Oscillatory Solutions Of Fourth- Order Delay Difference Equations With Damp

R. Kodeeswaran[a] and K. Saravanan[b]
[a] Department of Mathematics, Kandaswami Kandar's College, P.Velur, Namakkal, Tamilnadu, India.
[b]Department of Mathematics, Shree Amirtha College of Education, Namakkal, Tamilnadu, India.

Article History: Received: 10 January 2021; Revised: 12 February 2021; Accepted: 27 March 2021; Published online: 20 April 2021

Abstract: In this research work concerned with new oscillatory solutions of Fourth-order delay difference equations with damping term of the form

$$\Delta(r_l(\Delta^3 w_l)) + p_l\Delta^3 w_l + q_l w_{l-\sigma} = 0 \qquad l \geq l_0.$$

Using generalized Riccati transformationand new comparison principles, we establish new oscillation criteria for the equation. Give an example to improve the main results.

Keywords: Fourth- order, delay difference eqations, Riccati transformation.

1. Introduction

In this article, we consider the Fourth-order damped delay difference equations of the term

$$\Delta(r_l(\Delta^3 w_l)) + p_l\Delta^3 w_l + q_l w_{l-\sigma} = 0 \qquad l \geq l_0. \qquad (1.1)$$

The entire work, we ensure the following
(I_1). r is a positive sequence of integers, (I_2).
p and q are real positive sequences,
(I_3). σ is a positive integer.

By a solution of equation (1.1), we mean a non-trivial sequences w and satisfies the equation(1.1). The solution of equation (1.1) is called oscillatory if it has arbitrarily large zero, and otherwise it is called non-oscillatory.

The oscillation of higher and fourth order difference equation have been investigated by many authors, see for examples [5-12], the references cited there in. The result is obtained this article is compliment of existing one.

In [5], the authors discussed oscillation of a class of the fourth order nonlinear difference equation of the form

$$\Delta\left(a_n(\Delta b_n(c_n(\Delta x_n)^\gamma)^\beta)^\alpha\right) + d_n x^\lambda{}_{n+\tau} = 0,$$

where $\alpha, \beta, \gamma, \lambda$ are the ratios of odd positive integers and $\{a_n\}, \{b_n\}, \{c_n\}$ *and* $\{d_n\}$ are positive real sequences. The authors obtained the oscillatory properties of studied equations.

In [13], M. Vijaya and E. Thandapani, deals the oscillation theorem for certain Fourth order quasilinear difference equations of the form

$$\Delta 2(pn|\Delta 2xn|\alpha-1\Delta 2xn) + qn|xn+3|\beta-1xn+3 = 0,$$

where α *and* β are positive constant, $\{p_n\}$ *and* $\{q_n\}$ are positive real sequences.

In [3] the authors studied oscillation of third-order delay difference equation with negative damping term of the form

$$\Delta 3yn - pn\Delta yn+1 + qnf(yn-l) = 0,$$

where $\{p_n\}$ *and* $\{q_n\}$ are real sequences, f is real valued function.

The purpose of this work is to obtain a new oscillatory criteria of equation (1.1) by using Ricatti transformations and comparison principle under the condition

$$R_l = \sum_{s=l_0}^{l-1} \frac{1}{r_s}\exp\left(-\sum_{u=l_0}^{s-1}\frac{p_u}{r_u}\right) \qquad (1.2)$$

and

$$R_l \to \infty \quad as\ l \to \infty. \qquad (1.3)$$

An example included to illustrate the main results.

2. Preparatory Lemmas and Main Results.

The second part of this article, we listed some basic lemmas and obtain new oscillatory criteria for equation (1.1).

Lemma 2.1 Let $\delta \geq 1$ be a ratio of two members, where X and Y are constants. Then

$$X_w - Y^{\frac{\delta+1}{\delta}} \leq \frac{\delta^\delta}{(\delta+1)^{\delta+1}}\frac{X^{\delta+1}}{Y^\delta}, \qquad Y > 0$$

Lemma 2.2 Let $w_l > 0, \Delta w_l > 0, \Delta^2 w_l > 0$ *and* $\Delta^3 w_l > 0$ for every $l \geq l_0$, then and $\delta\epsilon(0,1)$ and some integer L, one has

$$\frac{w_{l+1}}{\Delta w_l} \geq \frac{l-L}{2} \geq \frac{\delta_l}{2}, \qquad l \geq l_0.$$

