Study of Certain Classes of Multivalent Functions Associated With Integral Operators

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Abstract: In the given article we studied the class Υ^{+} $(\propto, \beta, H, V, \gamma, \sigma)$ of multivalent functions with integral operator. We obtained necessary and sufficient condition for the functions in the class. Several geometric properties like coefficient bounds, closureness property, and integral mean are part of discussion. Extensions of the given class with several results are also pointed out.

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1. Introduction.

Multivalent function theory is unique branch of geometric function theory. The special class of this topic is B (p).It consist of multivalent functions in the form,

$$f(z) = z^p + \sum_{n=1}^{\infty} a_n z^{p+n}, \quad p \in N. \qquad (1)$$

Which are analytic in unit disc D= {z: |z|<1}.B^{+} (p) be the subclass of B (p) consist of all functions in the form (1.1) with positive coefficients.

$$g(z) = z^p + \sum_{n=1}^{\infty} b_n z^{p+n}, \qquad (b_n \geq 0, \ p \in N) \qquad (2)$$

[1-5] and many other studied the classes of multivalent convex, starlike, close to convex functions of order $\partial \geq 0$. Hadmad [8] introduced concept of (convolution) hadmad product.

Definition 1.6.*If* f (z) = $z^p + \sum_{n=1}^{\infty} a_n z^{p+n}$, $g(z) = z^p + \sum_{n=1}^{\infty} b_n z^{p+n}$ is an analytic function on unit disk D.The hadmad product is denoted as f* g.It is defined as follows

$$(f*g)(z) = z^p + \sum_{n=1}^{\infty} a_n b_n z^{p+n}. \qquad (3)$$

This product is strong tool in development of geometric function theory. It has been shown that many subclasses of univalent and multivalent functions are remains invariant under convolution. Several authors like Bhousty [3], Ruschweyh [13], Robertson [10], and Thange [14] studied this product. We generalized this product by introducing t th hadmad product. It is defined as follows.

Definition 1.7*.For $t \in \mathbb{R}$,* $f(z) = z^p + \sum_{n=1}^{\infty} a_n z^{p+n}$, $g(z) = z^p + \sum_{n=1}^{\infty} b_n z^{p+n}$ in B(p)*, t^{th} hadmad product (Convolution) is denoted by $*^t$.It is defined as $(f *^t g)$=* $z^p + \sum_{n=1}^{\infty} (a_n . b_n)^t z^{p+n}$. (4)

The 1 th hadmad product is equivalent to hadmad product.

[2][11][7][14] used following definition. It is utilized in subordination techniques to obtained geometric property of classes of univalent and multivalent function

Definition 1.3. Let f and g analytic in unit disc D, we say that the function f (z) is subordinate to g (z) in D and write f (z) $\prec$ g (z) (z ∈D) (5)

If there exit Schwarz function w (z), analytical in U with w (0) = 0 and $|w(z)| < 1$ such that f (z) = g (w (z)) (z ∈ D) (6)

[7] has introduced following lemma known as Littlewoods Subordination lemma.[1][7] [11] has used this lemma to prove integral mean and subordination results for classes of univalent and analytic functions.

Lemma1.1. Let f and g analytic in unit disc and suppose g $\prec$ f, then for $0 < t < \infty$

$$\int_0^{2\pi} |g(re^{i\emptyset})|^t d\theta \le \int_0^{2\pi} |f(re^{i\emptyset})|^t d\theta \quad (0\le r < 1,\ t > 0) \qquad (7)$$

Strict equality holds for $0 \le r < 1$ unless f is constant or w (z) = α z, $|\alpha| = 1$.

[16] has found the application of integral operator $I^\sigma g(z) = \frac{1}{z^2\tau(\sigma)}\int_0^z (log\frac{z}{t})^{\sigma-1}\ tg(t)dt$ for univalent meromorphic function g (z).Associated with this operator he defined subclass of univalent meromorphic function f(z) on unit disc satisfying $\left|\frac{[(V-H)\gamma+H][z^2(I^\sigma f(z))'+\beta z I^\sigma f(z)+(1-\beta)}{[z^2(I^\sigma f(z))'+\beta z I^\sigma f(z)][(V-H)\gamma+H]+(1-\beta)[(V-H)\gamma+H+1]}\right| < \propto$ *(8)*

Analogous to same operator we define new integral operator as defined bellow,

Definition 1.2.The integral operator for f ∈ B (p) is denoted by $I^{\sigma,r,s}$ and defined as follows

$$I^{\sigma,r,s}f(z) = \frac{z^r}{\tau(\sigma)}\int_0^z (log\frac{z}{t})^{\sigma-1} t^s f(t)dt \quad \text{where } \sigma \ge 0, s, r \in \mathbb{Z}. \quad (9)$$

[16] has used operator $I^{\sigma,-2,1}$ for meromorphic function.

Example1.3. Show that for f (z) ∈ $B(p)$, $I^{(\sigma,p-1,-p)}f(z) = z^p + \sum_{n=1}^{\infty}\frac{1}{(n+1)^\sigma}a_n z^{p+n}$ p∈ ℕ.

