Research Article

The Beckman-Quarles Theorem For Rational Spaces: Mapping Of Q^d To Q^d That Preserve Distance 1

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Article History: Received: 11 January 2021; Revised: 12 February 2021; Accepted: 27 March 2021; Published online: 16 April 2021

Abstract : Let R^d and Q^d denote the real and the rational d-dimensional space, respectively, equipped with the usual Euclidean metric. For a real number $\rho > 0$, a mapping $f: A \longrightarrow X,$ where X is either R^d or Q^d and $A \subseteq X$, is called ρ-distance preserving $\|x - y\| = \rho$ implies $\|f(x) - f(y)\| = \rho$, for all x,y in A.

Let G(Q^d,a) denote the graph that has Q^d as its set of vertices, and where two vertices x and y are connected by edge if and only if $\|x - y\| = a$. Thus, G(Q^d,1) is the unit distance graph. Let ω(G) denote the clique number of the graph G and let ω(d) denote ω(G(Q^d, 1)).

The Beckman-Quarles theorem [1] states that every unit- distance-preserving mapping from R^d into R^d is an isometry, provided $d \geq 2$.
The rational analogues of Beckman- Quarles theorem means that, for certain dimensions d, every unit- distance preserving mapping from Q^d into Q^d is an isometry.

A few papers [2, 3, 4, 5, 6, 8,9,10 and 11] were written about rational analogues of this theorem, i.e, treating, for some values of d, the property "Every unit- distance preserving mapping $f: Q^d \longrightarrow Q^d$ is an isometry".
The purpose of this thesis is to present all the results (see [3, 5, 6 and 7]) about the rational analogues of the Beckman-Quarles theorem, and to establish rational analogues of the Beckman-Quarles theorem, for all the dimensions $d, d \geq 5$.

1.1 Introduction:

Let R^d and Q^d denote the real and the rational d-dimensional space, respectively.
Let $\rho > 0$ be a real number, a mapping : $R^d \longrightarrow Q^d$, is called ρ- distance preserving if $\|x - y\| = \rho$ implies $\|f(x) - f(y)\| = \rho$.

The Beckman-Quarles theorem [1] states that every unit- distance-preserving mapping from R^d into R^d is an isometry, provided$d \geq 2$.
A few papers [4, 5, 6, 8,9,10 and 11] were written about the rational analogues of this theorem, i.e, treating, for some values of d, the property "every unit- distance preserving mapping $f: Q^d \longrightarrow Q^d$ is isometry".

We shall survey the results from the papers [2, 3, 4, 5, 6, 8,9,10 and 11] concerning the rational analogues of the Backman-Quarles theorem, and we will extend them to all the remaining dimensions , $d \geq 5$.

History of the rational analogues of the Backman-Quarles theorem:
We shall survey the results from papers [2, 3, 4, 5, 6, 8,9,10 and 11] concerning the rational analogues of the Backman-Quarles theorem.

1. A mapping of the rational space Q^d into itself, for d=2, 3 or 4, which preserves all unit- distance is not necessarily an isometry; this is true by W.Bens [2, 3] and H.Lenz [6].

2. W.Bens [2, 3] had shown the every mapping $f: Q^d \longrightarrow Q^d$ that preserves the distances 1 and 2 is an isometry, provided $d \geq 5$.

3. Tyszka [8] proved that every unit- distance preserving mapping $f: Q^8 \longrightarrow Q^8$ is an isometry; moreover, he showed that for every two points x and y in Q^8 there exists a finite set S_{xy} in Q^8 containing x and y such that every unit- distance preserving mapping $f: S_{xy} \longrightarrow Q^8$ preserves the distance between x and y. This is a kind of compactness argument, that shows that for every two points x and y in Q^d there exists a finite set S_{xy}, that contains x and y ("a neighborhood of x and y") for which already every unit- distance preserving mapping from this neighborhood of x and y to Q^d must preserve the distance from x to y. This implies that every unit preserving mapping from Q^d to Q^d must preserve the distance between every two points of Q^d.

4. J.Zaks [8, 9] proved that the rational analogues hold in all the even dimensions d of the form $d = 4k(k+1)$, for $k \geq 1$, and they hold for all the odd dimensions d of the form $d = 2n^2-1 = m^2$. For integers $n, m \geq 2$, (in [9]), or $d = 2n^2 - 1$, $n \geq 3$ (in [10]).

5. R.Connelly and J.Zaks [5] showed that the rational analogues hold for all even dimensions $d, d \geq 6$.

We wish to remark that during the preparation of this thesis, it was pointed out to us that an important argument, in the proof of the even dimensions $d, d \geq 6$, is missing. Here we propose a valid proof for all the cases of $d, d \geq 5$.

6. J.Zaks [11] had shown that every mapping $f: Q^d \longrightarrow Q^d$ that preserves the distances 1 and $\sqrt{2}$ is an isometry, provided $d \geq 5$.

New results:

Denote by $L[d]$ the set of $4 \cdot \binom{d}{2}$ Points in Q^d in which precisely two non-zero coordinates are equal to 1/2 or -1/2.

A "quadruple" in $L[d]$ means here a set $L_{ij}[d]$, $i \neq j \in I = \{1, 2, \dots, d\}$; contains four j points of $L[d]$ in which the non- zero coordinates are in some fixed two coordinates i and j; i.e.

$$L_{ij}[d] = (0,\dots 0, \overset{i}{\pm \tfrac{1}{2}}, 0\dots 0, \overset{j}{\pm \tfrac{1}{2}}, 0, \dots 0)$$

Our main results are the following:

Theorem 1:
Every unit- distance preserving mapping $f: Q^5 \longrightarrow Q^5$ is an isometry; moreover, dim $(aff(f(L[5]))) = 5$.

Theorem 2:
Every unit- distance preserving mapping $f: Q^6 \longrightarrow Q^6$ is an isometry; moreover, dim $(aff(f(L[6]))) = 6$.

