The Beckman-Quarles Theorem For Rational Spaces: Mapping Of Q^6 To Q^6 That Preserve Distance 1

By: Wafiq Hibi
Wafiq. hibi@gmail.com
The college of sakhnin - math department

Article History: Received: 11 January 2021; Revised: 12 February 2021; Accepted: 27 March 2021; Published online: 16 April 2021

Abstract: Let R^d and Q^d denote the real and the rational d-dimensional space, respectively, equipped with the usual Euclidean metric. For a real number $\rho > 0$, a mapping $f: A \longrightarrow X,$ where X is either R^d or Q^d and $A \subseteq X$, is called ρ-distance preserving $\|x - y\| = \rho$ implies $\|f(x) - f(y)\| = \rho$, for all x,y in A.

Let G(Q^d,a) denote the graph that has Q^d as its set of vertices, and where two vertices x and y are connected by edge if and only if $\|x - y\| = a$. Thus, G(Q^d,1) is the unit distance graph. Let ω(G) denote the clique number of the graph G and let ω(d) denote ω(G(Q^d, 1)).

The Beckman-Quarles theorem [1] states that every unit- distance-preserving mapping from R^d into R^d is an isometry, provided $d \geq 2$.
The rational analogues of Beckman- Quarles theorem means that, for certain dimensions d, every unit- distance preserving mapping from Q^d into Q^d is an isometry.

A few papers [2, 3, 4, 5, 6, 8,9,10 and 11] were written about rational analogues of this theorem, i.e, treating, for some values of d, the property "Every unit- distance preserving mapping $f: Q^d \longrightarrow Q^d$ is an isometry".
The propos of this paper is to prove the following:
Every unit- distance preserving mapping $f: Q^6 \longrightarrow Q^6$is an isometry; moreover, dim *(aff(f(L[6])))= 6.*

Mapping of Q^6 to Q^6 that preserve distance 1

1.1 Introduction:

Let R^d and Q^d denote the real and the rational d-dimensional space, respectively.
Let $\rho > 0$ be a real number, a mapping : $R^d \longrightarrow Q^d$, is called ρ- distance preserving if $\|x - y\| = \rho$ implies $\|f(x) - f(y)\| = \rho$.

The Beckman-Quarles theorem [1] states that every unit- distance-preserving mapping from R^d into R^d is an isometry, provided$d \geq 2$.
A few papers [4, 5, 6, 8,9,10 and 11] were written about the rational analogues of this theorem, i.e, treating, for some values of d, the property "every unit- distance preserving mapping $f: Q^d \longrightarrow Q^d$ is isometry".

We shall survey the results from the papers [2, 3, 4, 5, 6, 8,9,10 and 11] concerning the rational analogues of the Backman-Quarles theorem, and we will extend them to all the remaining dimensions , $d \geq 5$.

History of the rational analogues of the Backman-Quarles theorem:
We shall survey the results from papers [2, 3, 4, 5, 6, 8,9,10 and 11] concerning the rational analogues of the Backman-Quarles theorem.

1. A mapping of the rational space Q^d into itself, for d=2, 3 or 4, which preserves all unit- distance is not necessarily an isometry; this is true by W.Bens [2, 3] and H.Lenz [6].

2. W.Bens [2, 3] had shown the every mapping $f: Q^d \longrightarrow Q^d$ that preserves the distances 1 and 2 is an isometry, provided $d \geq 5$.

3. Tyszka [8] proved that every unit- distance preserving mapping $f: Q^8 \longrightarrow Q^8$ is an isometry; moreover, he showed that for every two points x and y in Q^8 there exists a finite set S_{xy} in Q^8 containing x and y such that every unit- distance preserving mapping $f: S_{xy} \longrightarrow Q^8$ preserves the distance between x and y. This is a kind of compactness argument, that shows that for every two points x and y in Q^d there exists a finite set $S_{xy,}$ that contains x and y ("a neighborhood of x and y") for which already every unit- distance preserving mapping from this neighborhood of x and y to Q^d must preserve the distance from x to y. This implies that every unit preserving mapping from Q^d to Q^d must preserve the distance between every two points of Q^d.

4. J.Zaks [8, 9] proved that the rational analogues hold in all the even dimensions d of the form $d = 4k(k+1)$, for $k \geq 1$, and they hold for all the odd dimensions d of the form $d = 2n^2-1 = m^2$. For integers $n, m \geq 2$, (in [9]), or $d = 2n^2 - 1$, $n \geq 3$ (in [10]).