By definition, $I^{(\sigma,p-1,-p)}f(z) = \frac{z^{p-1}}{\tau(\sigma)}\int_0^z (log\frac{z}{t})^{\sigma-1}\frac{1}{t^p}f(t)dt$

If $f(z) = z^p + \sum_{n=1}^{\infty} a_n z^{p+n}$

$$I^{(\sigma,p-1,-p)}f(z) = \frac{z^{p-1}}{\tau(\sigma)}\int_0^z (log\frac{z}{t})^{\sigma-1}\frac{1}{t^p}[\,t^p + \sum_{n=1}^{\infty} a_n t^{p+n}]dt\ .$$

$$= \frac{z^{p-1}}{\tau(\sigma)}\int_0^z (log\frac{z}{t})^{\sigma-1}[1 + \sum_{n=1}^{\infty} a_n t^n]dt.$$

$$= \frac{z^{p-1}}{\tau(\sigma)}\int_0^z (log\frac{z}{t})^{\sigma-1}dt + \sum_{n=1}^{\infty}\frac{z^{p-1}}{\tau(\sigma)}a_n\int_0^z (log\frac{z}{t})^{\sigma-1}t^n dt.$$

Put $\log(\frac{z}{t}) = x$

$$I^{(\sigma,p-1,-p)}f(z) = \frac{z^{p-1}}{\tau(\sigma)}\int_0^\infty z x^{\sigma-1}e^{-x}dx + \sum_{n=1}^{\infty}\frac{z^{p+n}}{\tau(\sigma)}a_n\int_0^\infty x^{\sigma-1}e^{-(n+1)}dx$$

$$= z^p + \sum_{n=1}^{\infty}\frac{1}{(n+1)^\sigma}a_n z^{p+n}$$

2. Class Y⁺ $(\propto,\boldsymbol{\beta},\boldsymbol{H},\boldsymbol{V},\boldsymbol{\gamma},\boldsymbol{\sigma})$, Y $(\propto,\boldsymbol{\beta},\boldsymbol{H},\boldsymbol{V},\boldsymbol{\gamma},\boldsymbol{\sigma})$.

In this section we introduced new classes $Y^+(\propto,\beta,H,V,\gamma,\sigma)$ of multivalent functions.We examined various geometric properties of this classes.We start with *the necessary condition for the class* Y $(\propto,\beta,H,V,\gamma,\sigma)$.

Definition 2.1. A function f (z) in B (p) is said to be in the class Y $(\propto, \beta, H, V, \gamma, \sigma)$ if it satisfies the condition

$$\left|\frac{[(V-H)\gamma+H][\frac{1}{z^{p-1}}\left(I^{(\sigma,p-1,-p)}\mathrm{f}(z)\right)'+\frac{\beta}{z^p}I^{(\sigma,p-1,-p)}\mathrm{f}(z)-(\beta+p)}{\left[\frac{1}{z^{p-1}}\left(I^{(\sigma,p-1,-p)}\mathrm{f}(z)\right)'+\frac{\beta}{z^p}I^{(\sigma,p-1,-p)}\mathrm{f}(z)\right][(V-H)\gamma+H]-(\beta+p)[(V-H)\gamma+H+1]}\right|<\propto \qquad (10)$$

$0\leq \beta<1,\ 0<\propto\leq 1, -1\leq H<V\leq 1, \frac{H}{H-V}< \ \leq 1, \sigma>0$

We write the class Y⁺ $(\propto, \beta, A, B, \gamma, \sigma)$ as bellow,

$$Y^+(\propto, \beta, H, V, \gamma, \sigma) = B^+(p) \cap Y(\propto, \beta, H, V, \gamma, \sigma) \qquad (11)$$

Example 2.2. If $f_3(z) = z^p + \frac{1}{(p+1+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{2}\right)^{\sigma}} z^{p+1} + \frac{\propto(\beta+p)-1}{(p+1+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{2}\right)^{\sigma}} z^{p+2}$

With $\propto(\beta+p)>1$.Then $f_3(z) \in Y^+(\propto, \beta, H, V, \gamma, \sigma)$.

Putting $a_1 = \frac{1}{(p+1+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{2}\right)^{\sigma}}$ $a_2 = \frac{\propto(\beta+p)-1}{(p+1+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{2}\right)^{\sigma}}$, $a_n = 0$ for $n > 2$.

Then, $\sum_{n=1}^{\infty}(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma} a_n = \propto(\beta+p)$

Hence $f_3(z) \in Y^+(\propto, \beta, H, V, \gamma, \sigma)$.

Theorem 2.3. If f (z) ∈ $B(p)$, satisfies

$$\sum_{n=1}^{\infty}(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}|a_n| \leq \propto(\beta+p) \qquad (12)$$

Then f (z) ∈ Y $(\propto, \beta, A, B, \gamma, \sigma)$.

Proof: Suppose f (z) satisfies condition (8)

$$I^{(\sigma,p-1,-p)}\mathrm{f}(z) = z^p + \sum_{n=1}^{\infty}\frac{1}{(n+1)^{\sigma}}a_n z^{p+n}$$

$$D\left(I^{(\sigma,p-1,-p)}\mathrm{f}(z)\right) = [(V-H)\gamma+A]\left[\frac{1}{z^{p-1}}\left(I^{(\sigma,p-1,-p)}\mathrm{f}(z)\right)' + \frac{\beta}{z^p}I^{(\sigma,p-1,-p)}\mathrm{f}(z) - (\beta+p)\right]$$

$$= \sum_{n=1}^{\infty}(p+n+\beta)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}|a_n|z^n.$$

$$E\left(I^{(\sigma,p-1,-p)}\mathrm{f}(z)\right) = [(V-H)\gamma+H]\left[\frac{1}{z^{p-1}}\left(I^{(\sigma,p-1,-p)}\mathrm{f}(z)\right)' + \frac{\beta}{z^p}I^{(\sigma,p-1,-p)}\mathrm{f}(z)\right] - (\beta+p)[(V-H)\gamma+H+1]$$

$$= -(\beta+p) + \sum_{n=1}^{\infty}(p+n+\beta)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}a_n z^n.$$

