The Beckman-Quarles Theorem For Rational Spaces: Mappings Of $Q^5$ To $Q^5$ That Preserve Distance 1

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Abstract: Let $R^d$ and $Q^d$ denote the real and the rational d-dimensional space, respectively, equipped with the usual Euclidean metric. For a real number $\rho > 0$, a mapping $f: A \rightarrow X$, where $X$ is either $R^d$ or $Q^d$ and $A \subseteq X$, is called $\rho$-distance preserving $\|x - y\| = \rho$ implies $\|f(x) - f(y)\| = \rho$, for all $x,y$ in $A$. Let $G(Q^d,a)$ denote the graph that has $Q^d$ as its set of vertices, and where two vertices $x$ and $y$ are connected by edge if and only if $\|x - y\| = a$. Thus, $G(Q^d,1)$ is the unit distance graph. Let $\omega(G)$ denote the clique number of the graph $G$ and let $\omega(d)$ denote $\omega(G(Q^d,1))$. The Beckman-Quarles theorem [1] states that every unit-distance-preserving mapping from $R^d$ into $R^d$ is an isometry, provided $d \geq 2$.

Every unit-distance preserving mapping $f: Q^5 \rightarrow Q^5$ is an isometry; moreover, $\dim(\text{aff}(L[5])) = 5$.

1.1 Introduction:
Let $R^d$ and $Q^d$ denote the real and the rational d-dimensional space, respectively. Let $\rho > 0$ be a real number, a mapping $f: R^d \rightarrow Q^d$, is called $\rho$-distance preserving if $\|x - y\| = \rho$ implies $\|f(x) - f(y)\| = \rho$.

The Beckman-Quarles theorem [1] states that every unit-distance-preserving mapping from $R^d$ into $R^d$ is an isometry, provided $d \geq 2$.

We shall survey the results from the papers [2, 3, 4, 5, 6, 8, 9, 10 and 11] concerning the rational analogues of the Beckman-Quarles theorem, and we will extend them to all the remaining dimensions, $d \geq 5$.

History of the rational analogues of the Beckman-Quarles theorem:

1. A mapping of the rational space $Q^d$ into itself, for $d = 2, 3$ or 4, which preserves all unit-distance is not necessarily an isometry; this is true by W.Bens [2, 3] and H.Lenz [6].

2. W.Bens [2, 3] had shown the every mapping $f: Q^d \rightarrow Q^d$ that preserves the distances 1 and 2 is an isometry, provided $d \geq 5$.

3. Tyszka [8] proved that every unit-distance preserving mapping $f: Q^d \rightarrow Q^d$ is an isometry; moreover, he showed that for every two points $x$ and $y$ in $Q^d$ there exists a finite set $S_{xy}$ in $Q^d$ containing $x$ and $y$ such that every unit-distance preserving mapping $f: S_{xy} \rightarrow Q^d$ preserves the distance between $x$ and $y$. This is a kind of compactness argument, that shows that for every two points $x$ and $y$ in $Q^d$ there exists a finite set $S_{xy}$ that contains $x$ and $y$ ("a neighborhood of $x$ and $y$") for which already every unit-distance preserving mapping from this neighborhood of $x$ and $y$ to $Q^d$ must preserve the distance from $x$ to $y$. This implies that every unit preserving mapping from $Q^d$ to $Q^d$ must preserve the distance between every two points of $Q^d$.  


4. J.Zaks [8, 9] proved that the rational analogues hold in all the even dimensions $d$ of the form $d = 4k (k+1)$, for $k \geq 1$, and they hold for all the odd dimensions $d$ of the form $d = 2n^2 - 1 = m^2$. For integers $n, m \geq 2$, (in [9]), or $d = 2n^2 - l$, $n \geq 3$ (in [10]).

5. R.Connelly and J.Zaks [5] showed that the rational analogues hold for all even dimensions $d, d \geq 6$.

We wish to remark that during the preparation of this thesis, it was pointed out to us that an important argument, in the proof of the even dimensions $d, d \geq 6$, is missing. Here we propose a valid proof for all the cases of $d, d \geq 5$.

6. J.Zaks [11] had shown that every mapping $f: Q^d \rightarrow Q^d$ that preserves the distances 1 and $\sqrt{2}$ is an isometry, provided $d \geq 5$.

New results:
Denote by $L[d]$ the set of $4 \cdot \left(\frac{d}{2}\right)$ Points in $Q^d$ in which precisely two non-zero coordinates are equal to 1/2 or -1/2. A "quadruple" in $L[d]$ means here a set $L_{ij}[d], i \neq j \in I = \{1, 2, \ldots, d\};$ contains four points of $L[d]$ in which the non-zero coordinates are in some fixed two coordinates $i$ and $j$; i.e.

$$L_{ij}[d] = (0, \ldots, 0, \pm \frac{1}{2}, 0, \ldots, \pm \frac{1}{2}, 0, \ldots, 0)$$

Our main results are the following:

**Theorem 1:**
Every unit-distance preserving mapping $f: Q^5 \rightarrow Q^5$ is an isometry; moreover, dim $(\text{aff}(fL[5])) = 5$.

