The Beckman-Quarles Theorem For Rational Spaces

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Abstract: Let R^d and Q^d denote the real and the rational d-dimensional space, respectively, equipped with the usual Euclidean metric. For a real number $\rho > 0$, a mapping $f: A \longrightarrow X$, where X is either R^d or Q^d and $A \subseteq X$, is called ρ-distance preserving $\|x-y\| = \rho$ implies $\|f(x)-f(y)\| = \rho$, for all x,y in A.

Let G(Q^d,a) denote the graph that has Q^d as its set of vertices, and where two vertices x and y are connected by edge if and only if $\|x-y\| = a$. Thus, G(Q^d,1) is the unit distance graph. Let ω(G) denote the clique number of the graph G and let ω(d) denote ω(G(Q^d, 1)).

The Beckman-Quarles theorem [1] states that every unit- distance-preserving mapping from R^d into R^d is an isometry, provided $d \geq 2$.

The rational analogues of Beckman- Quarles theorem means that, for certain dimensions d, every unit- distance preserving mapping from Q^d into Q^d is an isometry.

A few papers [2, 3, 4, 5, 6, 8,9,10 and 11] were written about rational analogues of this theorem, i.e, treating, for some values of d, the property "Every unit- distance preserving mapping $f: Q^d \longrightarrow Q^d$ is an isometry".

The purpose of this section is to prove the following Lemma
Lemma: If x and y are two points in $Q^d, d \geq 5$, so that:

$$\sqrt{2+\frac{2}{m-1}}-1 \leq \|x-y\| \leq \sqrt{2+\frac{2}{m-1}}+1$$

where $\omega(d) = m$, then there exists a finite set S(x,y), contains x and y such that $f(x) \neq f(y)$ holds for every unit-distance preserving mapping $f: S(x,y) \rightarrow Q^d$.

1.1 Introduction:
Let R^d and Q^d denote the real and the rational d-dimensional space, respectively.
Let $\rho > 0$ be a real number, a mapping : $R^d \longrightarrow Q^d$, is called ρ- distance preserving if $\|x-y\| = \rho$ implies $\|f(x)-f(y)\| = \rho$.

The Beckman-Quarles theorem [1] states that every unit- distance-preserving mapping from R^d into R^d is an isometry, provided$d \geq 2$.
A few papers [4, 5, 6, 8,9,10 and 11] were written about the rational analogues of this theorem, i.e, treating, for some values of d, the property "every unit- distance preserving mapping $f: Q^d \longrightarrow Q^d$ is isometry".

We shall survey the results from the papers [2, 3, 4, 5, 6, 8,9,10 and 11] concerning the rational analogues of the Backman-Quarles theorem, and we will extend them to all the remaining dimensions , $d \geq 5$.

History of the rational analogues of the Backman-Quarles theorem:
We shall survey the results from papers [2, 3, 4, 5, 6, 8,9,10 and 11] concerning the rational analogues of the Backman-Quarles theorem.

1. A mapping of the rational space Q^d into itself, for d=2, 3 or 4, which preserves all unit- distance is not necessarily an isometry; this is true by W.Bens [2, 3] and H.Lenz [6].

2. W.Bens [2, 3] had shown the every mapping $f: Q^d \longrightarrow Q^d$ that preserves the distances 1 and 2 is an isometry, provided $d \geq 5$.

3. Tyszka [8] proved that every unit- distance preserving mapping $f: Q^8 \longrightarrow Q^8$ is an isometry; moreover, he showed that for every two points x and y in Q^8 there exists a finite set S_{xy} in Q^8 containing x and y such that every unit- distance preserving mapping $f: S_{xy} \longrightarrow Q^8$ preserves the distance between x and y. This is a kind of compactness argument, that shows that for every two points x and y in Q^d there exists a finite set $S_{xy,}$ that contains x and y ("a neighborhood of x and y") for which already every unit- distance preserving mapping from this neighborhood of x and y to Q^d must preserve the distance from x to y. This implies that every unit preserving mapping from Q^d to Q^d must preserve the distance between every two points of Q^d.

4. J.Zaks [8, 9] proved that the rational analogues hold in all the even dimensions d of the form $d = 4k (k+1)$, for $k \geq 1$, and they hold for all the odd dimensions d of the form $d = 2n^2 - 1 = m^2$. For integers $n, m \geq 2$, (in [9]), or $d = 2n^2 - 1$, $n \geq 3$ (in [10]).

5. R.Connelly and J.Zaks [5] showed that the rational analogues hold for all even dimensions $d, d \geq 6$.

We wish to remark that during the preparation of this thesis, it was pointed out to us that an important argument, in the proof of the even dimensions $d, d \geq 6$, is missing. Here we propose a valid proof for all the cases of $d, d \geq 5$.