5. R.Connelly and J.Zaks [5] showed that the rational analogues hold for all even dimensions $d, d \geq 6$.

We wish to remark that during the preparation of this thesis, it was pointed out to us that an important argument, in the proof of the even dimensions $d, d \geq 6$, is missing. Here we propose a valid proof for all the cases of $d, d \geq 5$.

6. J.Zaks [11] had shown that every mapping $f: Q^d \longrightarrow Q^d$ that preserves the distances 1 and $\sqrt{2}$ is an isometry, provided $d \geq 5$.

New results:

Denote by $L[d]$ the set of $4 \cdot \binom{d}{2}$ Points in Q^d in which precisely two non-zero coordinates are equal to 1/2 or -1/2.

A "quadruple" in $L[d]$ means here a set $L_{ij}[d]$, $i \neq j \in I = \{1, 2, \ldots, d\}$; contains four j points of $L[d]$ in which the non- zero coordinates are in some fixed two coordinates i and j; i.e.

$$L_{ij}[d] = (0, \ldots 0, \overset{i}{\pm \tfrac{1}{2}}, 0 \ldots 0, \overset{j}{\pm \tfrac{1}{2}}, 0, \ldots 0)$$

Hibi Prove the following results:

Lemma:

If x and y are two points in $Q^d, d \geq 5$, so that:

$$\sqrt{2+\frac{2}{m-1}}-1 \leq \|x-y\| \leq \sqrt{2+\frac{2}{m-1}}+1$$

where $\omega(d) = m$, then there exists a finite set S(x,y), contains x and y such that $f(x) \neq f(y)$ holds for every unit- distance preserving mapping $f: S(x,y) \rightarrow Q^d$.

Theorem 1

Every unit- distance preserving mapping $f: Q^5 \longrightarrow Q^5$is an isometry; moreover, dim *(aff(f(L[5])))= 5.*

We will prove the following theorem:

Theorem 2:

Every unit- distance preserving mapping $f: Q^6 \longrightarrow Q^6$is an isometry; moreover, dim *(aff(f(L[6])))= 6.*

Mapping of Q^6 to Q^6 that preserve distance 1

The purpose of this section is to prove the following Theorem:

Theorem 2:

Every unit –distance preserving mapping $f: Q^6 \rightarrow Q^6$ is an isometry; moreover, dim (*aff(f(L[6]))*) = 6.

To prove Theorem 2, we prove first the following Theorem.

Theorem 2*:

if Z, W are any two points in Q^6, for which $\|Z-W\| = \sqrt{2}$, then there exists a finite set M_6, containing Z and W, such that for every unit –distance preserving mapping $f: M_6 \rightarrow Q^6$, the following equality holds:

$\|f(Z)-f(W)\| = \|Z-W\|$.

Proof of Theorem 2*:
Consider the 6 points $\{A_1, \ldots, A_6\}$, defined as follows:

$$A_1 = (\tfrac{1}{2}, \quad 0, \quad 0, \quad 0, \quad 0, \quad \tfrac{1}{2})$$
$$A_2 = (\tfrac{1}{2}, \quad 0, \quad 0, \quad 0, \quad 0, \quad -\tfrac{1}{2})$$
$$A_3 = (0, \quad \tfrac{1}{2}, \quad 0, \quad 0, \quad \tfrac{1}{2}, \quad 0)$$
$$A_4 = (0, \quad \tfrac{1}{2}, \quad 0, \quad 0, \quad -\tfrac{1}{2}, \quad 0)$$
$$A_5 = (0, \quad 0, \quad \tfrac{1}{2}, \quad \tfrac{1}{2}, \quad 0, \quad 0)$$
$$A_6 = (0, \quad 0, \quad \tfrac{1}{2}, \quad -\tfrac{1}{2}, \quad 0, \quad 0)$$

The points $\{A_1, \ldots, A_6\}$ form the vertices of a regular 5- simplex of edge length one in Q^6. Let the 6 points $B_1, B_2, \ldots, B_6$ of Q^6 be defined by $B_i = -A_i$, $1 \le i \le 6$, their mutual distances are one, so they form the vertices of a regular 5 – simplex of edge length one in Q^6. Let $T_6 = \{A_1, \ldots, A_6, B_1, \ldots, B_6\}$.
Fix a $k, 1 \le k \le 6$, by Lemma 1 and based on $\|Z - W\| = \|A_k - B_k\|$there exists a rational isometry $h: Q^6 \to Q^6$ for which $h(A_k) = Z := A^*{}_k$ and $h(B_k) = W := B^*{}_k$; denote $h(A_i) = A^*{}_i$ and $h(B_i) = B^*{}_i$ for all $1 \le i \le 6$.
Let $T^*{}_6 = \{A^*{}_1, \ldots, A^*{}_d, B^*{}_1, \ldots, B^*{}_6\}$; it is clear that $Z, W \in T^*{}_6$.
Define the set M_6 by: $M_6 = S(A^*{}_1, B^*{}_1) \cup S(A^*{}_2, B^*{}_2) \cup \ldots \cup S(A^*{}_6, B^*{}_6)$, where the sets S are given by Lemma 4.
Let $f, f: M_6 \to Q^6$ be any unit-distance preserving mapping.

