore we have the constraint

$k(k-1) = 0$. (12)

This equation , coming from the coefficient of the lower power of x , is called the indicial equations .

The indicial equation and its roots (or indices of the regular singular point of ODE) play a crucial role in our attempt to find the solutions .

We have two choices for k , $k = 0$ or $k = 1$. We see that a_1 is arbitrary if $k = 0$ and necessarily zero if $k = 1$. Thus we will set a_1 equal to zero .

Case $k = 0$:

We have the general term in the equation

$a_l l(l-1) + w^2 a_{l-2} = 0$. (13)

Since $a_0 \neq 0$ we have

$$a_2 \,.\, 2 \,.\, 1 + w^2 a_0 = 0$$

$$a_3 \,.\, 3 \,.\, 2 + w^2 a_1 = 0$$

$$a_4 \,.\, 4 \,.\, 3 + w^2 a_2 = 0$$

$$a_5 \,.\, 5 \,.\, 4 + w^2 a_3 = 0$$

Until

$$a_{j+2}\,(j+2)\,(j+1) + w^2 a_j = 0$$

Since $a_1 = 0$, then the above set of equations reduced as :

$$a_2 \,.\, 2 \,.\, 1 + w^2 a_0 = 0$$

$$a_4 \,.\, 4 \,.\, 3 + w^2 a_2 = 0$$

Until

$$a_{j+2}\,(j+2)\,(j+1) + w^2 a_j = 0$$

This gives a two-term recurrence relation for $k = 0$ case :

$a_{j+2} = -\frac{w^2}{(j+1)(j+2)} a_j$. (14)

Case $k = 1$:

We have the general term in the equation

$a_l l(l+1) + w^2 a_{l-2} = 0$. (15)

Since $a_0 \neq 0$ we have

$$a_2 \,.\, 2 \,.\, 3 + w^2 a_0 = 0$$

$$a_3 \,.\, 3 \,.\, 4 + w^2 a_1 = 0$$

$$a_4 \,.\, 4 \,.\, 5 + w^2 a_2 = 0$$

$$a_5 \,.\, 5 \,.\, 6 + w^2 a_3 = 0$$

Until

$$a_{j+2}\,(j+2)\,(j+3) + w^2 a_j = 0$$

Again since $a_1 = 0$, then the above set of equations reduced as :

$$a_2 . 2 . 3 + w^2 a_0 = 0$$

$$a_4 . 4 . 5 + w^2 a_2 = 0$$

Until

$$a_{j+2} \, (j+2) \, (j+3) + w^2 a_j = 0$$

This gives a two-term recurrence relation for $k = 1$ case :

$a_{j+2} = -\frac{w^2}{(j+1)(j+3)} a_j$. (16)

For $k = 0$ we have

$a_{2l} = \frac{(-1)^l w^{2l}}{2l!} a_0$. (17)

And our solution is

$$y(x)|_{k=0} = a_0 \left[1 - \frac{(wx)^2}{2!} + \frac{(wx)^4}{4!} - \frac{(wx)^6}{6!} + \cdots \right]$$

$= a_0 \, cos \, cos \, wx$. (18)

For $k = 1$ we have

$a_{2l} = \frac{(-1)^l w^{2l}}{(2l+1)!} a_0$. (19)

And then we get

$$y(x)|_{k=1} = a_0 x \left[1 - \frac{(wx)^2}{3!} + \frac{(wx)^4}{5!} - \frac{(wx)^6}{7!} + \cdots \right]$$

$$= \frac{a_0}{w} \left[wx - \frac{(wx)^3}{3!} + \frac{(wx)^5}{5!} - \frac{(wx)^7}{7!} + \cdots \right]$$

$= \frac{a_0}{w} \, sin \, sin \, wx$. (20)

Thus we have arrived at two independent series solutions of the linear oscillator equations using the method of generalized series substitution (Frobenius method) .

If $x_0 \neq 0$ we get

$y(x) = \sum_{l=0}^{\infty} \quad a_l (x - x_0)^{k+l}$, $a_0 \neq 0$. (21)

3. Explained the Frobenius method to solve DF

We can explain this method by take some notes and examples :

From above equations we get , if $x = x_0$ is regular point then the solution can be expression as :

$$y(x) = \sum_{l=0}^{\infty} \quad a_l (x - x_0)^{k+l}$$

And if $x = x_0$ is regular singular point then the solution can be expression as :

$$y(x) = \sum_{l=0}^{\infty} a_l x^{k+l}$$

Example 1 : find a solution of below DF by Frobenius series :

$$2xy'' + (x+1)y' + 3y = 0$$

Solution :