Lemma 2.3 Let w_l be defined for $l \geq l_0$and $w_l > 0$ with $\Delta^m w_l \leq 0$ for $l \geq l_0$ and not identically zero. Then there exists a large $l \geq l_0$ such that

$$w_l \geq \frac{1}{(m-1)!}(l-l_1)^{m-1}\Delta^{m-1} w(2^{m-k-1}(l)), \qquad l \geq l_1$$

Further if w_l is increasing then,

$$w_l \geq \frac{1}{(m-1)!}\left(\frac{l}{2^{m-1}}\right)^{m-1} \Delta^{m-1} w_l, \qquad l \geq 2^{m-1} l_1.$$

The proof is lemma 2.3 found in [1].

The second part of this section, we obtain new oscillation criteria for equation (1.1).

Theorem 2.4 Assume that (1.2) holds. If there exists positive sequences ρ and ε such that

$$\sum_{u=l_0}^{\infty} \frac{p_u q_u}{\Delta^2 w}(u-\sigma)^2 \frac{\mu}{2}\, \Delta^2 w_u - \frac{1}{4\rho_{u+1} r_{u+1}}\left[\frac{\Delta\rho_u}{\rho_{u+1}} - \frac{p_u}{r_u}\right] = \infty, \qquad (2.1)$$

for some $\mu\epsilon(0,1)$ and

$$\sum_{s=l_1}^{l-1}\left(\varepsilon_{s+1}\sum_{v=s}^{\infty}\frac{1}{r_v}\sum_{u=v}^{\infty} q_u\left(\frac{u-\sigma}{u}\right)^2\right) + \frac{1}{4\varepsilon_s}(\Delta\varepsilon_s)^2 = \infty, \qquad (2.2)$$

then any solution of (1.1) is oscillatory.

Proof: Suppose, let w be a non-oscillatory solution of (1.1), with no less of generality, we suppose that $l_1\epsilon[l_0, \infty)$ such that $w_l > 0$, $w_{l-\sigma} > 0$ for all $l \geq l_1$.

In view of (1.1) and (1.2) that

$\Delta(r_l\Delta^3 w_l) < 0$, there exists two possible cases:

(i) $\Delta^k w_l > 0$ for k = 0,1,2,3. (2.3)

(ii) $\Delta^k w_l > 0$ fork = 0,1, 3 and $\Delta^2 w_l < 0$. (2.4)

Assume that case(i) holds, we define

$$V_l = \frac{\rho_l\, r_l \Delta^3 w_l}{\Delta^2 w}, \qquad (2.5)$$

then $w_l > 0$.

$$\Delta V_l = \Delta\rho_l\left(\frac{r_{l+1}\,\Delta^3 w_{l+1}}{\Delta^2 w}\right) + \rho_l\left(\frac{\Delta r_l\,\Delta^3 w_l}{\Delta^2 w}\right) - \rho_l\left(\frac{r_{l+1}\,\Delta^3 w_{l+1}\Delta^3 w}{(\Delta^2 w)^2}\right) \qquad (2.6)$$

From (1.1), we see that

$$\Delta(r_l(\Delta^3 w_l) = -p_l\Delta^3 w_l - q_l w_{l-\sigma} \qquad (2.7)$$

In view of (2.6) and (2.7), we obtain

$$\Delta V_l = \frac{\Delta\rho_l\,(r_{l+1}\,\Delta^3 w_{l+1})}{\Delta^2 w} - \frac{\rho_l\, p_l\Delta^3 w_l}{\Delta^2 w} - \frac{\rho_l\, q_l w_{l-\sigma}}{\Delta^2 w} - \frac{\rho_{l+1}\, r_{l+1}\,(\Delta^3 w_{l+1})^2}{(\Delta^2 w)^2} \qquad (2.8)$$

From lemma 2.2, we find that

$$\frac{\Delta w_l}{w_l} \leq \frac{2}{l}.$$

Summing from $l-\sigma$ to $l-1$, we have

$$\frac{\Delta w_{l-\sigma}}{w_l} \geq \frac{(l-\sigma)^2}{l^2}$$

$$\Delta w_{l-\sigma} \geq \frac{(l-\sigma)^2}{l^2} w_l. \qquad (2.9)$$

It follows from lemma 2.3, that

$$w_l \geq \frac{\mu}{2} l^2 \Delta^2 w_l \qquad (2.10)$$

for all $\mu\epsilon(0,1)$.