Theorem 3:
For all the dimensions d, $d \geq 5$, every unit- distance preserving mapping $f: Q^d \longrightarrow Q^d$ is an isometry.

<u>Auxiliary Lemmas:</u>

We need the following Lemmas for our proofs of the Theorems 1 and 2.

Lemma 1: (due J.Zaks [10]).
If $v_1, \dots, v_n, w_1, \dots, w_m$ are points in Q^d, $n \leq m$ such that $\|v_i - v_j\| = \|w_r - w_s\|$,
for all $1 \leq i \leq j \leq n, 1 \leq r \leq s \leq m$ then there exists a congruence $f: Q^d \longrightarrow Q^d$, such that
$f(v_i) = w_i$ for all $1 \leq i \leq n$.

Lemma 2: (due to Chilakamarri [4]).
a. For even d, $\omega(d) = d+1$, if $d+1$ is a complete square; otherwise $\omega(d) = d$.
b. For odd d, $d \geq 5$, the value of $\omega(d)$ is as follows: if $d = 2n^2 - 1$, then $\omega(d) = d+1$; if $d \neq 2n^2-1$ and the Diophantine equation $dx^2 - 2(d-1)y^2 = z^2$ has a solution in which $x \neq 0$ then $\omega(d) = d$; otherwise $\omega(d) = d - 1$.

Lemma 3:
If a, b, c are three numbers that satisfy the triangle inequality and if a^2, b^2, c^2 are rational numbers then:
a. , and

$$b^2 - \frac{(b^2 - a^2 + c^2)^2}{4c^2}$$

b. The space $Q^d, d \geq 8$ contains a triangle *ABC*, having edge length: *AB*=*c*, *BC*=*a*, *AC*=*b*.

Proof of Lemma 3:
To prove (**a**), its suffices to prove that $4b^2c^2 - (b^2 - a^2 + c^2)^2 > 0$

$$4b^2c^2 - (b^2 - a^2 + c^2)^2 =$$

$$= [2bc + (b^2 - a^2 + c^2)] \cdot [2bc - (b^2 - a^2 + c^2)]$$
$$= [(b + c)^2 - a^2] \cdot [a^2 - (b - c)^2]$$
$$= (\mathrm{a + b + c})(\mathrm{b + c - a})(\mathrm{a + b - c})(\mathrm{a - b + c}) > 0.$$

The triangle inequality implies that the expression in the previous line on the left is positive; it appears also in Heron's formula.

To prove (**b**): Let *a*, *b*, *c* be three numbers that satisfy the triangle inequality, and so that a^2, b^2, c^2 are rational numbers.
The number $c^2/4$ is positive and rational, hence there exist, according to Lagrange Four Squares theorem [8], rational numbers $\alpha, \beta, \gamma, \delta$ such that $c^2/4 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2$.

By part (a), the following holds: $b^2 - \frac{(b^2 - a^2 + c^2)^2}{4c^2} > 0$, therefore there exist by Lagrange Theorem rational numbers: *x*, *y*, *z*, *w*, such that:

$$b^2 - \frac{(b^2 - a^2 + c^2)^2}{4c^2} = x^2 + y^2 + z^2 + w^2.$$

Consider the following points:
$A = (-\alpha, -\beta, -\gamma, -\delta, 0, \dots, 0)$
$B = (\alpha, \beta, \gamma, \delta, 0, \dots, 0)$
$C = (\frac{b^2 - a^2}{c^2}\alpha, \frac{b^2 - a^2}{c^2}\beta, \frac{b^2 - a^2}{c^2}\gamma, \frac{b^2 - a^2}{c^2}\delta, x, y, z, w, 0, \dots, 0)$

The points *A*,*B* and *C* satisfy:

$$\|A - B\| = \sqrt{4(\alpha^2 + \beta^2 + \delta^2 + \gamma^2} = c$$

$$\|A - C\| = \sqrt{\left[\frac{b^2 - a^2}{c^2} + 1\right]^2 (\alpha^2 + \beta^2 + \delta^2 + \gamma^2) + x^2 + y^2 + z^2 + w^2}$$

$$= \sqrt{\frac{(b^2 - a^2 + c^2)^2}{4c^2} + b^2 - \frac{(b^2 - a^2 + c^2)^2}{4c^2}} = b,$$

and:

$$\|B - C\| = \sqrt{\left[\frac{b^2 - a^2}{c^2} - 1\right]^2 (\alpha^2 + \beta^2 + \delta^2 + \gamma^2) + x^2 + y^2 + z^2 + w^2}$$
$$= \sqrt{\frac{(b^2 - a^2 - c^2)^2}{4c^2} - \frac{(b^2 - a^2 + c^2)^2}{4c^2} + b^2} =$$
$$= \sqrt{\frac{-4(b^2 - a^2)c^2 + 4b^2c^2}{4c^2}} = a$$

This completes the proof of Lemma 3.

Corollary 1:

If a, b, 1 satisfy the triangle inequality and if a^2, b^2 are rational numbers, then the space Q^5 contains the vertices of a triangle which has edge lengths *a, b, 1*.

Proof:
Consider the following points:

$$A = (\frac{1}{2}, 0,0,0,0)$$
$$B = (-\frac{1}{2}, 0,0,0,0)$$
$$C = ((b^2 - a^2)^{\frac{1}{2}}, \alpha, \beta, \gamma, \delta)$$

Where $\alpha, \beta, \gamma, \delta$ are the rational numbers that exist according to Lagrange theorem, for which:

$$b^2 - \frac{(b^2 - a^2 + 1)^2}{4} = \alpha^2 + \beta^2 + \delta^2 + \gamma^2$$

From the proof of Lemma 2 the triangle, *ABC* has the edge length *a*, *b*, *1*.

Corollary 2:
If t is a number such that $\sqrt{2 + \frac{2}{m-1}} - 1 \leq t \leq \sqrt{2 + \frac{2}{m-1}} + 1$, $t^2 \in Q$
Where $m \geq 4$ is a natural number, then the space $Q^d, d \geq 5$, contains a triangle *ABC* having edge length *1*,*t*, $\sqrt{2 + \frac{2}{m-1}}$.