Now to show result we claim that $\left|\frac{D\left(I^{(\sigma,p-1,-p)}\mathrm{f}(z)\right)}{E\left(I^{(\sigma,p-1,-p)}\mathrm{f}(z)\right)}\right| < \propto$

Given that $\sum_{n=1}^{\infty}(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}|a_n| \leq \propto(\beta+p)$

Hence,

$$\sum_{n=1}^{\infty}(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}|a_n||z^n| \leq \propto(\beta+p)$$

$\sum_{n=1}^{\infty}(p+n+\beta)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}|a_n||z^n| \leq -\propto \sum_{n=1}^{\infty}(p+n+\beta)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}|a_n||z^n| - \propto (\beta+p)$.

$$\frac{\sum_{n=1}^{\infty}(p+n+\beta)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}|a_n||z^n|}{(\beta+p)-\sum_{n=1}^{\infty}(p+n+\beta)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}|a_n||z^n|} \leq \propto$$

$$\left|\frac{\mathrm{D}\left(I^{(\sigma,p-1,-p)}\mathrm{f(z)}\right)}{\mathrm{E}\left(I^{(\sigma,p-1,-p)}\mathrm{f(z)}\right)}\right| = \frac{|\sum_{n=1}^{\infty}(p+n+\beta)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}a_n z^n|}{|(\beta+p)-\sum_{n=1}^{\infty}(p+n+\beta)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}a_n z^n|}$$

$$\leq \frac{\sum_{n=1}^{\infty}(p+n+\beta)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}|a_n||z^n|}{(\beta+p)-\sum_{n=1}^{\infty}(p+n+\beta)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}|a_n||z^n|}$$

$$< \propto.$$

$\therefore$ f (z) $\in$ Y$(\propto,\beta,H,V,\gamma,\sigma)$.

Theorem 2.4. f (z) ∈ $\mathrm{Y}^{+}(\propto,\beta,H,V,\gamma,\sigma)$ iff

$$\sum_{n=1}^{\infty}(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}a_n \leq \propto (\beta+p) \qquad (13)$$

Proof: Assume inequality (9) holds. Therefore by theorem (8) f (z) ∈ $\mathrm{Y}^{+}(\propto,\beta,H,V,\gamma,\sigma)$.

Conversely suppose f (z) ∈ $\mathrm{Y}^{+}(\propto,\beta,H,V,\gamma,\sigma)$.

$$\left|\frac{[(V-H)\gamma+H][\frac{1}{z^{p-1}}\left(I^{(\sigma,p-1,-p)}\mathrm{f(z)}\right)'+\frac{\beta}{z^{p}}I^{(\sigma,p-1,-p)}\mathrm{f(z)}-(\beta+p)}{\left[\frac{1}{z^{p-1}}\left(I^{(\sigma,p-1,-p)}\mathrm{f(z)}\right)'+\frac{\beta}{z^{p}}I^{(\sigma,p-1,-p)}\mathrm{f(z)}\right][(V-H)\gamma+H]-(\beta+p)[[(V-H)\gamma+H]+1]}\right| < \propto$$

$$\left|\frac{\sum_{n=1}^{\infty}(p+n+\beta)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}a_n z^n}{(\beta+p)-\sum_{n=1}^{\infty}(p+n+\beta)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}a_n z^n}\right| < \propto$$

Now Re {z} $\leq |z|$

$$Re\left\{\frac{\sum_{n=1}^{\infty}(p+n+\beta)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}a_n z^n}{(\beta+p)-\sum_{n=1}^{\infty}(p+n+\beta)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}a_n z^n}\right\} < \propto$$

Allowing $z \to 1^{-}$ through positive real axis.

$$\frac{\sum_{n=1}^{\infty}(p+n+\beta)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}a_n}{(\beta+p)-\sum_{n=1}^{\infty}(p+n+\beta)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}a_n} < \propto$$

$$\sum_{n=1}^{\infty}(p+n+\beta)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}a_n <$$

$$\propto (\beta+p) - \propto \sum_{n=1}^{\infty}(p+n+\beta)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}a_n$$

$$\sum_{n=1}^{\infty}(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}a_n \leq \propto (\beta+p)$$

Hence f (z) ∈ $\mathrm{Y}^{+}(\propto,\beta,H,V,\gamma,\sigma)$

In next corollary we find coefficient bounds for functions in $\mathrm{Y}^{+}(\propto,\beta,H,V,\gamma,\sigma)$

Corollary 2.5. $f(z)= z^p+ \sum_{n=1}^{\infty} a_n z^{p+n} \in Y^+(\propto,\beta,H,V,\gamma,\sigma)$ Then

$$a_n \le \frac{\propto(\beta+p)}{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}}$$

Equality occurs for the function,

$$f_n(z) = z^p + \frac{\propto(\beta+p)}{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}}\ z^{p+n}.$$

Proof. Given that, $f(z)= z^p+ \sum_{n=1}^{\infty} a_n z^{p+n} \in\ Y+(\propto,\beta,H,V,\gamma,\sigma)$.

$$\sum_{n=1}^{\infty}(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma} a_n \le \propto(\beta+p)$$

$$\therefore (p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma} a_n \le \propto(\beta+p)$$

$$\therefore a_n \le \frac{\propto(\beta+p)}{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}}$$

With equality occurs for the function,

$$f_n(z) = z^p + \frac{\propto(\beta+p)}{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}}\ z^{p+n}.$$

Further, we prove closurness property for class $Y^+(\propto,\beta,H,V,\gamma,\sigma)$

Theorem 2.6.The class $Y^+(\propto,\beta,H,V,\gamma,\sigma)$ is closed under convex combination.