From 2.8, 2.9 and 2.10, we see that

$$\Delta V_l = \frac{\Delta\rho_l\, V_{l+1}}{\rho_{l+1}} - \frac{p_l V_l}{r_l} - \frac{\rho_l\, q_l}{\Delta^2 w}\frac{(l-\sigma)^2}{l^2}\frac{l^2\mu}{2}\Delta^2 w_l - \frac{(w_{l+1})^2}{\rho_{l+1}\, r_{l+1}}$$

$$\leq \frac{\Delta\rho_l V_{l+1}}{\rho_{l+1}} - \frac{p_l V_l}{r_l} - \frac{\rho_l\, q_l}{\Delta^2 w}(l-\sigma)^2 \frac{\mu}{2}\Delta^2 w_l - \frac{(w_{l+1})^2}{\rho_{l+1}\, r_{l+1}}$$

We obtain

$$\Delta V_l \;\leq \left(\frac{\Delta\rho_l}{\rho_{l+1}} - \frac{p_l}{r_l}\right) V_{l+1} - \frac{-\rho_l\, q_l}{\Delta^2 w}(l-\sigma)^2 \frac{\mu}{2}\Delta^2 w_l - \frac{(w_{l+1})^2}{\rho_{l+1}\, r_{l+1}} \qquad (2.11)$$

Applying Lemma 2.1, by denote

$$X = \frac{\Delta\rho_l}{\rho_{l+1}} - \frac{p_l}{r_l} \quad and\; Y = \frac{1}{\rho_{l+1}\, r_{l+1}} \quad with\; \beta = 1 .$$

we get

$$\Delta V_l \;\leq -\frac{\rho_l\, q_l}{\Delta^2 w}(l-\sigma)^2 \frac{\mu}{2}\Delta^2 w_l + \frac{1}{4\rho_{l+1}\, r_{l+1}}\left(\frac{\Delta\rho_l}{\rho_{l+1}} - \frac{p_l}{r_l}\right)^2$$

This implies that

$$\sum_{u=l_1}^{l-1}\left(\frac{\rho_u\, q_u}{\Delta^2 w}(u-\sigma)^2 \frac{\mu}{2}\Delta^2 w_u - \frac{1}{4\rho_{u+1}\, r_{u+1}}\left(\frac{\Delta\rho_u}{\rho_{u+1}} - \frac{p_u}{r_u}\right)^2\right) \leq w_{l_1}. \qquad (2.12)$$

for some $\mu\epsilon(0,1)$, which contradiction to (2.2)
Now, assume that the case (ii) holds.
Define

$$U_{l=}\, \varepsilon_l \frac{\Delta w_l}{w_l}, \qquad (2.13)$$

then $U_l > 0$ for $l \geq l_1$

$$\Delta U_l \leq \frac{\Delta\varepsilon_l}{\varepsilon_l} U_l + \varepsilon_{l+1} \frac{\Delta^2 w_l}{w_l} - \frac{{U_l}^2}{\varepsilon_l} \qquad (2.14)$$

Summing (1.1) from l to $u-1$, we find

$$ru\Delta 3wu - rl\Delta 3wl + \sum_{s=l}^{u-1} qsws{-}\sigma \leq 0.$$

This by virtue of $\Delta w_l > 0, w_l > 0$ *and* $\Delta^2 w_l < 0$ and by (2.9), we obtain

$$r_u\Delta^3 w_u - r_l\Delta^3 w_l + \sum_{s=l}^{u-1} q_s \frac{(s-\sigma)^2}{s^2} w_s \leq 0 .$$

Its follows from $\Delta w_l > 0$ that

$$r_u\Delta^3 w_u - r_l\Delta^3 w_l + w_l \sum_{s=l}^{u-1} q_s \frac{(s-\sigma)^2}{s^2} w_s \leq 0 .$$

By letting $u \to \infty$, we reach at

$$-r_l\Delta^3 w_l + w_l \sum_{s=l}^{u-1} q_s \frac{(s-\sigma)^2}{s^2} w_s \leq 0 .$$