Proof:
According to Lemma 2, the numbers 1,t, $\sqrt{2 + \frac{2}{m-1}}$ satisfy the triangle inequality, and the result follows from Corollary 1.

Lemma 4:
If x and y are two points in $Q^d, d \geq 5$, so that:

$$\sqrt{2 + \frac{2}{m-1}} - 1 \leq \|x - y\| \leq \sqrt{2 + \frac{2}{m-1}} + 1$$

where $\omega(d) = m$, then there exists a finite set S(*x*,*y*), contains x and y such that *f(x)≠f(y)* holds for every unit-distance preserving mapping *f: S(x,y)→* Q^d.

Proof of Lemma 4:
Let x and y be points in $Q^d, d \geq 5$, for which,
$\sqrt{2 + \frac{2}{m-1}} - 1 \leq \|x - y\| \leq \sqrt{2 + \frac{2}{m-1}} + 1$ where $\omega(d) = m$.
The real numbers $\|x\text{-}y\|$, $\sqrt{2 + \frac{2}{m-1}}$ and *1* satisfy the triangle inequality, hence by Corollary 2 there exist three points A, B, C such that $\|A\text{-}B\| = \|x\text{-}y\|$,
$\|A\text{-}C\| = \sqrt{2 + \frac{2}{m-1}}$ and $\|B\text{-}C\| = 1$. It follows by two rational reflections that there exists a rational point **z** for which $\|y\text{-}z\|$=1 and $\|x\text{-}z\| = \sqrt{2 + \frac{2}{m-1}}$, (see Figure 1).
Let $\{v_0, \dots, v_{m-1}\}$ be a maximum clique *in* G(Q^d,*1)*, and let w_0 be the reflection of v_0 with respect to the rational hyperplane passing through the points $\{v_1, \dots, v_{m-1}\}$ it follows that $\|v_0 - w_0\| = \sqrt{2 + \frac{2}{m-1}}$, (see Figure 2).

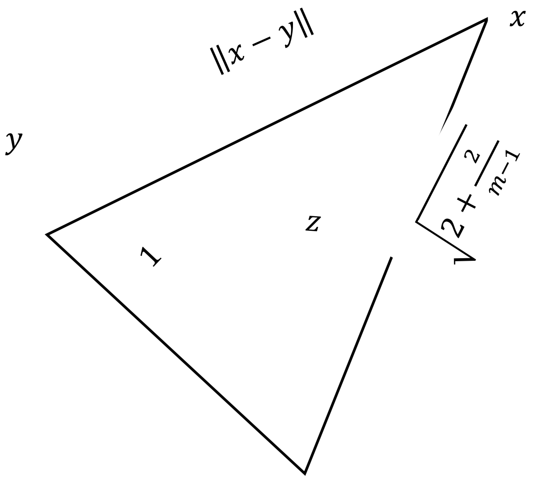

Figure 1

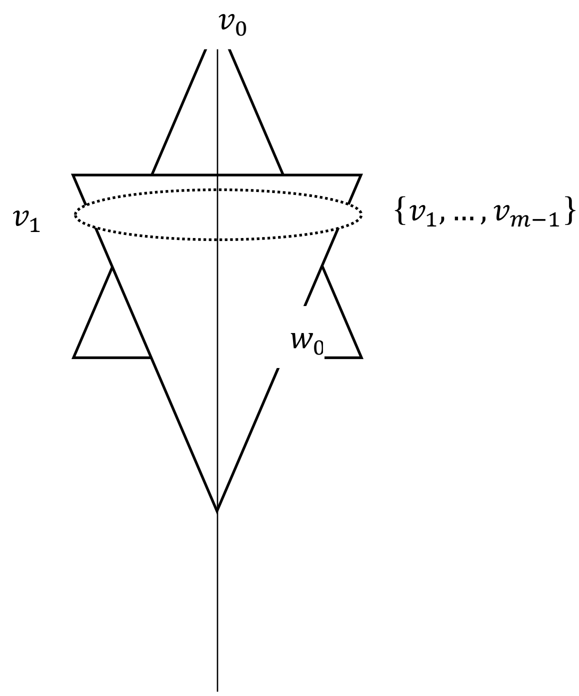

Figure 2

Based on $\|x\text{-}z\| = \|v_0 - w_0\|$ and lemma 1, there exist a rational translation h for which $h(v_0)= x$ and $h(w_0)=z$. Denote $g\ (h(v_i))= V_i$ for all $1\le i\le m\text{-}1$, (see Figure 3).

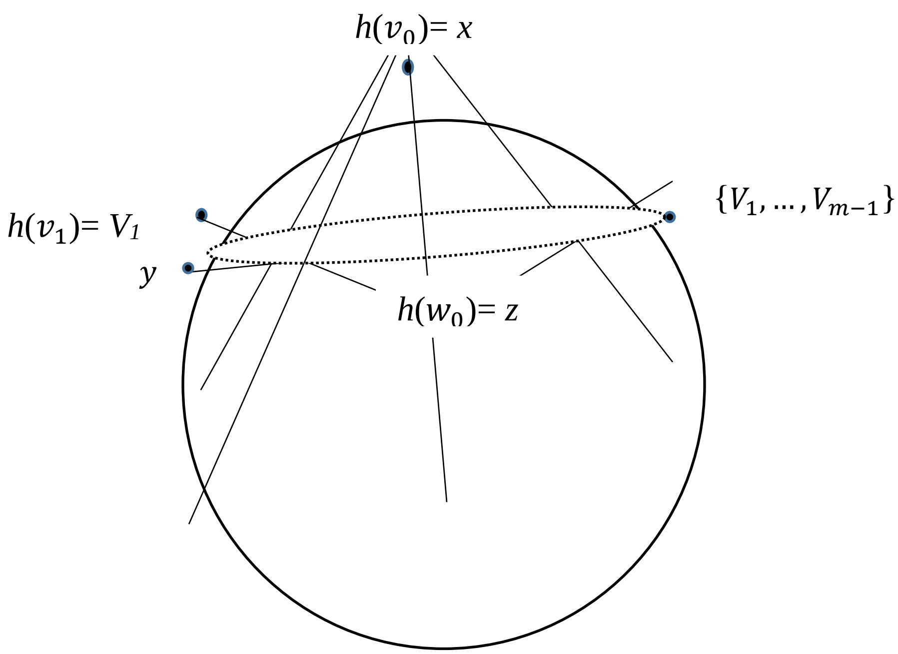

Figure 3

Denote $S(x, y) = \{x, y, z, v_1, \ldots, v_{m-1}\}$. Suppose that $f(x)= f(y)$ holds for some unit- distance preserving mapping $f: S(x,y)\rightarrow Q^d$.