Claim 3:
If x and y are two points in $T^*{}_6$, then $f(x) \ne f(y)$.
Proof of Claim 3:
Computing the mutual distances of the points in $T^*{}_6$ show that:
$\|A^*{}_i - A^*{}_j\| = \|B^*{}_i - B^*{}_j\| = \|A^*{}_i - B^*{}_j\| = 1$, for all $1 \le i < j \le 6$, and
$\|A^*{}_i - B^*{}_i\| = \sqrt{2}$, for all $1 \le i \le 6$.
All of the distances above are between $\sqrt{2 + \frac{2}{m-1}} - 1$ and $\sqrt{2 + \frac{2}{m-1}} + 1$.
where $m = \omega(d) = 6$ for $d = 6$.
Therefore if $\|x - y\| = 1$, then $\|f(x) - f(y)\| = 1$, hence $f(x) \ne f(y)$;
if $\|x - y\| = \sqrt{2}$ there is an $i, 1 \le i \le 6$, such that $x = A^*{}_i, y = B^*{}_i$ and
$\|A^*{}_i - B^*{}_i\| = \sqrt{2}$.
By Lemma 4, applied to $A^*{}_i$ and $B^*{}_i$, there exists a set $S(A^*{}_i, B^*{}_i)$, that contains $A^*{}_i$ and $B^*{}_i$, for which every unit-distance preserving mapping $g: S(A^*{}_i, B^*{}_i) \to Q^d$satisfies $g(A^*{}_i) \ne g(B^*{}_i)$.
In particular, this holds for the mapping $g = f/S(A^*{}_i, B^*{}_i)$, therefore$f(A^*{}_i) \ne f(B^*{}_i)$.

Claim 4:
The mapping f preserves all the distances$\sqrt{2}$, between $A^*{}_i$ and $B^*{}_i$ for all $i = 1,2, \ldots, 6$. In particular $\|f(Z) - f(w)\| = \sqrt{2}$.
Proof of Claim 4:
Consider the following (4) points:

$$\Delta_1 = \{f(A_3^*), f(B_4^*), f(B_5^*), f(B_6^*)\}.$$

All of their mutual distances are one, since f preserves distance one, so they form the vertices of a regular 3-simplex of edge length one in Q^6. The intersection of the 4 unit spheres, centered at the vertices of this simplex, is a 2-sphere of radius$t = \sqrt{\frac{5}{8}}$, centered at the center O_1 of Δ_1; let $S^2_{(O_1,t)}$ denote this 2-sphere.
Let Δ_2 be defined by:

$$\Delta_2 = \{f(A_4^*), f(B_3^*), f(B_5^*), f(B_6^*)\}.$$

In the similar way we obtain the 2-spheres $S^2_{(O_2,t)}$, having her center at O_2, which is also the center of Δ_2.
The four points $f(A_1^*), f(A_2^*), f(B_1^*)$ and $f(B_2^*)$ are in the intersection of the two 2-spheres $S^2_{(O_j,t)}$, $j = 1,2$.
By claim 3, the two simplices Δ_1, and Δ_2 are different, but they have vertices $f(B_5^*), and\ f(B_6^*)$ in common.

We will prove that $O_1 \neq O_2$:
Assume that $O_1 = O_2 = O$. (See figure 6)