Summing from $l\; to\; \infty$, we find

$$\Delta^2 w_l + \; w_l \sum_{v=l}^{\infty} \frac{1}{r_v} \sum_{l=v}^{\infty} q_l \frac{(l-\sigma)^2}{l^2} \leq 0 .$$

We see that

$$\frac{\Delta^2 w_l}{w_l} \leq -\left[\sum_{v=l}^{\infty} \frac{1}{r_v} \sum_{l=v}^{\infty} q_l \frac{(l-\sigma)^2}{l^2}\right] \qquad (2.15)$$

By using (2.15) in (2.14), we see that

$$\Delta U_l \leq -\varepsilon_{l+1}\left[\sum_{v=l}^{\infty} \frac{1}{r_v} \sum_{l=v}^{\infty} q_l \frac{(l-\sigma)^2}{l^2}\right] + \frac{\Delta\varepsilon_l}{\varepsilon_l} U_l - \frac{{U_l}^2}{\varepsilon_l} \qquad (2.16)$$

Again by applying lemma (2.1), we taking $\beta = 1$,

$$X = \frac{\Delta \varepsilon_l}{\varepsilon_l} \ and \ Y = \frac{1}{\varepsilon_l},$$

we obtain

$$\Delta U_l \leq -\varepsilon_{l+1}\left[\sum_{v=l}^{\infty}\frac{1}{r_v}\sum_{l=v}^{\infty} q_l \frac{(l-\sigma)^2}{l^2}\right] + \frac{1}{4\varepsilon_l}(\Delta\varepsilon_l)^2$$

Summing $l_1 \ to \ l-1$, we obtain

$$\sum_{s=l_1}^{l-1} \varepsilon_{s+1}\left[\sum_{v=l}^{\infty}\frac{1}{r_v}\sum_{u=v}^{\infty} q_u \frac{(u-\sigma)^2}{u^2}\right] + \frac{1}{4\varepsilon_s}(\Delta\varepsilon_s)^2 \leq \Delta U_l$$

Which is contradiction with(2.2). This completes the proof.

Next. We establish criteria for oscillation of (1.1), by comparison with lower order difference equation.

Let us consider the well known second order difference equations see [1]

$$\Delta(r_l(\Delta w_l)) + q_l w_l = 0, \qquad l \geq l_0$$

where $r \ and \ q$ are non negative real sequences. Now, we develop a comparison theorem that the properties of solution of (1.1) with second order difference equation (2.16).

The necessary and sufficient condition for the nonoscilltory solutions of (2.16) that, there exists $l \geq l_0$ and a real sequence u satisfies

$$\Delta u_l + r^{-1} u^2{}_l q_l \leq 0 \qquad (2.17)$$

Lemma 2.5 [1]. Let $\sum_{s=l_0}^{\infty}\frac{1}{r_s} = \infty$ on the condition $liminf \left(\sum_{s=l_0}^{\infty}\frac{1}{r_s}\right)\sum_{s=l}^{\infty} q_s > \frac{1}{4}$, are satisfied, then equation (2.16) is oscillatory.

Theorem 2.6. Let (1.2) and (1.3) hold, and assume that

$$\Delta(r_l(\Delta w_l)) + q_l \frac{\mu}{2}(l-\sigma)^2 w_l = 0 \qquad (2.18)$$

and

$$\Delta^2 w_l + \left(\sum_{v=l}^{\infty}\frac{1}{r_v}\sum_{s=v}^{\infty} q_s \left(\frac{s-\sigma}{s}\right)^2\right) w_l = 0 \qquad (2.19)$$

are oscillatory then any solution of (1.1) is oscillatory.

Proof: As the proof of theorem (2.4), if we take $\rho_l = 1$ in (2.11), we obtain

$$\Delta w_l + \frac{p_l}{r_l}V_{l+1} + \frac{q_l(l-\sigma)^2}{\Delta^2 w}\frac{\mu}{2}\Delta^2 V_l \frac{(w_{l+1})^2}{r_{l+1}} \leq 0 \quad for \ any \ \mu \in (0,1) \qquad (2.20)$$

In view of (2.17) and (2.20) that the equation (2.18) has non oscillatory solution. Which is a contradiction.