The assumption *f(x) = f(y)* and $\|y\text{-}z\|=1$ imply that $\|f(y)\text{ - }f(z)\|=1=\|f(x)=f(z)\|$, hence the set $\{f(x), f(z), f(v_1), \dots, f(v_{m-1})\}$, forms a clique in G($Q^d$,1) of size *m+1*,which is a contradiction. It follows that *f(x) ≠ f(y)* holds for every unit- distance preserving mapping *f: S(x,y)*→ Q^d.
This completes the proof of Lemma 4.

Corollary 3:
If *x* and *y* are two points in $Q^d, d \geq 5$, such that $\|x\text{-}y\|=\sqrt{2}$, then every unit- distance preserving mapping *f:* $Q^d \rightarrow Q^d$ satisfies *f(x)≠f(y).*

Mappings of Q^5 to Q^5 that preserve distance 1

The purpose of this section is to prove the following Theorem.

Theorem 1:
Every unit- distance preserving mapping *f*: $Q^5 \rightarrow Q^5$ is an isometry; moreover, *dim(aff(f(L[5])))=5.*
To prove Theorem 1, we prove first the following Theorem.

Theorem 1*:
If *Z, W* are two points in Q^5, for which $\|Z-W\| = \sqrt{2}$, then there exists a finite set M_5, containing *Z* and *W*, such that for every unit- distance preserving mapping *f*: $M_5 \rightarrow Q^5$, the following equality holds:
$\|f(Z)\text{-}f(W)\| = \|Z\text{-}W\|$

Proof of Theorem 1*:
Let *Z, W* are any two points in Q^5, for which $\|Z-W\| = \sqrt{2}$.
Denote by *L[5]* the set of $4 \cdot \binom{5}{2} = 40$ points in Q^d in which precisely two coordinates are non- zero and are equal to 1/2 or -1/2 .
A "quadruple" in *L[5]* means a set $L_{ij}[5], i \neq j \epsilon I = \{1, 2, 3, 4, 5\}$, containing four points of *L[5]* in which the non-zero coordinates are in some fixed two, the *i*-th and the *j*-th coordinates; i.e.

$$L_{ij}[5] = \{\left(0, \pm\frac{1}{2}, 0, \pm\frac{1}{2}, 0\right)\} \qquad 1 \quad i \quad . \quad j \quad 5$$

If ρ is a distance between any two points of the set *L[5]* then $\rho \epsilon \{\sqrt{0.5}, 1, \sqrt{1.5}, \sqrt{2}\}$.
Fix a quadruple $L_{ij}[5]$ let *x, y* two points in $L_{ij}[5]$ such that $\|x\text{-}y\|=\sqrt{2}$.
By Lemma 1 and based on $\|Z\text{-}W\|=\|x\text{-}y\|$, there exists a rational isometry *h*: $Q^5 \rightarrow Q^5$ for which *h(x) =:Z=x** and *h(y)=W:=y** ; denote *h(l)=l** for all lϵ *L[5].*
Let *L*[5] = {l* = h(l)* for all lϵ *L[5]*}; it is clear that *Z, W* ϵ *L*[5]*, and to simplify terminology we will denote *L*[5] =* $\{l^*_i\}$ when *i* ϵ {1, 2, …, 40}.
Define the set M_5 by: $M_5 = \cup \{ S(l^*_i, l^*_j) \cup S(l^*_n, l^*_m) \cup S(l^*_s, l^*_t) \}$;
for all *i, j, n, m, s ,t*ϵ {1, 2, …, 40} when $\|l^*_i - l^*_j\| = \sqrt{0.5}$,
$\|l^*_n - l^*_n\| = \sqrt{1.5}$ and $\|l^*_s - l^*_t\| = \sqrt{2}$; where the sets *S* are given by Lemma 4.

Let *f, f*: $M_5 \rightarrow Q^5$ be any unit- distance preserving mapping.

Claim 1:
If *x* and *y* are two points in *L*[5]* for which $\|x\text{-}y\|=1, \sqrt{2}$ then *f(x) ≠ f(y).*
Proof of Claim 1:
Clearly, if $\|x\text{-}y\|=1$, then $\|f(x)\text{ - }f(y)\|=1$, hence *f(x) ≠ f(y).*
The distance $\sqrt{2}$ is between $\sqrt{2+\frac{2}{m-1}} - 1$ and $\sqrt{2+\frac{2}{m-1}} + 1$.
Where *m=ω(d)*=4 for *d*=5.

Therefore, if $\|x\text{-}y\|=\sqrt{2}$, then there exist an i and j, $1\leq i\neq j\leq 40$, such that $x=l^*_i$, $y=l^*_j$ and $\|l^*_i - l^*_j\| = \sqrt{2}$. (l^*_i and l^*_j on the same quadruple).
By Lemma 4, applied to l^*_i *and* l^*_j, there exists a set S(l^*_i, l^*_j), that contains l^*_i *and* l^*_j, for which every unit-distance preserving mapping g: S(l^*_i, l^*_j)→ Q^5 satisfies
g(l^*_i)≠ g(, l^*_j).
In particular this holds for the mapping g= f / S(l^*_i, l^*_j), therefore f(l^*_i)≠ f(, l^*_j).