Since $\frac{(x+1)x}{2x} = \frac{1}{2}, \frac{3x^2}{2x} = 0$

Then $x = 0$ is regular singular point and

$$y(x) = \sum_{l=0}^{\infty} a_l x^{k+l}$$

Implies that

$$y' = \sum_{l=0}^{\infty} a_l (k+l) x^{k+l-1}$$

and

$$y'' = \sum_{l=0}^{\infty} a_l (k+l)(k+l-1) x^{k+l-2}$$

By substitute in DF we get

$$2x \sum_{l=0}^{\infty} a_l (k+l)(k+l-1) x^{k+l-2} + (x+1) \sum_{l=0}^{\infty} a_l (k+l) x^{k+l-1} + 3 \sum_{l=0}^{\infty} a_l x^{k+l} = 0$$

And then we get

$2a_l(k+l)(k+l-1) + a_l(k+l) + a_{l-1}(l+k+2) = 0 \quad \ldots$ (a)

If we put $l = 0$, $a_{-1} = 0$ we get

$$a_0(k)(2k - 2 + 1) = 0$$

Then

$$a_0 k(2k - 1) = 0$$

Since $a_0 \neq 0$ then $k(2k-1) = 0$ this is indicial equation and either $k = 0$ or $l = \frac{1}{2}$. And by Eq. a , we get

$$a_l = \frac{-(k+l+2)}{(k+l)(2k+2l-1)} a_{l-1} \ , n \geq 1$$

If $k = 0$ we get

$$a_l = \frac{-(l+2)}{(l)(2l-1)} a_{l-1}$$

And the we get

$$a_1 = -3a_0$$

$$a_2 = \frac{-2}{3} a_1 = 2a_0$$

$$a_3 = \frac{-1}{3} a_2 = -\frac{2}{3} a_0 \quad , \ldots$$

And if $k = \frac{1}{2}$ we get

$$a_l = \frac{-\left(l + \frac{5}{2}\right)}{\left(\frac{1}{2} + l\right)(2l)} a_{l-1}$$

We get

$$a_1 = \frac{-\frac{7}{2}}{3} a_0 = \frac{-7}{6} a_0$$

$$a_2 = \frac{-9}{20} a_1 = \frac{21}{40} a_0$$

$$a_3 = \frac{-11}{42} a_2 = -\frac{11}{80} a_0 \quad , \ldots$$

Let that $a_0 = 1$ we get

$$y = A_1 \left[1 - 3x + 2x^2 - \frac{2}{3}x^3 + \cdots\right] + A_2[1 - \frac{7}{6}x + \frac{21}{40}x^2 - \frac{11}{80}x^3 + \cdots]$$

Note : In any DF we have three cases of two roots (as a , b) of the indicial equation :

Case1 : $a - b = c/d$ such that $c, d \in Z$ where Z is integer number and $d \neq 0, d \neq 1$.

Case 2 : $a - b = 0$

Case 3 : $a - b = c$, such that $c \in Z$.

In case 1 we solved above Example 1 , and now we will take another example

Example 2 : find a solution of below DF by Frobenius series :

$$2x(1-x)y'' + (1-x)y' + 3y = 0$$

Solution :

Since $\frac{(1-x)x}{2x(1-x)} = \frac{1}{2}, \frac{3x^2}{2x(1-x)} = 0$

Then $x = 0$ is regular singular point and

$$y(x) = \sum_{l=0}^{\infty} a_l x^{k+l}$$

Implies that

$$y' = \sum_{l=0}^{\infty} a_l (k+l) x^{k+l-1}$$

and

$$y'' = \sum_{l=0}^{\infty} a_l (k+l)(k+l-1)x^{k+l-2}$$

By substitute in DF we get

$$2x(1-x)\sum_{l=0}^{\infty} a_l(k+l)(k+l-1)x^{k+l-2} + (1-x)\sum_{l=0}^{\infty} a_l(k+l)x^{k+l-1} + 3\sum_{l=0}^{\infty} a_l x^{k+l} = 0$$

And then we get

$$2a_l(k+l)(k+l-1) + a_l(k+l) - 2a_{l-1}(k+l-1)(k+l-2) - a_{l-1}(k+l-1) + 3a_{l-1} = 0$$

Implies that

$a_l(k+l)(2k+2l-1) - a_{l-1}\big((k+l-1)(2k+2l-5)+3\big) = 0 \quad \ldots$ (b)

If we put $l = 0$, and $a_{-1} = 0$ we get

$$a_0(k)(2k-2+1) = 0$$

Then

$$a_0 k(2k-1) = 0$$

Since $a_0 \neq 0$ then $k(2k-1) = 0$ this is indicial equation and either $k = 0$ or $k = \frac{1}{2}$. And by Eq. b , we get

$$a_l = \frac{(k+l-1)(2k+2l-5)+3}{(k+l)(2k+2l-1)} a_{l-1} \;, n \geq 1$$

If $k = 0$ we get

$$a_l = \frac{(l-1)(2l-5)+3}{(l)(2l-1)} a_{l-1}$$

And the we get

$$a_1 = 3a_0$$

$$a_2 = \frac{1}{3}a_1 = a_0$$

$$a_3 = \frac{1}{3}a_2 = \frac{1}{3}a_0 \;, \ldots$$

And if $k = \frac{1}{2}$ we get

$$a_l = \frac{\left(l-\frac{1}{2}\right)(2l-4)+3}{\left(\frac{1}{2}+l\right)(2l)} a_{l-1}$$