Proof. Let $f_i(z) = z^p + \sum_{n=1}^{\infty} a_{n,i} z^{p+n}$ $\quad 1 \le i \le t$

is in $Y^+(\propto,\beta,H,V,\gamma,\sigma)$

$$\therefore \sum_{n=1}^{\infty}(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma} a_{n,i} \le \propto(\beta+p).$$

Let $G(z) =\sum_{i=1}^{t} \lambda_i f_i(z)$, where $\sum_{i=1}^{t} \lambda_i=1$.

$$= \sum_{i=1}^{t} \lambda_i \left(z^p +\sum_{n=1}^{\infty} a_{n,i} z^{p+n}\right).$$

$$= z^p + \sum_{i=1}^{t} \lambda_i \left(\sum_{n=1}^{\infty} a_{n,i} z^{p+n}\right)$$

$$= z^p + \sum_{n=1}^{\infty}. \left(\sum_{i=1}^{t} \lambda_i a_{n,i}\right) z^{p+n}.$$

$$= z^p + \sum_{n=1}^{\infty}. T_n z^{p+n} \qquad \text{where } T_n = \sum_{i=1}^{t} \lambda_i a_{n,i}$$

$$\sum_{n=1}^{\infty}(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma} T_n$$

$$= \sum_{n=1}^{\infty}(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma} \sum_{i=1}^{t} \lambda_i a_{n,i}.$$

$$= \sum_{i=1}^{t} \lambda_i\left(\sum_{n=1}^{\infty}(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma} a_{n,i}\right)$$

$$\le \sum_{i=1}^{t} \lambda_i \propto(\beta+p)$$

$$\le \propto(\beta+p)$$

Therefore G (z) ∈ Y⁺ $(\propto, \beta, H, V, \gamma, \sigma)$

Theorem 2.7. The function of the form f (z) = $z^p + \sum_{n=1}^{\infty} a_n z^{p+n}$ is in Y⁺ $(\propto, \beta, H, V, \gamma, \sigma)$ and

g (z)= $z^p + \sum_{n=1}^{\infty} \frac{a_n}{n^\sigma} z^{p+n}$ then,

$$\sum_{n=1}^{\infty} \frac{a_n}{n^\sigma} \leq \frac{2^\sigma \propto(\beta+p)}{(p+1+\beta)(1+\propto)[(V-H)\gamma+H]}$$

Proof. f (z)= $z^p + \sum_{n=1}^{\infty} a_n z^{p+n}$ is in Y⁺$(\propto, \beta, H, V, \gamma, \sigma)$

$$\sum_{n=1}^{\infty}(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^\sigma a_n \leq \propto(\beta+p)$$

$$\left(\frac{1}{2n}\right)^\sigma \leq \left(\frac{1}{n+1}\right)^\sigma$$

$$(p+1+\beta)(1+\propto)[(V-H)\gamma+H]\sum_{n=1}^{\infty}\left(\frac{1}{2n}\right)^\sigma a_n. \leq \sum_{n=1}^{\infty}(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^\sigma a_n \leq \propto(\beta+p).$$

$$\sum_{n=1}^{\infty} \frac{a_n}{n^\sigma}. \leq \frac{2^\sigma \propto(\beta+p)}{(p+1+\beta)(1+\propto)[(V-H)\gamma+H]}$$

Theorem 2.8. If f (z) = $z^p + \sum_{n=1}^{\infty} a_n z^{p+n}$ ∈ Y⁺$(\propto, \beta, H, V, \gamma, \sigma)$ and g(z)= $z^p + \sum_{n=1}^{\infty} \frac{a_n}{n^\sigma} z^{p+n}$

$$r - \frac{2^\sigma \propto(\beta+p)}{(p+1+\beta)(1+\propto)[(V-H)\gamma+H]} r \leq |g(z)| \leq r + \frac{2^\sigma \propto(\beta+p)}{(p+1+\beta)(1+\propto)[[(V-H)\gamma+H]]} r.$$

Proof. Given that f (z) = $z^p + \sum_{n=1}^{\infty} a_n z^{p+n}$ ∈ Y⁺$(\propto, \beta, H, V, \gamma, \sigma)$

Therefore by theorem (2.7) $\sum_{n=1}^{\infty} \frac{a_n}{n^\sigma}. \leq \frac{2^\sigma \propto(\beta+p)}{(p+1+\beta)(1+\propto)[(V-H)\gamma+H]}$

$$|g(z)| = \left| z^p + \sum_{n=1}^{\infty} \frac{a_n}{n^\sigma} z^{p+n} \right|$$

$$\leq |z^p| + \sum_{n=1}^{\infty} \frac{a_n}{n^\sigma} |z^{p+n}|$$

$$\leq r + \sum_{n=1}^{\infty} \frac{a_n}{n^\sigma} r.$$

$$\leq r + \frac{2^\sigma \propto(\beta+p)}{(p+1+\beta)(1+\propto)[(V-H)\gamma+H]} r$$