Claim 2:
The mapping f preserves all the distances$\sqrt{2}$. In particular $\|f(Z)\text{-}f(W)\|=\sqrt{2}$.
Proof of Claim 2:
Consider the graph P of unit distances among the points of *L*[5]*; it is isomorphic to the famous Petersen's graph, by substituting a 4-cycle for each vertex of P.
(See figure 4).

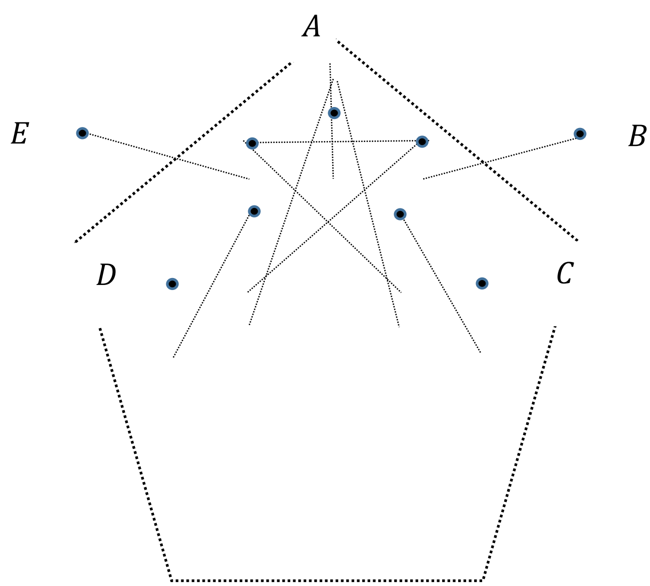

Figure 4

We prove that the affine dimension of the *f*- image of each quadruple, i.e., the image of the four points that correspond to one vertex of P must be 2. Indeed, by claim 1 this dimension is at least 2, since $f(l^*_i) \neq f(l^*_j)$ for all l^*_i and l^*_j on *L*[5]*
(In particular, this holds for all l^*_i and l^*_j on the same quadruple).
Suppose, by contradiction, that $dim(aff(f(A)))\geq 3$, for some quadruple *A*, let the quadruple *B*, *C*, *D*, and *E* correspond to vertices of P so that A, B, C, D and *E* is a cycle in P.
All the points of *f(B)* and *f(E)* must be at unit distance from those of *f(A)*, so all the points of *f(B)* and *f(E)* lie on a circle, say circle S with enter O.
This means that *f(B)* and *f(C)* are two squares inscribed in S. it follows that all the points of *f(C)* and *f(D)* must lie on the 3-flat that is perpendicular to 2-flat determined by S and passes through O.
But this cannot happen, since the points of *f(C)* span a flat of dimension at least 2 in this 3-flat, which then forces the points of *f(D)* to lie on a line, which is impossible.
It follows that the points of any *f*(F) lie on the intersection of some unit-distance spheres and a 2-flat which is a circle; when F={a, b, c, d} is a given block,
such that $\|a\text{-}b\| = \|b\text{-}c\| = \|c\text{-}d\| = \|d\text{-}a\| = 1$ *and* $\|a\text{-}c\| = \|b\text{-}d\| = \sqrt{2}$.
Thus *f(a), f(b), f(c),*and *f(d)* form the vertex set of quadrangle, of edge length one that lies in a circle. (See figure 5).

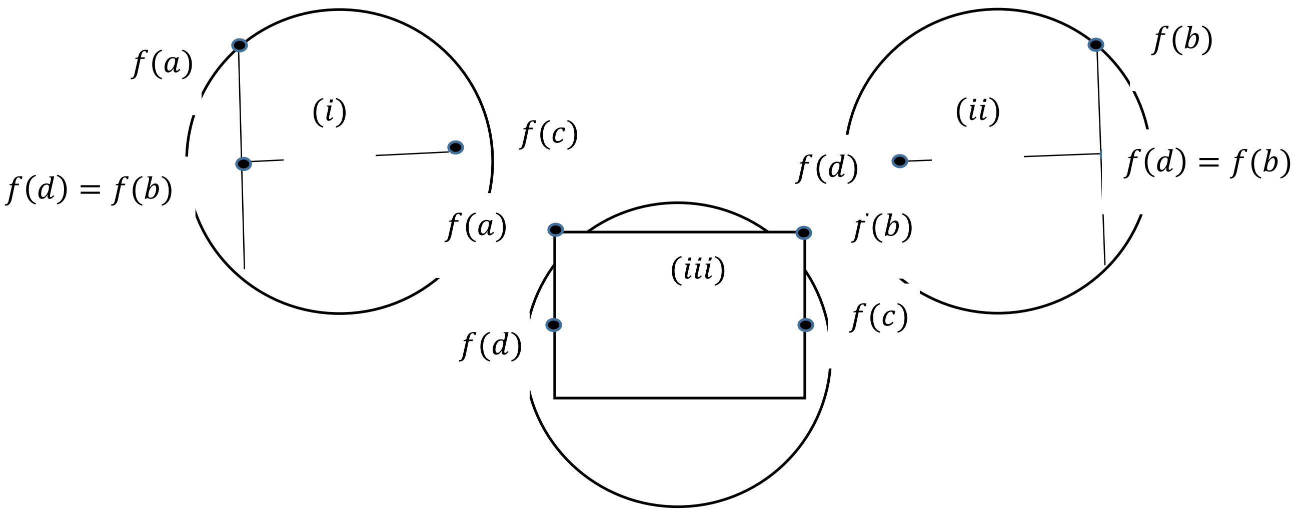

Figure 5

The situations (*i*) and (*ii*) are impossible since $f(l^*_i) \neq f(l^*_j)$ for all l^*_i and l^*_j on $L^*[5]$.
It follows that *f(a), f(b), f(c),* and *f(d)* form vertex set of a square in circle of diameter $\sqrt{2}$, implying: $\|f(a)-f(c)\| = \|f(b)-f(d)\| = \sqrt{2}$.
Hence, the distance $\sqrt{2}$, within each quadrangle are preserved. In particular
$\|f(Z)-f(W)\| = \sqrt{2}$.
This completes the proof of Theorem 1*.