If we take $\varepsilon_l = 1$in (2.16) that, we find

$$\Delta w_l + \left(\sum_{v=l}^{\infty}\frac{1}{r_v}\sum_{l=v}^{\infty} q_l \left(\frac{l-\sigma}{l}\right)^2\right) + U_l{}^2 \leq 0. \qquad (2.21)$$

Hence equation (2.19) is nonoscillatory. This contradiction completes the proof.

3. Examples

In this section, we give an example to improve the main result.

Example 3.1 Consider the fourth order delay difference equation

$$\Delta\left(\frac{1}{8l}(\Delta^3 w_l)\right) + \frac{1}{8}(\Delta^3 w_l) + 2l w_{l-1} = 0 \qquad (3.1)$$

Here $r_l = \frac{1}{8l}, p_l = \frac{1}{8}, q_l = 2l \ and \ \sigma = 1$

All the conditions of Theorem 2.4 are satisfied. Also $R_l = 8l - 1 \to \infty \ as \ l \to \infty$. Further, let $p_l = \varepsilon_l = 1$, then

$$\sum_{u=l_0}^{\infty}\left(p_u q_u (u-\sigma)^2 \frac{\mu}{2} - \frac{1}{4p_{u+1}r_{u+1}}\left(\frac{\Delta\rho_u}{\rho_{u+1}} - \frac{p_u}{r_u}\right)\right) = \infty,$$

and

$$\sum_{s=l_0}^{\infty}\left(\varepsilon_{s+1}\sum_{v=s}^{\infty}\frac{1}{r_u}\sum_{u=v}^{\infty}q_u\left(\frac{u-\sigma}{u}\right)^2-\frac{\Delta\varepsilon_s}{4\varepsilon_s}\right)=\infty$$

Hence by theorem 2.4, any solution of equation 3.1 is oscillatory.

References

1. Agarwal. R.P., Grace.S.R., O'Regan.D., (2013), Oscillation theory for difference and functional differential equations, Springer Science and Business Media.
2. Agarwal.R.P., Bohner.M., Grace.S.R., and O'Regan.D., Discrete oscillation Theory, Hindawi Publishing Corporation, New York, 2005.
3. Bohner.M., Geeatha.S., Selvarangam.S., and Thandapani.E., Oscillation of Third-order delay difference equations with negative damping Annales University Mariae Curie - Sklodowska Lublin- Polonia., Vol. LXX II, No.1 2018. Pp19-28.
4. Dharuma.C., Srinivasan.R., Thandapani.E., Oscillation criteria for Third-order Non-linear Functional Difference Equation with Damping, Appl. Math. Int. Sci, 11, No.3, 1-8 (2017).
5. Dosla and Krejcova., Oscillation of a class of the fourth order non-linear difference of equations, Advances in Difference Equations.2012, 2012:99.
6. Dosla and Krejcova., Asymptotic and Oscillatory properties of fourth order nonlinear difference equations, App/.Math. Comput.,249(2014), 164-173.
7. Selvaraj.B and Mohammed Alizaffer. I., Oscillatory Property of Fourth-order Neutral Delay Difference Equations., J.Comp & Math. Sci., Vol. 1 (3), (2010), 364-373.
8. Thandapani. E., Selvaraj.B., Oscillatory and nonoscillatory behavior of fourth order quasilinear difference Equations, Far East Jour. Appl. Math. 17 (2004), 287-307.
9. Thandapani. E., Selvraj.B., Oscillation of Fourth-order quasilinear Difference Equations, Fasc. Math., 37 (2007) 109-119.
10. Thandapani.E., Arokiasamy. I.M., Oscillation and asymptotic properties of solutions of nonlinear fourth order difference equations, Glasnik mat., 37(2003), 119-131.
11. Thandapani. E., Graef. J. R, Oscillation and asymptotic behavior of fourth- order nonlinear delay difference equations. Fasc. Math., 31 (2001),23-36.
12. Taylor W.E., Fourth order Difference Equations; Oscillation and nonoscillation, Rocky Mountain J. Math., 23 (1993), 8781-795.
13. Vijaya. M., and Thandapani. E., Oscillation theorem for certain fourth order quasilinear difference equations, IOSR Journal of Mathematics, Vol.12, issue 5 ver. VII (2016) pp.41-46.