Research Article

Power Bondage Number For Some Classes Of Graphs

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Abstract: The power bondage number is an essential specification of graphs which is standard upon the well-known power domination number. The power bondage number $b_p(G)$ of a nonempty undirected graph G is the least count of edges whose removal from the given graph G. In this article we discuss about the power bondage number of some classes of graphs.

Keywords: Bondage number; Sunlet graph; Line graph of sunlet graph; Polygonal snake graph.

Section 1: Introduction

The topic of Power Domination Set(PDS) was started by Haynes et al. A set S is known as a power dominating set (PDS) of G if $m(S)=v(G)$. Then $\gamma p(G)$is the lowest cardinality of a PDS of G. A PDS of G with the lowest cardinality is called a γ_p(G). Since every dominating set is a power dominating set, $1 \leq \gamma p(G) \leq \gamma(G)\ \forall\ G(V,E)$.

Among several problems related with the power domination number, many focus on graph and their results on the PD number. Here we are discussed with a specific graph modification, the cancellation of edges from a graph. Graphs with power domination numbers changed by the deletion of an edge were first explored by Walikar and Acharya in 1979. A graph is called edge domination-critical graph if γ_p (G−e) > γ_p(G) for every edge e ∈ E(G).

The edge domination-critical graph was were characterized by Bauer et al. in 1983, that is, a graph is edge domination-critical if and only if it is the union of stars. Suppose that S is a γ_p-set of G. Then every vertex of degree at least two must be in S, and no two vertices in S can be adjacent. Hence G is a union of stars.

For many number of graphs, the power domination number is exceed the range of an edge removal. It is immediate that γ_p(H) > γ_p(G) for any spanning sub graph H of G. Every graph G has a spanning forest T with γ_p(G) = γ_p(T) and so, in general, a graph will have a nonempty set of edges F ∈ E(G) for which γ_p(G − F) = γ_p(G). It is basic alternation to generalized the cancellation of several edges, which is just over to maximize the domination number.

Section 2: Preliminaries

If anyone vertex of a graph G is adjacent with two or more pendant vertices, then $b_p(G)$ = 1. Bauer et al. [1983] found that the star graph is the unique graph with the property that the bondage number is 1 and the deletion of anyone edge results in the domination number maximizing the result. A graph G is called to be equally bonded if it has bondage number b and the cancellation of any number of edge results in a graph with increased domination number.

Definition 2.1: Triangular snake

The triangular snake T_n is formed from the path P_n by replacing every edge of the path by C_3.

Definition 2.2: Quadrilateral snake

A quadrilateral snake Q_n is formed from a path P_n by joining V_i and V_{i+1} to new vertices u_i and w_i respectively and connecting the vertices u_i and w_i for i=1,2,….. n-1. That is every edge of a path is obtained by C_4.

Definition 2.3:

Let $P_{n\,,}$ is the path with the length n (n ≥ 1). If we join every vertex of $P_{n\,,}$ with new two edges, we obtain a new graph which is a tree of size m_1 = 3n + 2 and order n_1 = 3n + 3 called 2-centipede and symbolized by C_n

Definition 2.4:

The graph obtained from a path by attaching exactly two pendant edges to each internal vertex of the path is called a Twig and it is denoted by T(n).

Lemma 2.1: Let H be a spanning subgraph of a nonempty graph G. If $\gamma_p(H) = \gamma_p(G)$ then $b_p(H) \leq b_p(G)$.
Lemma 2.2: If G is a nonempty graph, then $b(G) < \Delta(G) + 1$.
Lemma 2.3: If G has edge connectivity k, then $b(G) < \Delta(G) + k - 1$.
Lemma 2.4: Let H be a spanning subgraph obtained by removing k edges from a graph G. Then $b(G) \leq b(H) + k$.

Section 3: Bondage number of some graph families

For a vertex $v \in V(G)$. Let $N_G(v)$ be the set of all neighbors of and $N_G[v] = N[v] = N_G(v) \cup \{v\}$ be the set of neighbors of v. For a subset $V \subset V(G)$, $N_G(V) = (U_{v\in V} N_G(v)) \cap \bar{V}$, $N_G[V] = N_G(V) \cup \bar{V}$, where $\bar{V} = V(G) \setminus V$. Let E_v be The edge set incident with $v \in V(G)$, that is, $E_v = \{vw \in E(G)) : w \in N_G(v)\}$. We notate the degree of v by $deg_G(x) = |E_v|$. The highest and the lowest degree of G are denoted by $\Delta(G)$ and $\delta(G)$ respectively.

Definition 3.1:

If $N_G[S] = V(G)$, then a subset S in V (G) is called a dominating set of a graph G, i.e. All the vertex v in S has at least one adjacent vertex in S. The domination number of a graph G, declared by γ(G), is the lowest cardinality among all dominating sets. That is,

$$\gamma_p(G) = \text{Min}\{|S| : S \subseteq V(G), N[S] = V(G)\}.$$

Definition 3.2:

A graph is called the critical graph of edge domination if $\gamma_p(G - e_i) > \gamma_p(G)\ \forall\ e_i \in E(G)$. that is, a graph is edge domination- critical if and only if it is the association of stars.
Definition 3.3:

The bondage number b(G) of a graph G(V,E) is the lowest tally of edges whose removal from G in a graph with high domination number. The bondage number is defined by $b(G) = \min\{|B| : B \subseteq E(G), \gamma_p(G - B) > \gamma_p(G)\}$.