Proof of Theorem 1:
Let *f* be a unit distance preserving mapping $f{:}Q^5 \to Q^5$. By Theorem 1^* the unit distance preserving mapping f preserves the distance $\sqrt{2}$ **.**
Our result follows by using a Theorem of J. Zaks [8], which states that if a mapping
$g{:}Q^d \to Q^d$ preserves the distances 1 and $\sqrt{2}$**,** then *g* is an isometry, provided $d \geq 5$.
Moreover, dim(*aff(f(L[5]))*) = 5:
The mapping f is an isometry, hence it suffices to provide that dim(*aff (L[5])*) = 5.
To show this, notice that:

$$\frac{1}{2}\left(\frac{1}{2},\frac{1}{2},0,0,0\right)+\frac{1}{2}\left(\frac{1}{2},-\frac{1}{2},0,0,0\right)=\frac{1}{2}(1,0,0,0,0)$$
$$\frac{1}{2}\left(\frac{1}{2},\frac{1}{2},0,0,0\right)+\frac{1}{2}\left(-\frac{1}{2},\frac{1}{2},0,0,0\right)=\frac{1}{2}(0,1,0,0,0)$$
$$\frac{1}{2}\left(0,0,\frac{1}{2},\frac{1}{2},0\right)+\frac{1}{2}\left(0,0,\frac{1}{2},-\frac{1}{2},0\right)=\frac{1}{2}(0,0,1,0,0)$$
$$\frac{1}{2}\left(0,0,\frac{1}{2},\frac{1}{2},0\right)+\frac{1}{2}\left(0,0,-\frac{1}{2},\frac{1}{2},0\right)=\frac{1}{2}(0,0,0,1,0)$$
$$\frac{1}{2}\left(0,0,0,\frac{1}{2},\frac{1}{2}\right)+\frac{1}{2}\left(0,0,0,-\frac{1}{2},\frac{1}{2}\right)=\frac{1}{2}(0,0,0,0,1)$$

Hence all the major unit vectors in R^5 when multiplied by $\frac{1}{2}$, are convex combinations of points in *L[5]*.
This completes the proof of Theorem 1.

Mapping of Q^6 to Q^6 that preserve distance 1

The purpose of this section is to prove the following Theorem:
Theorem 2:

Every unit –distance preserving mapping $f: Q^6 \to Q^6$ is an isometry; moreover, dim (*aff(f(L[6]))*) = 6.

To prove Theorem 2, we prove first the following Theorem.

Theorem 2*:

if Z,W are any two points in Q^6, for which $\|Z\text{-}W\| = \sqrt{2}$, then there exists a finite set M_6, containing Z and W, such that for every unit –distance preserving mapping f: $M_6 \to Q^6$, the following equality holds: $\|f(Z)\text{-}f(W)\| = \|Z\text{-}W\|$.

Proof of Theorem 2*:

Consider the 6 points $\{A_1, \ldots, A_6\}$, defined as follows:

$$A_1 = (\tfrac{1}{2}, \quad 0, \quad 0, \quad 0, \quad 0, \quad \tfrac{1}{2})$$
$$A_2 = (\tfrac{1}{2}, \quad 0, \quad 0, \quad 0, \quad 0, \quad -\tfrac{1}{2})$$
$$A_3 = (0, \quad \tfrac{1}{2}, \quad 0, \quad 0, \quad \tfrac{1}{2}, \quad 0)$$
$$A_4 = (0, \quad \tfrac{1}{2}, \quad 0, \quad 0, \quad -\tfrac{1}{2}, \quad 0)$$
$$A_5 = (0, \quad 0, \quad \tfrac{1}{2}, \quad \tfrac{1}{2}, \quad 0, \quad 0)$$
$$A_6 = (0, \quad 0, \quad \tfrac{1}{2}, \quad -\tfrac{1}{2}, \quad 0, \quad 0)$$

The points $\{A_1, \ldots, A_6\}$ form the vertices of a regular 5- simplex of edge length one in Q^6. Let the 6 points $B_1, B_2, \ldots, B_6$ of Q^6 be defined by $B_i = -A_i$, $1 \le i \le 6$, their mutual distances are one, so they form the vertices of a regular 5 – simplex of edge length one in Q^6. Let $T_6 = \{A_1, \ldots, A_6, B_1, \ldots, B_6\}$.

Fix a $k, 1 \le k \le 6$, by Lemma 1 and based on $\|Z - W\| = \|A_k - B_k\|$ there exists a rational isometry $h: Q^6 \to Q^6$ for which $h(A_k) = Z := A^*{}_k$ and $h(B_k) = W := B^*{}_k$; denote $h(A_i) = A^*{}_i$ and $h(B_i) = B^*{}_i$ for all $1 \le i \le 6$.

Let $T^*{}_6 = \{A^*{}_1, \ldots, A^*{}_d, B^*{}_1, \ldots, B^*{}_6\}$; it is clear that $Z, W \in T^*{}_6$.

Define the set M_6 by: $M_6 = S(A^*{}_1, B^*{}_1) \cup S(A^*{}_2, B^*{}_2) \cup \ldots \cup S(A^*{}_6, B^*{}_6)$, where the sets S are given by Lemma 4.

Let $f, f: M_6 \to Q^6$ be any unit-distance preserving mapping.

Claim 3:

If x and y are two points in $T^*{}_6$, then $f(x) \ne f(y)$.

Proof of Claim 3:

Computing the mutual distances of the points in $T^*{}_6$ show that:

$\|A^*{}_i - A^*{}_j\| = \|B^*{}_i - B^*{}_j\| = \|A^*{}_i - B^*{}_j\| = 1$, for all $1 \le i < j \le 6$, and

$\|A^*{}_i - B^*{}_i\| = \sqrt{2}$, for all $1 \le i \le 6$.