**Hibi prove the following lemma:**
If $x$ and $y$ are two points in $Q^d, d \geq 5$, so that:

$$\sqrt{2 + \frac{2}{m-1}} - 1 \leq \|x - y\| \leq \sqrt{2 + \frac{2}{m-1} + 1}$$

where $\omega(d) = m$, then there exists a finite set $S(x,y)$, contains $x$ and $y$ such that $f(x) \neq f(y)$ holds for every unit-distance preserving mapping $f: S(x,y) \rightarrow Q^d$.

**Corollary:**
If $x$ and $y$ are two points in $Q^d, d \geq 5$, such that $\|x - y\| = \sqrt{2}$, then every unit-distance preserving mapping $f: Q^d \rightarrow Q^d$ satisfies $f(x) \neq f(y)$.

**Mappings of $Q^5$ to $Q^5$ that preserve distance 1**

The purpose of this section is to prove the following Theorem.

**Theorem 1:**
Every unit-distance preserving mapping $f: Q^5 \rightarrow Q^5$ is an isometry; moreover, dim $(\text{aff}(fL[5])) = 5$.

To prove Theorem 1, we prove first the following Theorem.

**Theorem 1*:**
If $Z, W$ are two points in $Q^5$, for which $\|Z - W\| = \sqrt{2}$, then there exists a finite set $M_5$, containing $Z$ and $W$, such that for every unit-distance preserving mapping $f: M_5 \rightarrow Q^5$, the following equality holds:

$$\|f(Z) - f(W)\| = \|Z - W\|$$

**Proof of Theorem 1***:
Let $Z, W$ are any two points in $Q^5$, for which $\|Z - W\| = \sqrt{2}$.

Denote by $L[5]$ the set of $4 \cdot \left(\frac{5}{2}\right) = 40$ points in $Q^d$ in which precisely two coordinates are non-zero and are equal to 1/2 or -1/2.

A "quadruple" in $L[5]$ means a set $L_{ij}[5], i \neq j \in I = \{1, 2, 3, 4, 5\}$, containing four points of $L[5]$ in which the non-zero coordinates are in some fixed two, the $i$-th and the $j$-th coordinates; i.e.
\[ L_{ij}[5] = \left\{ \left( 0, \pm \frac{1}{2}, 0, \pm \frac{1}{2}, 0 \right) \right\} \]

If \( \rho \) is a distance between any two points of the set \( L[5] \) then \( \rho \in \{ \sqrt{0.5}, 1, \sqrt{1.5}, \sqrt{2} \} \).

Fix a quadruple \( L_{ij}[5] \) let \( x, y \) two points in \( L_{ij}[5] \) such that \( \|x-y\| = \sqrt{2} \) and \( h(x) =: Z = x^* \) and \( h(y) =: W = y^* \); denote \( h(l) = l^* \) for all \( l \in L[5] \).

Let \( L^*[5] = \{ l^* = h(l) \text{ for all } l \in L[5] \} \); it is clear that \( Z, W \in L^*[5] \), and to simplify terminology we will denote \( L^*[5] = \{ l^*_i \} \) when \( i \in \{1, 2, ..., 40\} \).

Define the set \( M_5 \) by:
\[
M_5 = \cup \{ S(l^*_i, l^*_j) \cup S(l^*_n, l^*_m) \cup S(l^*_s, l^*_t) \} \)
\[
\text{for all } i, j, n, m, s, t \in \{1, 2, ..., 40\} \text{ when } \|l^*_i - l^*_j\| = \sqrt{0.5}, \quad \|l^*_n - l^*_m\| = \sqrt{1.5} \text{ and } \|l^*_s - l^*_t\| = \sqrt{2} \text{; where the sets } S \text{ are given by Lemma 4.}
\]

Let \( f, f: M_5 \rightarrow Q^5 \) be any unit-distance preserving mapping.

Claim 1:
If \( x \) and \( y \) are two points in \( L^*[5] \) for which \( \|x-y\| = 1, \sqrt{2} \) then \( f(x) \neq f(y) \).

Proof of Claim 1:
Clearly, if \( \|x-y\| = 1 \), then \( \|f(x) - f(y)\| = 1 \); hence \( f(x) \neq f(y) \).

The distance \( \sqrt{2} \) is between \( \sqrt{2 + \frac{2}{m-1}} - 1 \) and \( \sqrt{2 + \frac{2}{m-1}} + 1 \).

Where \( m = \omega(d) = d = 5 \).

Therefore, if \( \|x-y\| = \sqrt{2} \), then there exist an \( i \) and \( j \), \( 1 \leq i \neq j \leq 40 \), such that \( x = l^*_i, y = l^*_j \) and \( \|l^*_i - l^*_j\| = \sqrt{2} \). (\( l^*_i \) and \( l^*_j \) on the same quadruple).

By Lemma 4, applied to \( l^*_i \) and \( l^*_j \), there exists a set \( S(l^*_i, l^*_j) \), that contains \( l^*_i \) and \( l^*_j \), for which every unit-distance preserving mapping \( g: S(l^*_i, l^*_j) \rightarrow Q^5 \) satisfies \( g(l^*_i) \neq g(l^*_j) \).