Similarly, $|g(z)| = \left| z^p + \sum_{n=1}^{\infty} \frac{a_n}{n^\sigma} z^{p+n} \right| \geq r - \frac{2^\sigma \propto(\beta+p)}{(p+1+\beta)(1+\propto)[(V-H)\gamma+H]} r.$

In next theorem we will discuss the extreme points of the class Y⁺ $(\propto, \beta, H, V, \gamma, \sigma)$

Theorem 2.9.The extreme points of the class Y⁺ $(\propto, \beta, H, V, \gamma, \sigma)$ are f_n (z) where,

$$f_0(z) = z^p, \quad f_n(z) = z^p + \frac{\propto(\beta+p)}{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^\sigma} z^{p+n} \quad n \in \mathbb{N}.$$

Proof. Suppose f (z) = $z^p + \sum_{n=1}^{\infty} a_n z^{p+n}$ ∈ Y⁺ $(\propto, \beta, H, V, \gamma, \sigma)$

$$\therefore \sum_{n=1}^{\infty}(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^\sigma a_n \leq \propto(\beta+p).$$

$\therefore \sum_{n=1}^{\infty} \frac{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}}{\propto(\beta+p)} a_n \leq 1.$

Put $\sigma_n = \frac{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}}{\propto(\beta+p)} a_n$

$0 \leq \sum_{n=1}^{\infty} \sigma_n \leq 1$

$\therefore 0 \leq 1- \sum_{n=1}^{\infty} \sigma_n \leq 1.$

Put $\sigma_0 = 1- \sum_{n=1}^{\infty} \sigma_n$

$f(z) = z^p + \sum_{n=1}^{\infty} a_n z^{p+n}$

$= z^p + \sum_{n=1}^{\infty} \frac{\propto(\beta+p)}{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}} \sigma_n z^{p+n}$

$= z^p + \sum_{n=1}^{\infty}(f_n(z) - z^p)\sigma_n$

$= z^p + \sum_{n=1}^{\infty}(f_n(z)\sigma_n - \sum_{n=1}^{\infty} z^p \sigma_n.$

$= z^p (1- \sum_{n=1}^{\infty} \sigma_n) + \sum_{n=1}^{\infty}(f_n(z)\sigma_n$

$= \sigma_0 f_0 (z) + \sum_{n=1}^{\infty}(f_n(z)\sigma_n$

$= \sum_{n=0}^{\infty} f_n(z)\sigma_n.$

Hence $f_n(z)$ is extreme point.

Conversely assume $f(z) = \sum_{n=0}^{\infty} f_n(z)\sigma_n$ where $\sum_{n=0}^{\infty} f_n = 1.$

$= f_0(z)\sigma_0 + \sum_{n=1}^{\infty} f_n(z)\sigma_n$

$= f_0(z)[1 - \sum_{n=1}^{\infty} \sigma_n] + \sum_{n=1}^{\infty} \sigma_n \left(z^p + \frac{\propto(\beta+p)}{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}} z^{p+n}\right)$

$= z^p + \sum_{n=1}^{\infty} T_n z^{p+n}$ where $T_n = \frac{\propto(\beta+p)}{(p+n+\beta)(1+\propto)[(B-A)\gamma+A]\left(\frac{1}{n+1}\right)^{\sigma}} \sigma_n$

$= \sum_{n=1}^{\infty} \frac{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma} T_n}{\propto(\beta+p)}$

$= \sum_{n=1}^{\infty} \left(\frac{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}}{\propto(\beta+p)}\right)\left(\frac{\propto(\beta+p)}{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}} \sigma_n\right)$

$= \sum_{n=1}^{\infty} \sigma_n$

$= 1- \sigma_0$

$\leq 1.$

$\therefore \sum_{n=1}^{\infty}(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma} T_n \leq \propto (\beta+p)$

$\therefore f(z) \in Y^+ (\propto, \beta, H, V, \gamma, \sigma).$

Next we proved that under certain condition, $Y^+ (\propto, \beta, H, V, \gamma, \sigma)$ is closed under convolution.

Theorem 2.10.If $f, g \in Y^+(\propto, \beta, H, V, \gamma, \sigma)$

Where, f (z) = $z^p + \sum_{n=1}^{\infty} a_n z^{p+n}$ *and* , g(z) $= z^p + \sum_{n=1}^{\infty} b_n z^{p+n}$

Then $f *^{\frac{1}{2}} g \in WAG^+(\propto, \beta, A, B, \gamma, \sigma)$.