All of the distances above are between $\sqrt{2 + \frac{2}{m-1}} - 1$ and $\sqrt{2 + \frac{2}{m-1}} + 1$.

where $m = \omega(d) = 6$ for $d = 6$.

Therefore if $\|x - y\| = 1$, then $\|f(x) - f(y)\| = 1$, hence $f(x) \ne f(y)$;

if $\|x - y\| = \sqrt{2}$ there is an $i, 1 \le i \le 6$, such that $x = A^*{}_i, y = B^*{}_i$ and $\|A^*{}_i - B^*{}_i\| = \sqrt{2}$.

By Lemma 4, applied to $A^*{}_i$ and $B^*{}_i$, there exists a set $S(A^*{}_i, B^*{}_i)$, that contains $A^*{}_i$ and $B^*{}_i$, for which every unit-distance preserving mapping $g: S(A^*{}_i, B^*{}_i) \to Q^d$ satisfies $g(A^*{}_i) \ne g(B^*{}_i)$.

In particular, this holds for the mapping $g = f / S(A^*{}_i, B^*{}_i)$, therefore $f(A^*{}_i) \ne f(B^*{}_i)$.

Claim 4:

The mapping f preserves all the distances $\sqrt{2}$, between $A^*{}_i$ and $B^*{}_i$ for all $i = 1,2, \ldots, 6$. In particular $\|f(Z) - f(w)\| = \sqrt{2}$.

Proof of Claim 4:

Consider the following (4) points:

$$\Delta_1 = \{f(A_3^*), f(B_4^*), f(B_5^*), f(B_6^*)\}.$$

All of their mutual distances are one, since f preserves distance one, so they form the vertices of a regular 3-simplex of edge length one in Q^6. The intersection of the 4 unit spheres, centered at the vertices of this simplex, is a 2-sphere of radius $t = \sqrt{\frac{5}{8}}$, centered at the center O_1 of Δ_1; let $S^2_{(O_1,t)}$ denote this 2-sphere.
Let Δ_2 be defined by:

$$\Delta_2 = \{f(A_4^*), f(B_3^*), f(B_5^*), f(B_6^*)\}.$$

In the similar way we obtain the 2-spheres $S^2_{(O_2,t)}$, having her center at O_2, which is also the center of Δ_2.
The four points $f(A_1^*), f(A_2^*), f(B_1^*)$ and $f(B_2^*)$ are in the intersection of the two 2-spheres $S^2_{(O_j,t)}$, $j = 1,2$.
By claim 3, the two simplices Δ_1, and Δ_2 are different, but they have vertices $f(B_5^*), and\ f(B_6^*)$ in common.
We will prove that $O_1 \neq O_2$:
Assume that $O_1 = O_2 = O$. (See figure 6)

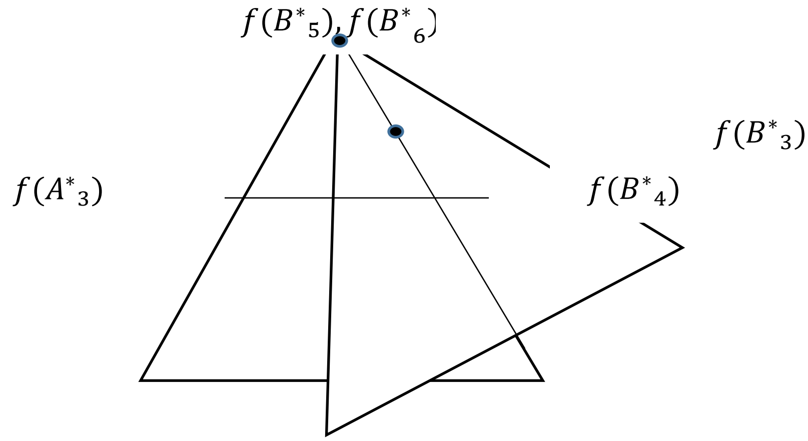

Figure 6

It follows that $\|f(B_j^*) - O\| = \|f(A_i^*) - O\|$=t, i=3, 4, and j=3,4, 5, 6.
In particular, the point O the center of the simplex $\{f(B_3^*), f(B_4^*), f(B_5^*), f(B_6^*)\}$, so
O = $\frac{1}{4}(f(B_3^*) + f(B_4^*), + f(B_5^*) + f(B_6^*))$, but point O is also the center of the simplex Δ_1 so O = $\frac{1}{4}(f(A_3^*) + f(A_4^*), + f(A_5^*) + f(A_6^*))$.
It follows that $f(A_3^*) = f(B_3^*)$, a contradiction to Claim 3, thus $O_1 \neq O_2$.
Therefore the 2- spheres $S^2_{(O_j,t)}$, $j = 1,2$, are different.
They have the same radius $t = \sqrt{\frac{5}{8}}$ and they have a non-empty intersection. It follows that there two 2-spheres intersect in a one-dimensional sphere, which is a circle.
Thus $f(A_1^*), f(A_2^*), f(B_1^*)$ and $f(B_2^*)$ form the vartex set of a quadrangle, of edge length one, that lies in a circle. (See figure 5).
It follows as the previous case that $f(A_1^*), f(A_2^*), f(B_1^*)$ and $f(B_2^*)$ form the vartex set of a square in a circle of diameter $\sqrt{2}$, implying:
$\| f(A_1^*) - f(B_1^*)\| = \|f(A_2^*) - f(B_2^*)\| = \sqrt{2}$ since $f(A_i^*) \neq f(B_i^*)$ for $i = 1,2$.
It follows by Lemma 1 that the mapping f preserves the distance $\sqrt{2}$ between A_i^* and B_i^* for all $i = 1,2,\dots,6$. In particular $\| f(Z) - f(W)\| = \sqrt{2}$.

This completes the proof of Theorem 2*.