In particular this holds for the mapping \( g \neq f / S(l^*_i, l^*_j) \), therefore \( f(l^*_i) \neq f(l^*_j) \).

Claim 2:
The mapping \( f \) preserves all the distances \( \sqrt{2} \). In particular \( \|f(Z)-f(W)\| = \sqrt{2} \).

Proof of Claim 2:
Consider the graph \( P \) of unit distances among the points of \( L^*[5] \); it is isomorphic to the famous Petersen’s graph, by substituting a 4-cycle for each vertex of \( P \).
(See figure 4).
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We prove that the affine dimension of the $f$- image of each quadruple, i.e., the image of the four points that correspond to one vertex of $P$ must be 2. Indeed, by claim 1 this dimension is at least 2, since $f(l_i') \neq f(l_j')$ for all $l_i', l_j'$ on $L^*[5]$. 

(In particular, this holds for all $l_i'$ and $l_j'$ on the same quadruple).

Suppose, by contradiction, that $\dim(\text{aff}(f(A))) \geq 3$, for some quadruple $A$, let the quadruple $B, C, D$ and $E$ correspond to vertices of $P$ so that $A, B, C, D$ and $E$ is a cycle in $P$.

All the points of $f(B)$ and $f(E)$ must be at unit distance from those of $f(A)$, so all the points of $f(B)$ and $f(E)$ lie on a circle, say circle $S$ with enter $O$.

This means that $f(B)$ and $f(C)$ are two squares inscribed in $S$. it follows that all the points of $f(C)$ and $f(D)$ must lie on the 3-flat that is perpendicular to 2-flat determined by $S$ and passes through $O$.

But this cannot happen, since the points of $f(C)$ span a flat of dimension at least 2 in this 3-flat, which then forces the points of $f(D)$ to lie on a line, which is impossible.

It follows that the points of any $f(F)$ lie on the intersection of some unit-distance spheres and a 2-flat which is a circle; when $F=\{a, b, c, d\}$ is a given block, such that $\|a-b\| = \|b-c\| = \|c-d\| = \|d-a\| = 1$ and $\|a-c\| = \|b-d\| = \sqrt{2}$.

Thus $f(a), f(b), f(c),$ and $f(d)$ form the vertex set of quadrangle, of edge length one that lies in a circle. (See figure 5).

The situations (i) and (ii) are impossible since $f(l_i') \neq f(l_j')$ for all $l_i'$ and $l_j'$ on $L^*[5]$.

It follows that $f(a), f(b), f(c),$ and $f(d)$ form vertex set of a square in circle of diameter $\sqrt{2}$, implying: $\|f(a)-f(c)\| = \|f(b)-f(d)\| = \sqrt{2}$.

Hence, the distance $\sqrt{2}$, within each quadrangle are preserved. In particular $\|f(Z)-f(W)\| = \sqrt{2}$.

This completes the proof of Theorem 1*.

**Proof of Theorem 1:**

Let $f$ be a unit distance preserving mapping $f:Q^5 \to Q^5$. By Theorem 1* the unit distance preserving mapping $f$ preserves the distance $\sqrt{2}$.

Our result follows by using a Theorem of J. Zaks [8], which states that if a mapping $g:Q^d \to Q^d$ preserves the distances 1 and $\sqrt{2}$, then $g$ is an isometry, provided $d \geq 5$.

Moreover, $\dim(\text{aff}(L[5])) = 5$:

The mapping $f$ is an isometry, hence it suffices to provide that $\dim(\text{aff}(L[5])) = 5$.

To show this, notice that:

$$\frac{1}{2}\left(\frac{1}{\sqrt{2}}, 0, 0\right) + \frac{1}{2}\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0\right) = \frac{1}{2}(1, 0, 0, 0, 0)$$
\[
\frac{1}{2}(\frac{1}{2},\frac{1}{2},0,0,0) + \frac{1}{2}(-\frac{1}{2},\frac{1}{2},0,0,0) = \frac{1}{2}(0,1,0,0,0)
\]
\[
\frac{1}{2}(0,0,\frac{1}{2},\frac{1}{2},0) + \frac{1}{2}(0,0,-\frac{1}{2},\frac{1}{2},0) = \frac{1}{2}(0,0,1,0,0)
\]
\[
\frac{1}{2}(0,0,\frac{1}{2},\frac{1}{2},0) + \frac{1}{2}(0,0,0,-\frac{1}{2},\frac{1}{2}) = \frac{1}{2}(0,0,0,0,1)
\]
Hence all the major unit vectors in \(\mathbb{R}^5\) when multiplied by \(\frac{1}{2}\), are convex combinations of points in \(L[5]\).

This completes the proof of Theorem 1.

References

11. J.Zaks: On mapping of \(\mathbb{Q}^4\) to \(\mathbb{Q}^3\) that preserve distances 1 and \(\sqrt{2}\). and the Beckman-Quarles theorem. J of Geom. 82 (2005), 195-203.