6. J.Zaks [11] had shown that every mapping $f: Q^d \longrightarrow Q^d$ that preserves the distances 1 and $\sqrt{2}$ is an isometry, provided $d \geq 5$.

New results:

Denote by $L[d]$ the set of $4 \cdot \binom{d}{2}$ Points in Q^d in which precisely two non-zero coordinates are equal to 1/2 or -1/2.

A "quadruple" in $L[d]$ means here a set $L_{ij}[d]$, $i \neq j \in I = \{1, 2, \ldots, d\}$; contains four j points of $L[d]$ in which the non- zero coordinates are in some fixed two coordinates i and j; i.e.

$$L_{ij}[d] = (0, \ldots 0, \overset{i}{\pm \tfrac{1}{2}}, 0 \ldots 0, \overset{j}{\pm \tfrac{1}{2}}, 0, \ldots 0)$$

Our main results are the following:

Lemma: If x and y are two points in $Q^d, d \geq 5$, so that:

$$\sqrt{2 + \frac{2}{m-1}} - 1 \leq \|x - y\| \leq \sqrt{2 + \frac{2}{m-1}} + 1$$

where $\omega(d) = m$, then there exists a finite set $S(x,y)$, contains x and y such that $f(x) \neq f(y)$ holds for every unit-distance preserving mapping $f: S(x,y) \longrightarrow Q^d$.

Auxiliary Lemmas:

We need the following Lemmas for our proofs of the Theorems 1 and 2.

Lemma 1: (due J.Zaks [10]).

If $v_1, \ldots, v_n, w_1, \ldots, w_m$ are points in Q^d, $n \leq m$ such that $\|v_i - v_j\| = \|w_r - w_s$,
for all $1 \leq i \leq j \leq n, 1 \leq r \leq s \leq m$ then there exists a congruence $f: Q^d \longrightarrow Q^d$, such that $f(v_i) = w_i$ for all $1 \leq i \leq n$.

Lemma 2: (due to Chilakamarri [4]).

a. For even d, $\omega(d) = d+1$, if $d+1$ is a complete square; otherwise $\omega(d) = d$.

b. For odd d, $d \geq 5$, the value of $\omega(d)$ is as follows: if $d = 2n^2 - 1$, then $\omega(d) = d+1$; if $d \neq 2n^2-1$ and the Diophantine equation $dx^2 - 2(d-1)y^2 = z^2$ has a solution in which $x \neq 0$ then $\omega(d) = d$; otherwise $\omega(d) = d - 1$.

Lemma 3:
If *a*, *b*, *c* are three numbers that satisfy the triangle inequality and if a^2, b^2, c^2 are rational numbers then:
a. $b^2 - \frac{(b^2 - a^2 + c^2)^2}{4c^2}$, and

b. The space Q^d, $d \geq 8$ contains a triangle *ABC*, having edge length: *AB*=*c*, *BC*=*a*, *AC*=*b*.

Proof of Lemma 3:
To prove (**a**), its suffices to prove that $4b^2c^2 - (b^2 - a^2 + c^2)^2 > 0$

$$4b^2c^2 - (b^2 - a^2 + c^2)^2 =$$

$$= [2bc + (b^2 - a^2 + c^2)] \cdot [2bc - (b^2 - a^2 + c^2)]$$
$$= [(b + c)^2 - a^2] \cdot [a^2 - (b - c)^2]$$
$$= (a + b + c)(b + c - a)(a + b - c)(a - b + c) > 0.$$

The triangle inequality implies that the expression in the previous line on the left is positive; it appears also in Heron's formula.

To prove (**b**): Let *a*, *b*, *c* be three numbers that satisfy the triangle inequality, and so that a^2 ,b^2 ,c^2 are rational numbers.
The number $c^2/4$ is positive and rational, hence there exist, according to Lagrange Four Squares theorem [8], rational numbers $\alpha, \beta, \gamma, \delta$ such that $c^2/4 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2$.

By part (a), the following holds: $b^2 - \frac{(b^2 - a^2 + c^2)^2}{4c^2} > 0$, therefore there exist by Lagrange Theorem rational numbers: *x*, *y*, *z*, *w*, such that:

$$b^2 - \frac{(b^2 - a^2 + c^2)^2}{4c^2} = x^2 + y^2 + z^2 + w^2.$$

Consider the following points:
$A = (-\alpha, -\beta, -\gamma, -\delta, 0, \dots, 0)$
$B = (\alpha, \beta, \gamma, \delta, 0, \dots, 0)$
$C = (\frac{b^2 - a^2}{c^2}\alpha, \frac{b^2 - a^2}{c^2}\beta, \frac{b^2 - a^2}{c^2}\gamma, \frac{b^2 - a^2}{c^2}\delta, x, y, z, w, 0, \dots, 0)$