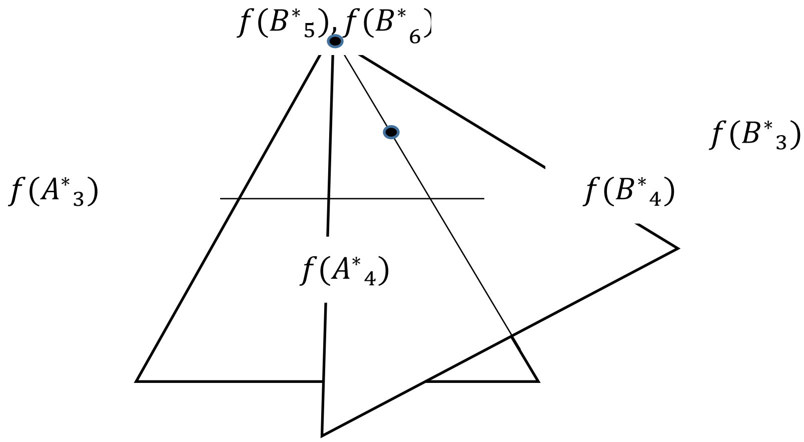

Figure 6

It follows that $\left\| f\left(B_j^*\right) - O \right\| = \left\| f(A_i^*) - O \right\|$=t, i=3, 4, and j=3,4, 5, 6.
In particular, the point O the center of the simplex $\{f(B_3^*), f(B_4^*), f(B_5^*), f(B_6^*)\}$, so
O = $\frac{1}{4}\left(f(B_3^*) + f(B_4^*), +f(B_5^*) + f(B_6^*)\right)$, but point O is also the center of the simplex Δ_1 so O = $\frac{1}{4}\left(f(A_3^*) + f(A_4^*), +f(A_5^*) + f(A_6^*)\right)$.
It follows that $f(A_3^*) = f(B_3^*)$, a contradiction to Claim 3, thus $O_1 \neq O_2$.
Therefore the 2- spheres $S^2_{(O_j,t)}$, $j = 1,2$, are different.
They have the same radius $t = \sqrt{\frac{5}{8}}$ and they have a non-empty intersection. It follows that there two 2-spheres intersect in a one-dimensional sphere, which is a circle.
Thus $f(A_1^*), f(A_2^*), f(B_1^*)$ and $f(B_2^*)$ form the vartex set of a quadrangle, of edge length one, that lies in a circle. (See figure 5).
It follows as the previous case that $f(A_1^*), f(A_2^*), f(B_1^*)$ and $f(B_2^*)$ form the vartex set of a square in a circle of diameter $\sqrt{2}$, implying:
$\| f(A_1^*) - f(B_1^*) \| = \| f(A_2^*) - f(B_2^*) \| = \sqrt{2}$ since $f(A_i^*) \neq f(B_i^*)$ for $i = 1,2$.
It follows by Lemma 1 that the mapping f preserves the distance $\sqrt{2}$ between A_i^* and B_i^* for all $i = 1,2,\dots,6$. In particular $\| f(Z) - f(W) \| = \sqrt{2}$.

This completes the proof of Theorem 2*.

Proof of Theorem
Let f be a unit distance preserving mapping $f: Q^6 \to Q^6$. By Theorem 2* the unit distance preserving mapping $f$preserves the distance $\sqrt{2}$.
Our result follows by using a Theorem of J.Zaks [8], which states that if a mapping $g: Q^d \to Q^d$ preserves the distance 1 and $\sqrt{2}$, then g is an isometry, provided $d \geq 5$.
The proof that dim(*aff(L[6])*) = 6 is similar to the proof that dim(*aff(L[5])*) = 5 that appeared in of Theorem 1, hence it is omitted.

This completes the proof of Theorem

References

1. F.S Beckman and D.A Quarles: On isometries of Euclidean spaces, Proc. Amer. Math. Soc. 4, (1953), 810-815.
2. W.Benz, An elementary proof of the Beckman and Quarles, Elem.Math. 42 (1987), 810-815
3. W.Benz, Geometrische Transformationen, B.I.Hochltaschenbucher, Manheim 1992.
4. Karin B. Chilakamarri: Unit-distance graphs in rational n-spaces Discrete Math. 69 (1988), 213-218.
5. R.Connelly and J.Zaks: The Beckman-Quarles theorem for rational d-spaces, d even and d≥6. Discrete Geometry, Marcel Dekker, Inc. New York (2003) 193-199, edited by Andras Bezdek.
6. H.Lenz: Der Satz von Beckman-Quarles in rationalen Raum, Arch. Math. 49 (1987), 106-113.
7. I.M.Niven, H.S.Zuckerman, H.L.Montgomery: An introduction to the theory of numbers, J. Wiley and Sons, N.Y., (1992).
8. A.Tyszka: A discrete form of the Beckman-Quarles theorem for rational eight- space. Aequationes Math. 62 (2001), 85-93.
9. J.Zaks: A distcrete form of the Beckman-Quarles theorem for rational spaces. J. of Geom. 72 (2001), 199-205.
10. J.Zaks: The Beckman-Quarles theorem for rational spaces. Discrete Math. 265 (2003), 311-320.
11. J.Zaks: On mapping of Q^d to Q^d that preserve distances 1 and $\sqrt{2}$. and the Beckman-Quarles theorem. J of Geom. 82 (2005), 195-203.