Moreover if $\sqrt{a_k b_k} < 1$, with $a_{k,} b_k > 0$, then f* g $\in Y^+(\propto, \beta, H, V, \gamma, \sigma)$

Proof. Given that $f, g \in Y^+(\propto, \beta, H, V, \gamma, \sigma)$

$$\therefore \sum_{n=1}^{\infty}(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma} a_n \le \propto(\beta+p)$$

$$\sum_{n=1}^{\infty}(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma} b_n \le \propto(\beta+p)$$

We know that if $\sum_{n=1}^{\infty} T_n^2$ and $\sum_{n=1}^{\infty} L_n^2$ is convergent then,

$$\sum_{n=1}^{\infty} T_n L_n \le \left(\sum_{n=1}^{\infty} T_n^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} L_n^2\right)^{\frac{1}{2}}$$

Put $T_n = (t_n a_n)^{\frac{1}{2}}$ and $L_n = (t_n b_n)^{\frac{1}{2}}$, where $t_n = (p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}$

Hence $\sum_{n=1}^{\infty}(t_n a_n t_n b_n)^{\frac{1}{2}}. \le \left(\sum_{n=1}^{\infty} t_n a_n\right)^{\frac{1}{2}}. \left(\sum_{n=1}^{\infty} t_n b_n\right)^{\frac{1}{2}} \le \propto(\beta+p)$.

$$\therefore \sum_{n=1}^{\infty}(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma} (a_n b_n)^{\frac{1}{2}} \le \propto(\beta+p)$$

$\therefore f *^{\frac{1}{2}} g \in Y^+(\propto, \beta, H, V, \gamma, \sigma)$

By assumption $\sqrt{a_n b_n} < 1 \Rightarrow a_n b_n \le \sqrt{a_n b_n} < 1$

$$\therefore \sum_{n=1}^{\infty}(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma} a_n b_n \le \propto(\beta+p)$$

$\therefore$ f* g $\in Y^+(\propto, \beta, H, V, \gamma, \sigma)$

Theorem 2.11. Let f (z) = $z^p + \sum_{n=1}^{\infty} a_n z^{p+n}$ is a function in $Y^+(\propto, \beta, H, V, \gamma, \sigma)$

Consider the function $f_j(z) = z^p + \frac{\propto(\beta+p)}{(p+j+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{j+1}\right)^{\sigma}}$.

If analytic function w (z) is given by

$$(w(z))^{j-1} = \frac{(p+j+\beta)(1+\propto)[(V-H)\gamma+H]}{\propto(\beta+p)} \sum_{n=1}^{\infty}\left(\frac{1}{n+1}\right)^{\sigma} a_n z^n.$$

Then for $z = re^{i\theta}$, $0 < r < 1$.

$$\int_0^{2\pi} |I^{(\sigma,p-1,-p)} f(z)|^p d\theta \le \int_0^{2\pi} |I^{(\sigma,p-1,-p)} f_j(z)|^p d\theta.$$

Where $I^{(\sigma,p-1,-p)}$ is differential operator defined in (4)

Proof. For f (z) = $z^p + \sum_{n=1}^{\infty} a_n z^{p+n}$ *in* $Y^+(\propto, \beta, H, V, \gamma, \sigma)$

$$\therefore (p+j+\beta)(1+\propto)[(V-H)\gamma+H]\sum_{n=1}^{\infty}\left(\frac{1}{n+1}\right)^{\sigma} a_n$$

$$\leq \sum_{n=1}^{\infty}(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma} a_n \leq \propto(\beta+p)$$

$$\therefore \sum_{n=1}^{\infty}\left(\frac{1}{n+1}\right)^{\sigma} a_n \leq \frac{(p+j+\beta)(1+\propto)}{\propto(\beta+p)[(V-H)\gamma+H]}$$

$$I^{(\sigma,p-1,-p)}\text{f (z)} = z^p + \sum_{n=1}^{\infty}\frac{1}{(n+1)^{\sigma}} a_n z^{p+n}.$$

We set $\left|(w(z))^{j-1}\right| = \left|\frac{(p+j+\beta)(1+\propto)[(V-H)\gamma+H]}{\propto(\beta+p)}\sum_{n=1}^{\infty}\left(\frac{1}{n+1}\right)^{\sigma} a_n z^n\right|$

$$\leq |z|\left|\frac{(p+j+\beta)(1+\propto)}{\propto(\beta+p)}\sum_{n=1}^{\infty}\left(\frac{1}{n+1}\right)^{\sigma} a_n\right|$$

$$< \frac{(p+j+\beta)(1+\propto)[(V-H)\gamma+H]}{\propto(\beta+p)}\frac{\propto(\beta+p)}{(p+j+\beta)(1+\propto)[[(V-H)\gamma+H]]}$$

$$< |z|$$

$$< 1.$$

$\therefore |w(z)| < 1$

$$\therefore 1+\frac{\propto(\beta+p)}{(p+j+\beta)(1+\propto)[(B-A)\gamma+A]}(w(z))^{j-1} = 1+\sum_{n=1}^{\infty}\left(\frac{1}{n+1}\right)^{\sigma} a_n z^n$$

$$\therefore 1+\sum_{n=1}^{\infty}\left(\frac{1}{n+1}\right)^{\sigma} a_n z^n \prec 1+\frac{\propto(\beta+p)}{(p+j+\beta)(1+\propto)[(B-A)\gamma+A]}(w(z))^{j-1}.$$

∴ By Littlewood Subordintion Theorem,

$$\int_0^{2\pi}|1+\sum_{n=1}^{\infty}\left(\frac{1}{n+1}\right)^{\sigma} a_n z^n|^p d\theta \leq \int_0^{2\pi} 1+\frac{\propto(\beta+p)}{(p+j+\beta)(1+\propto)[(B-A)\gamma+A]} z^{j-1}|^p d\theta.$$

$$\therefore \int_0^{2\pi}|I^{(\sigma,p-1,-p)}\text{f(z)}|^p d\theta \leq \int_0^{2\pi}|I^{(\sigma,p-1,-p)}f_j\text{(z)}|^p d\theta.$$