The points *A*,*B* and *C* satisfy:

$$\|A - B\| = \sqrt{4(\alpha^2 + \beta^2 + \delta^2 + \gamma^2} = c$$

$$\|A - C\| = \sqrt{\left[\frac{b^2 - a^2}{c^2} + 1\right]^2 (\alpha^2 + \beta^2 + \delta^2 + \gamma^2) + x^2 + y^2 + z^2 + w^2}$$

$$= \sqrt{\frac{(b^2 - a^2 + c^2)^2}{4c^2} + b^2 - \frac{(b^2 - a^2 + c^2)^2}{4c^2}} = b,$$

and:

$$\|B - C\| = \sqrt{\left[\frac{b^2 - a^2}{c^2} - 1\right]^2 (\alpha^2 + \beta^2 + \delta^2 + \gamma^2) + x^2 + y^2 + z^2 + w^2}$$

$$= \sqrt{\frac{(b^2 - a^2 - c^2)^2}{4c^2} - \frac{(b^2 - a^2 + c^2)^2}{4c^2} + b^2} =$$

$$= \sqrt{\frac{-4(b^2 - a^2)c^2 + 4b^2c^2}{4c^2}} = a$$

This completes the proof of Lemma 3.

Corollary 1:
If a, b, 1 satisfy the triangle inequality and if a^2, b^2 are rational numbers, then the space Q^5 contains the vertices of a triangle which has edge lengths *a*, *b*, *1*.

Proof:
Consider the following points:

$$A = (\frac{1}{2},0,0,0,0)$$
$$B = (-\frac{1}{2},0,0,0,0)$$
$$C = ((b^2 - a^2)\frac{1}{2},\alpha,\beta,\gamma,\delta)$$

Where $\alpha, \beta, \gamma, \delta$ are the rational numbers that exist according to Lagrange theorem, for which:

$$b^2 - \frac{(b^2 - a^2 + 1)^2}{4} = \alpha^2 + \beta^2 + \delta^2 + \gamma^2$$

From the proof of Lemma 2 the triangle, *ABC* has the edge length *a*, *b*, *1*.

Corollary 2:
If t is a number such that $\sqrt{2+\frac{2}{m-1}} - 1 \leq t \leq \sqrt{2+\frac{2}{m-1}} + 1$, $t^2 \in Q$
Where $m \geq 4$ is a natural number, then the space $Q^d, d \geq 5$, contains a triangle *ABC* having edge length *1*,*t*, $\sqrt{2+\frac{2}{m-1}}$.

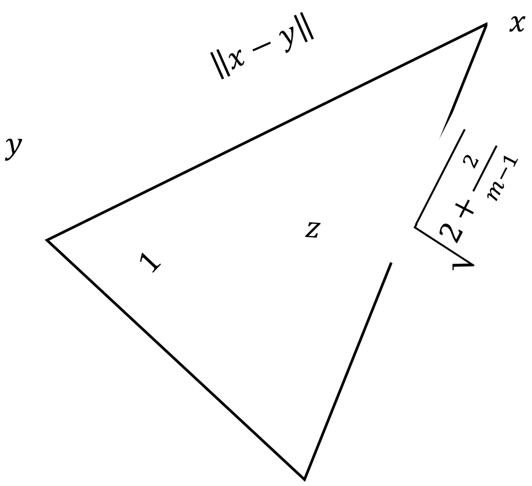
Proof:
According to Lemma 2, the numbers 1,t, $\sqrt{2+\frac{2}{m-1}}$ satisfy the triangle inequality, and the result follows from Corollary 1.

Lemma 4:
If x and y are two points in $Q^d, d \geq 5$, so that:

$$\sqrt{2+\frac{2}{m-1}} - 1 \leq \|x-y\| \leq \sqrt{2+\frac{2}{m-1}} + 1$$

where $\omega(d) = m$, then there exists a finite set S(*x*,*y*), contains x and y such that *f(x)≠f(y)* holds for every unit-distance preserving mapping *f: S(x,y)→* Q^d.

Proof of Lemma 4:
Let x and y be points in $Q^d, d \geq 5$, for which,
$\sqrt{2+\frac{2}{m-1}} - 1 \leq \|x-y\| \leq \sqrt{2+\frac{2}{m-1}} + 1$ where $\omega(d) = m$.
The real numbers $\|x\text{-}y\|$, $\sqrt{2+\frac{2}{m-1}}$ and *1* satisfy the triangle inequality, hence by Corollary 2 there exist three points A, B, C such that $\|A\text{-}B\| = \|x\text{-}y\|$,
$\|A\text{-}C\| = \sqrt{2+\frac{2}{m-1}}$ and $\|B\text{-}C\|$ *=1.* It follows by two rational reflections that there exists a rational point **z** for which $\|y\text{-}z\|$=1 and $\|x\text{-}z\| = \sqrt{2+\frac{2}{m-1}}$, (see Figure 1).
Let $\{v_0, \dots, v_{m-1}\}$ be a maximum clique *in* G*(*Q^d*,1)*, and let w_0 be the reflection of v_0 with respect to the rational hyperplane passing through the points $\{v_1, \dots, v_{m-1}\}$ it follows that $\|v_0 - w_0\| = \sqrt{2+\frac{2}{m-1}}$, (see Figure 2).