Proof of Theorem 2
Let f be a unit distance preserving mapping $f: Q^6 \to Q^6$. By Theorem 2* the unit distance preserving mapping

f preserves the distance $\sqrt{2}$.
Our result follows by using a Theorem of J.Zaks [8], which states that if a mapping $g: Q^d \rightarrow Q^d$ preserves the distance 1 and $\sqrt{2}$, then g is an isometry, provided $d \geq 5$.
The proof that dim(*aff(L[6])*) = 6 is similar to the proof that dim(*aff(L[5])*) = 5 that appeared in of Theorem 1, hence it is omitted.

This completes the proof of Theorem 2.

Mapping of Q^d to Q^d that preserve distance 1

The purpose of this section is to prove the following Theorem:
Theorem 3:
For all the dimensions, $d, d \geq 5$, every unit- distance preserving mapping $f: Q^d \rightarrow Q^d$ is an isometry.
To prove Theorem 3, we prove first the following Theorem in which $L[d]$ and quadruples are defined in a way, similar to the one that appeared in the proof of Theorem 1* in page 11.
Theorem 3*:
For every value of $d, d \geq 5$ if $g: L[d] \rightarrow R^d$ is a mapping that preserves unit distances, for which $g(x) \neq g(y)$ holds for any two points x, y such that $\|x - y\| = \sqrt{2}$, then the following holds:
a. For every quadruple T of $L[d]$, g(T) is the vertex set of a planar unit square.
b. dim(*aff(g(L[d]))*) = d
Proof of Theorem 3*:
It is clear that Theorem 3* holds for $d = 5$ and $d = 6$ from Theorems1 and 2.
Suppose, inductively on d, that the assertion holds for d and for $d + 1, d \geq 5$, and let $f: L[d + 2] \rightarrow R^{d+2}$ be any unit- preserving mapping such that $f(x) \neq f(y)$ for any two points x, y of $L[d + 2]$ satisfying $\|x - y\| = \sqrt{2}$.
Let $T = L_{ij}[d + 2]$ be any quadruple in $L[d + 2]$, which we may assume, without loss of generality, that it is the quadruple:
$T = L_{d+1,d+2}[d + 2] = \{(0, \ldots ,0, \pm 1/2, \pm 1,2) \subset R^{d+2}\}$.
By assumption we know that $f(x) \neq f(y)$ for any two points x, y such that $\|x - y\| = \sqrt{2}$, (in particular for any two points x, y such that $\|x - y\| = \sqrt{2}$ in the quadruple T).
Consider the subset $K[d + 2]$ of $L[d + 2]$, consisting of all the points of $L[d + 2]$ in which the last two coordinates vanish. Notice that the set $K[d + 2]$ is, of course, congruent to the last set L[d].

<u>To show that $f(T)$ has affine dimension 2:</u>
Assume, for contradiction, that dim(aff(f(T)))$\geq$ 3.
We restrict our attention to the set $f(T \cup K[d + 2])$.
The image $f(K[d + 2])$ lies in the intersection of the unit spheres centered at the points of $f(T)$, and since dim(*aff(f(T))*)$\geq$ 3 it follows that the dimension of the intersection of these four $(d + 1)$-spheres is at most $d - 2$, and it lies in an affine flat, say F, of dimension at most $d - 1$.
Let $h: F \rightarrow R^d$ be an isometric embedding, and consider the composition $h^\circ f: K[d + 2] \rightarrow R^d$. By an inductive assumption on the dimension d,
dim(aff($h_0 f$: $(K[d + 2])$))= dim(*aff*($h^\circ f(L[d])$))= d.
This is a contradiction, since $f(K[d + 2])$ lies in the affine flat F which is of dimension at most $d - 1$.
<u>To show that dim(aff($f(L[d + 2])$)) $= d + 2$:</u>
It follows by part (a) that dim(*aff*($f(T)$)) $= 2$, and $f(T)$ forms the vartex set of some planar unit square.
Assume, by contradiction that dim(*aff*($f(L[d + 2])$)) $\leq d + 1$, and consider the effect of the mapping f on the set $T \cup K[d + 2]$; as in the previous case, all the points of $K[d + 2]$ are at unit distance from all those of T, therefore all the points of $f(K[d + 2])$ are at unit distance from all the points of $f(T)$, hence the affine hull of $f(K[d + 2])$ is orthogonal to the affine hull of $f(T)$, thus:
$d + 1 \geq$ dim $(aff(f((L[d + 2]))))$ $\geq$ dim $(aff(f(T)))$ + dim(*aff*($f(K[d + 2])$))=
= 2 + dim(*aff*(f)$L[d]$)) = d+2, which is a contradiction.
It follows that dim(aff($f(L[d + 2])$)) $= d + 2$.

This completes the proof of Theorem 3*.

Proof of Theorem 3:
We will prove first the following Claim:
Claim 5:
Every unit-distance preserving mapping $f: Q^d \to Q^d$ preserves the distance $\sqrt{2}$, for all $d \geq 5$.
Proof of Claim 5:
Let $d \geq 5$ and let $f: Q^d \to Q^d$ be a unit distance-preserving mapping.
By Corollary 3 it follows that $f(x) \neq f(y)$ holds for every two points x and y in Q^d, for which $\|x - y\| = \sqrt{2}$.
Let $i: Q^d \to R^d$ be the natural inclusion isometry, and consider the combined mapping $i^\circ f:: Q^d \to R^d$.
By Theorem 3*, the distance $\sqrt{2}$ of opposite vertices in T preserved by $i^\circ f$, hence it is preserved by f.
It follows by Lemma 1 that for every pair of points x and y, if $\|x - y\| = \sqrt{2}$, then
$\|\mathrm{f}(x) - f(y)\| = \sqrt{2}$, i.e, the mapping f preserves the distance $\sqrt{2}$.
Let d be an integer,$d \geq 5$, and let f be a unit distance preserving mapping $f: Q^d \to Q^d$. By Claim 5 the unit distance preserving mapping f preserves the distance $\sqrt{2}$. Our result follows by using a Theorem of J.Zaks [11] which states that if a mapping g: $Q^d \to Q^d$ preserves the distances 1 and $\sqrt{2}$, then g is an isometry, provided$d \geq 5$.

This completes the proof of Theorem 3.

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