3. Class t-sqrt Υ^+ **($\propto, \boldsymbol{\beta}, \boldsymbol{H}, \boldsymbol{V}, \boldsymbol{\gamma}, \boldsymbol{\sigma}$)**

In this section we introduced another class t-sqrt Υ^+ $(\propto,\beta,H,V,\gamma,\sigma)$which is extension class of

Υ^+ $(\propto,\beta,H,V,\gamma,\sigma)$

Definition 3.1. A function f(z) in B(p) is said to be in the class t sqr- Υ^+ $(\propto,\beta,H,V,\gamma,\sigma)$if it satisfies the condition ,

$$\sum_{n=1}^{\infty}\frac{1}{t}\left(\frac{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{2}}{\propto(\beta+p)}\right)^{2} a_n{}^2 \leq 1 \qquad t\in\mathbb{N}. \quad (7)$$

$0\leq \beta<1$, $0 <\propto\leq 1$,$-1\leq A < B \leq 1$, $\frac{A}{A-B} < \ \leq 1$, $\sigma > 0$

Theorem 3.2. Υ^+ $(\propto,\beta,H,V,\gamma,\sigma) \subseteq$ t-sqrt Υ^+ $(\propto,\beta,H,V,\gamma,\sigma)$

Proof. Suppose f (z) ∈ Υ^+ $(\propto,\beta,H,V,\gamma,\sigma)$

$$\therefore \sum_{n=1}^{\infty}(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma} a_n \leq \propto(\beta+p)$$

$$\sum_{n=1}^{\infty}\frac{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma} a_n}{\propto(\beta+p)} \leq 1.$$

$\therefore\ \frac{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma} a_n}{\propto(\beta+p)} \leq 1.$

$\therefore\ \left(\frac{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma} a_n}{\propto(\beta+p)}\right)^2 \leq \frac{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma} a_n}{\propto(\beta+p)} \leq 1.$

For any t ≥ 1,

$\frac{1}{t}\left(\frac{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma} a_n}{\propto(\beta+p)}\right)^2 \leq .\frac{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma} a_n}{\propto(\beta+p)} \leq 1.$

$\sum_{n=1}^{\infty} \frac{1}{t}\left(\frac{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma} a_n}{\propto(\beta+p)}\right)^2 \leq 1$

Hence f (z) ∈ t-sqrt Y⁺ $(\propto, \beta, A, B, \gamma, \sigma)$

∴ Y⁺ $(\propto, \beta, A, B, \gamma, \sigma)$ ⊆ t-sqrt Y⁺ $(\propto, \beta, A, B, \gamma, \sigma)$

Theorem 3.3. If 1-sqrt Y⁺ $(\propto, \beta, H, V, \gamma, \sigma) is$ closed under $*^{\frac{1}{2}}$ convolution then 1-sqrt Y⁺ $(\propto, \beta, H, V, \gamma, \sigma) is\ \ closed\ under$ convex combination.

Proof. Given that 1-sqrt Y⁺ $(\propto, \beta, H, V, \gamma, \sigma)$ *is* closed under $*^{\frac{1}{2}}$ convolution.

∴ f, g ∈ 1-sqrt Y⁺ $(\propto, \beta, H, V, \gamma, \sigma)$ ⇒ f $*^{\frac{1}{2}}$ g ∈ 1-sqrt Y⁺ $(\propto, \beta, H, V, \gamma, \sigma)$

f (z)= $z^p + \sum_{n=1}^{\infty} a_n z^{p+n}$, g(z) $= z^p + \sum_{n=1}^{\infty} b_n z^{p+n}$

(f $*^{\frac{1}{2}}$ g)(z) = $z^p + \sum_{n=1}^{\infty} (a_n b_n)^{\frac{1}{2}} z^{p+n}$

$\therefore \sum_{n=1}^{\infty} \left(\frac{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}}{\propto(\beta+p)}\right)^2 a_n b_n \leq 1$

Now we will show that 1-sqrt Y⁺ $(\propto, \beta, H, V, \gamma, \sigma)$is closed under convex combination.

Now for f, g ∈ 1-sqrt Y⁺ $(\propto, \beta, H, V, \gamma, \sigma)$we have

$\sum_{n=1}^{\infty} \left(\frac{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}}{\propto(\beta+p)} a_n\right)^2 \leq 1$

$\sum_{n=1}^{\infty} \left(\frac{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}}{\propto(\beta+p)} b_n\right)^2 \leq 1.$

Let g (z) = (1- λ) f (z) + $\lambda g(z)$

= (1- λ) [$z^p + \sum_{n=1}^{\infty} a_n z^{p+n}] + \lambda[z^p + \sum_{n=1}^{\infty} a_n z^{p+n}]$

= $z^p + \sum_{n=1}^{\infty}[(1-\lambda)a_n + \lambda b_n] z^{p+n}$

$\sum_{n=1}^{\infty} \left(\frac{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}}{\propto(\beta+p)}\right)^2 [(1-\lambda)a_n + \lambda b_n]^2$

$= (1-\lambda)^2 \sum_{n=1}^{\infty} \left(\frac{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}}{\propto(\beta+p)} a_n\right)^2 + (\lambda)^2 \sum_{n=1}^{\infty} \left(\frac{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}}{\propto(\beta+p)} b_n\right)^2$

$+2\lambda(1-\lambda) \sum_{n=1}^{\infty} \left(\frac{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}}{\propto(\beta+p)}\right)^2 a_n b_n$

$\leq (1-\lambda)^2 + (\lambda)^2 + 2\lambda(1-\lambda)$

=1

∴g (z) ∈ 1-sqrt Y⁺ $(\propto, \beta, H, V, \gamma, \sigma)$

Theorem 3.4. If f (z) ∈ t-sqrt Y⁺ $(\propto, \beta, H, V, \gamma, \sigma)$. C is any real number such that q+p > 0.