We get

$$a_1 = \frac{2}{3}a_0$$

$$a_2 = \frac{3}{10}a_1 = \frac{1}{5}a_0$$

$$a_3 = \frac{8}{21}a_2 = \frac{8}{105}a_0 \;, \ldots$$

Let that $a_0 = 1$ we get

$$y = A_1\left[1 + 3x + x^2 + \frac{1}{3}x^3 + \cdots\right] + A_2[1 + \frac{2}{3}x + \frac{1}{5}x^2 + \frac{8}{105}x^3 + \cdots]$$

In case 2 , if the two roots are equal then the first solution is $y_1 = f(xk)$ and then $y_2 = \frac{\partial y_1(x,k)}{\partial k}$ at the regular singular point .

We take a below example on case 2 :

Example 3 : find a solution of below DF by Frobenius series :

$$x^2y'' + 3xy' + (1 - 2x)y = 0$$

Solution :

Since $\frac{(3x)x}{x^2} = 3$, $\frac{(1-2x)x^2}{x^2} = 1$

Then $x = 0$ is regular singular point and

$$y(x) = \sum_{l=0}^{\infty} a_l x^{k+l}$$

Implies that

$$y' = \sum_{l=0}^{\infty} a_l(k + l)x^{k+l-1}$$

and

$$y'' = \sum_{l=0}^{\infty} a_l(k + l)(k + l - 1)x^{k+l-2}$$

By substitute in DF we get

$$x^2\sum_{l=0}^{\infty} a_l(k + l)(k + l - 1)x^{k+l-2} + 3x\sum_{l=0}^{\infty} a_l(k + l)x^{k+l-1} + (1 - 2x)\sum_{l=0}^{\infty} a_l x^{k+l} = 0$$

And then we get

$a_l(k + l)(k + l - 1) + 3a_l(k + l) + a_l - 2a_{l-1} = 0$ … (c)

If we put $l = 0$, $a_{l-1} = 0$ we get

$$a_0\big((k)(k + 2) + 1\big) = 0$$

Then

$$a_0(k + 1)^2 = 0$$

Since $a_0 \neq 0$ then $(k + 1)^2 = 0$ this is indicial equation and either $k = -1$ or $k = -1$. And by Eq. c , we get

$$a_l = \frac{2}{(l + k + 1)^2}a_{l-1} \; , n \geq 1$$

We can find a_l by independent of k , as follows :

$$a_1 = \frac{2}{(k+2)^2} a_0$$

$$a_2 = \frac{2}{(k+3)^2} a_1 = \frac{2}{(k+3)^2} \frac{2}{(k+2)^2} a_0 = \frac{2^2}{\left((k+2)(k+3)\right)^2} a_0$$

$$a_3 = \frac{2}{(k+4)^2} a_2 = \frac{2^3}{\left((k+2)(k+3)(k+4)\right)^2} a_0 \quad , \ldots$$

And then we get

$$a_l = \frac{2^k}{\left((k+2)(k+3)(k+4)\ldots(k+l+1)\right)^2}$$

And then we have

$$y_1(x,k) = x^k \left[1 + \frac{2}{(k+2)^2} x + \frac{2^2}{\left((k+2)(k+3)\right)^2} x^2 + \cdots\right]$$

Implies that

$$y_1(x,-1) = x^{-1} \left[1 + 2x + x^2 + \frac{2}{9} x^3 + \cdots\right]$$

Now let that $y_2 = \frac{\partial y_1}{\partial k}$ at $k = -1$, we get :

$$y_2(x,k) = \frac{\partial y_1(x,k)}{\partial k}$$

$$= ln\ ln\ x\ x^k \left[1 + \frac{2}{(k+2)^2} x + \frac{2^2}{\left((k+2)(k+3)\right)^2} x^2 + \cdots\right] + x^k \left[-\frac{4}{(k+2)^3} x - 4\left(\frac{2^2}{(k+2)^2(k+3)} + \frac{2^2}{(k+2)^2(k+3)^3}\right) x^2 + \cdots\right]$$

$$y_2(x,-1) = x^{-1}\ ln\ ln\ x\ \left[1 + 2x + x^2 + \frac{2}{9} x^3 + \cdots\right] + x^{-1} \left[-4x - 4\left(\frac{2}{8} + \frac{4}{8}\right) x^2 + \cdots\right]$$