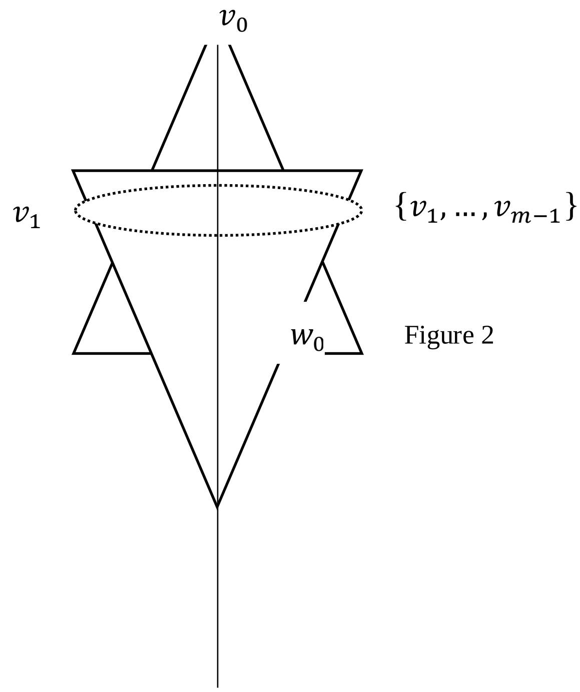


Figure 2

Based on $\|x\text{-}z\| = \|v_0 - w_0\|$ and lemma 1, there exist a rational translation h for which $h(v_0)= x$ and $h(w_0)=z$. Denote $g\ (h(v_i))= V_i$ for all $1\leq i\leq m\text{-}1$, (see Figure 3).

Figure 1

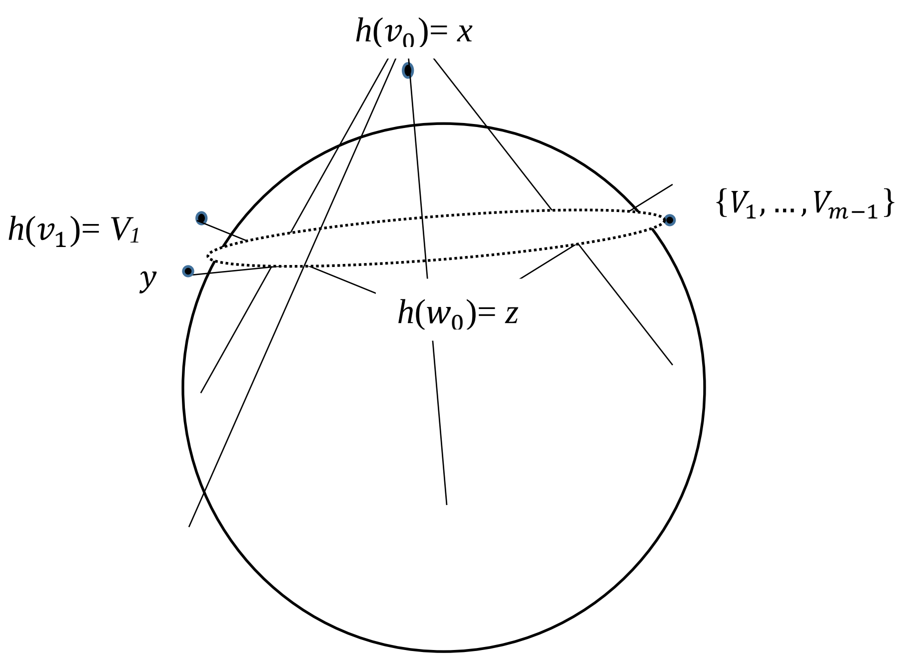


Figure 3

Denote S(x, y) = {x, y, z,$v_1, \dots, v_{m-1}$}. Suppose that *f(x)*= *f(y)* holds for some unit- distance preserving mapping *f: S(x,y)*→ Q^d.
The assumption *f(x) = f(y)* and ‖*y-z*‖=*1* imply that ‖*f(y) - f(z)*‖=*1*=‖*f(x) = f(z)*‖, hence the set
$\{f(x), f(z), f(v_1), \dots, f(v_{m-1})\}$, forms a clique in G($Q^d$,1) of size $m+1$,which is a contradiction. It follows that *f(x)* ≠ *f(y)* holds for every unit- distance preserving mapping *f: S(x,y)*→ Q^d.
This completes the proof of Lemma 4.

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