L (z) = $\frac{q+p}{z^q} \int_0^z (s)^{q-1} f(s)ds.$

Then L (z) ∈ t-sqrt Y⁺ $(\propto, \beta, H, V, \gamma, \sigma)$

Proof: Let f (z) = $z^p + \sum_{n=1}^{\infty} a_n z^{p+n}$.

L (z) = $\frac{q+p}{z^q} \int_0^z (s)^{q-1} f(s)ds = \frac{q+p}{z^q} \int_0^z (s)^{q-1} [s^p + \sum_{n=1}^{\infty} a_n s^{p+n}]ds$

$= \frac{q+p}{z^q} \left[\frac{s^{p+q}}{q+p} + \sum_{n=1}^{\infty} a_k \frac{s^{p+q+n}}{q+p+n}\right]_{s=0}^{s=z}$

$= z^p + \sum_{n=1}^{\infty} a_n \frac{p+q}{q+p+n} z^{p+n}$

Given that f(z) ∈ t-sqrt Y⁺ $(\propto, \beta, H, V, \gamma, \sigma)$. $\left(\frac{p+q}{q+p+k}\right)$

$\sum_{n=1}^{\infty} \frac{1}{t} \left(\frac{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma} a_n}{\propto(\beta+p)}\right)^2 \leq 1$

$\sum_{n=1}^{\infty} \frac{1}{t} \left(\frac{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma}\left(\frac{p+q}{q+p+k}\right) a_n}{\propto(\beta+p)}\right)^2$

$< \sum_{n=1}^{\infty} \frac{1}{t} \left(\frac{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma} a_n}{\propto(\beta+p)}\right)^2$ as $\left(\frac{p+q}{q+p+k}\right) < 1$

< 1

∴ L(z) ∈ t-sqrt Y⁺ $(\propto, \beta, H, V, \gamma, \sigma)$.

Corollary 3.5.If f (z) ∈ Y⁺ $(\propto, \beta, H, V, \gamma, \sigma)$ then G (z) ∈ t-sqrt Y⁺ $(\propto, \beta, H, V, \gamma, \sigma)$.

Theorem 3.6. If f (z) ∈ t-sqrt Y⁺ $(\propto, \beta, H, V, \gamma, \sigma)$

N_h(z) = (1 – h) z^p + hp $\int_0^z \frac{f(s)}{s}$ ds (h≥ 0)

Then N_h ∈ t-sqrt Y⁺ $(\propto, \beta, H, V, \gamma, \sigma)$. 0 ≤h ≤ $\frac{p+n}{p}$

Proof. f (z) = $z^p + \sum_{n=1}^{\infty} a_n z^{p+n}$.

$N_h(z) = (1-h)\, z^p + hp \int_0^z \frac{s^p + \sum_{n=1}^{\infty} a_n s^{p+n}}{s}\, ds$

$= (1-h)\, z^p + hp\left(\int_0^z \frac{s^p}{s}\, ds + \int_0^z \frac{\sum_{n=1}^{\infty} a_n s^{p+n}}{s}\, ds\right)$

$= (1-h)\, z^{p} + ph.\left(\frac{s^p}{s} + \sum_{n=1}^{\infty} a_n \frac{s^{p+n}}{p+n}\right)_{s=0}^{s=z}$

$= z^p + \sum_{n=1}^{\infty} \frac{ph}{p+n} a_n z^{p+n}$

For f (z) = $z^p + \sum_{n=1}^{\infty} a_n z^{p+n} \in$ t-sqrt Y⁺ $(\propto, \beta, H, V, \gamma, \sigma)$

$\therefore \sum_{n=1}^{\infty} \frac{1}{t} \left(\frac{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma} a_n}{\propto(\beta+p)}\right)^2 \leq 1$

$\sum_{n=1}^{\infty} \frac{1}{t} \left(\frac{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma} \frac{ph}{p+n} a_n}{\propto(\beta+p)}\right)^2$

$< \sum_{n=1}^{\infty} \frac{1}{t} \left(\frac{(p+n+\beta)(1+\propto)[(V-H)\gamma+H]\left(\frac{1}{n+1}\right)^{\sigma} \frac{ph}{p+n} a_n}{\propto(\beta+p)}\right)^2$ (as $\frac{ph}{p+n} < 1$)

$< 1.$

$\therefore L_h \in$ t-sqrt Y⁺ $(\propto, \beta, H, V, \gamma, \sigma)$.

Corollary3.7. If f (z) ∈ Y⁺ $(\propto, \beta, H, V, \gamma, \sigma)$ then $F_h \in$ t-sqrt Y⁺ $(\propto, \beta, H, V, \gamma, \sigma)$.

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