Then the general solution is

$$y = A_1 x^{-1} \left[1 + 2x + x^2 + \frac{2}{9} x^3 + \cdots\right] + A_2 \left(x^{-1}\ ln\ ln\ x\ \left[1 + 2x + x^2 + \frac{2}{9} x^3 + \cdots\right] + x^{-1}[-4x - 3x^2 + \cdots]\right)$$

In case 3 , assume that $y_1 = y(x,k,a_0)$, at k equal to minimum value of two roots . and $y_2 = \frac{\partial y(x,k,a_0)}{\partial k}$, at k equal to minimum value of two roots . and we suppose that $a_l = b_l - (minimum\ value\ of\ two\ roots\ )$.

Now we take a below example on case 3 :

Example 4 : find a solution of below DF by Frobenius series :

$$xy'' - 3y' + xy = 0$$

Solution :

Since $\frac{-3x}{x} = -3$, $\frac{x^3}{x} = 0$

Then $x = 0$ is regular singular point and

$$y(x) = \sum_{l=0}^{\infty} a_l x^{k+l}$$

Implies that

$$y' = \sum_{l=0}^{\infty} a_l (k+l) x^{k+l-1}$$

and

$$y'' = \sum_{l=0}^{\infty} a_l (k+l)(k+l-1) x^{k+l-2}$$

By substitute in DF we get

$$x\sum_{l=0}^{\infty} a_l (k+l)(k+l-1) x^{k+l-2} - 3\sum_{l=0}^{\infty} a_l (k+l) x^{k+l-1} + x\sum_{l=0}^{\infty} a_l x^{k+l} = 0$$

And then we get

$a_l(k+l)(k+l-1) - 3a_l(k+l) + a_{l-2} = 0$ … (d)

If we put $l = 0$, a_{l-2} we get

$$a_0(k)(k-4) = 0$$

Since $a_0 \neq 0$ then $(k)(k-4)$ this is indicial equation and either $k = 0$ or $k = 4$. And by Eq. d , we get

$$a_l = \frac{-1}{(k+l)(k+l-4)} a_{l-2} \ , n \geq 2$$

It is clear that $a_1 = 0$ and all $a_{2l+1} = 0$. Now we have

$$a_2 = \frac{-1}{(k+2)(k-2)} a_0$$

$$a_4 = \frac{-1}{(k+4)k} a_2 = \frac{(-1)^2}{(k-2)k(k+2)(k+4)} a_0$$

$$a_6 = \frac{-1}{(k+6)(k+2)} a_2 = \frac{(-1)^3}{(k-2)k(k+2)^2(k+4)(k+6)} a_0$$

Then

$$y(x,k,a_0) = a_0 x^k \left[1 - \frac{1}{(k+2)(k-2)} x^2 + \frac{1}{(k-2)k(k+2)(k+4)} x^4 - \frac{1}{(k-2)k(k+2)^2(k+4)(k+6)} x^6 + \cdots\right]$$

Since if $k = 0$ we cannot find a coefficient of above series the we consider that

$a_0 = b_0(k-0)$, we get

$$y(x,k,b_0) = b_0 x^k \left[k - \frac{k}{(k+2)(k-2)} x^2 + \frac{1}{(k-2)(k+2)(k+4)} x^4 - \frac{1}{(k-2)(k+2)^2(k+4)(k+6)} x^6 + \cdots\right]$$

Implies that

$$y_1 = y(x,0,b_0) = b_0\left[-\frac{1}{16}x^4 + \frac{1}{192}x^6 - \cdots\right]$$

And

$$\frac{\partial y(x,k,b_0)}{\partial k} = y(x,k,b_0)\ ln\ ln\ x$$

$$+b_0x^k\left[1-\left(\frac{1}{(k+2)(k-2)} - \frac{k}{(k+2)^2(k-2)}\frac{k}{(k+2)(k-2)^2}\right)x^2 - \left(\frac{1}{(k-2)^2(k+2)(k+4)} + \frac{1}{(k-2)(k+2)^2(k+4)} + \frac{1}{(k-2)(k+2)(k+4)}\right)x^4 + \cdots\right]$$

Then we have

$$y_2 = \frac{\partial y(x,0,b_0)}{\partial k} = y_1\ ln\ ln\ x\ + b_0\left[1 + \frac{1}{4}x^2 + \frac{1}{64}x^4 + \cdots\right]$$

Hence the general solution is

$$y = A_1\left[-\frac{1}{16}x^4 + \frac{1}{192}x^6 - \cdots\right] + A_2\left(\left[-\frac{1}{16}x^4 + \frac{1}{192}x^6 - \cdots\right] ln\ ln\ x\ + \left[1 + \frac{1}{4}x^2 + \frac{1}{64}x^4 + \cdots\right]\right)$$

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