Section 3.1:Power bondage number of sunlet graph and its line graph
Theorem 3.1.1: For a m-sunlet graph the bondage number

$$b_p(S_m) = \begin{cases} 3 & \text{if } m \equiv 1(\text{mÒd } 3); \\ 2 & \text{otherwise;} \end{cases}$$

Proof:

The vertices of the m-sunlet graph is labeled by the following method the vertices of the cycle C_m is labeled by $u_1, u_2, \ldots u_m$ and all the pendant vertices are declared by $v_1, v_2, \ldots v_m$ such that $(u_iv_i) \in E(S_n)\ \forall\ 1 \leq i \leq m$. Here $d(u_i) = 3$ and $d(v_i) = 1$.

We claim that $\gamma_p(S_m) = \lceil \frac{m}{3} \rceil ; m \geq 3$
Let H is a spanning subgraph of m-sunlet graph attained by cancelling fewer than $\lceil \frac{m}{3} \rceil$ edges from it. Then the H includes a vertex with degree $m - 1$, it can dominate all the remaining vertices.
If $m \equiv 0,2(\text{mÒd } 3)$Then the spanning subgraph H obtained by cancelling either two adjacent edges or tw or non adjacent edges from G. Then consists of three isolated vertex and a path of order $m - 1$. Thus,

$\gamma_p(H) = 1 + \gamma_p(P_{m-1})$
$= 1 + \lceil \frac{m-1}{3} \rceil$
$= 1 + \lceil \frac{m}{3} \rceil \quad = 1 + \gamma(S_m)$

hence $b_p(S_m) \leq 2$ this results shows the power bondage number for $b_p(S_m) = 2$.
If $m \equiv 1(\text{mÒd } 3)$ the cancellation of three edges from S_m produces a graph H consisting of paths P_{m1}, P_{m2}, P_{m3} and $m_1 + m_2 + m_3 = m$. Then all $m_i \equiv 1(\text{mod } 3)$(i= 1 to 3)
So that,

$\gamma_p(H) = \gamma_p(P_{mi})$
$= \sum_{i=1}^{3} \lceil \frac{mi}{3} \rceil$

$= \lceil \frac{m1+1}{3} \rceil + \lceil \frac{m2+1}{3} \rceil + \lceil \frac{m3+1}{3} \rceil = \lceil \frac{m+1}{3} \rceil = \lceil \frac{m}{3} \rceil$

Let H be the spanning graph got from the cancellation of three consecutive edges of S_m. Then H includes the path of order is $n-2$. Thus,

γ_p (H) = 2 + $\lceil \frac{m-2}{3} \rceil$

$= 2 + \lceil \frac{m-2}{3} \rceil$

$= 2 + \lceil \frac{m}{3} \rceil - 1$

$= 1 + \gamma_p(S_m)$

so that $b_p\ [L(S_m)] \leq 3$ Thus, $b_p\ (L(S_m)) = 3$. Hence proved the result.

Example 3.1.2: For a 7-sunlet graph the power bondage number is 3.

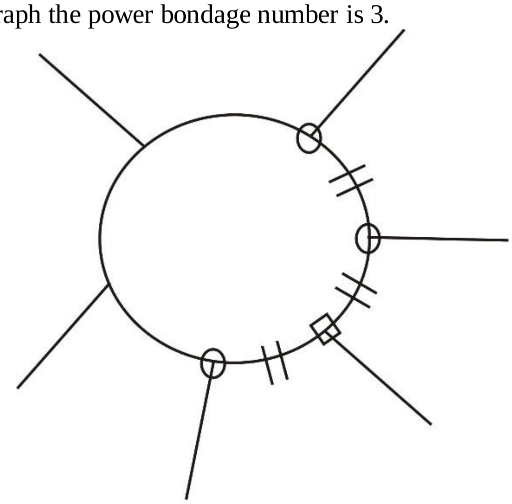

Figure 1: 7-Sunlet graph

Theorem 3.1.3: For a line graph of m-sunlet graph the power bondage number

$$b_p[(L(S_m)] = \begin{cases} 6; \text{ if } m \equiv 1 (\text{mod } 3); \\ 4; \text{ otherwise}; \end{cases}$$

Proof:

Label the vertices of line graph of sunlet graph $L(S_m)$ in the following manner:

Let the vertex set of our line of sunlet graph be { $v_1, v_2, \ldots, v_n$ } . As per the condition of line graph, set of all vertices in a line graph of sunlet graph is equal to the set of all edges in sunlet graph.

That is $V[L(S_m)] = E(S_m)$ then the labeling of vertices and edges of $L(S_m)$ are $v_1, v_2, \ldots, v_n$ and $e_1, e_2, \ldots, e_n$ respectively.

Here $P = 2(n+2)$; and $q = \frac{1}{2}(n+3)(n-2)$ *where* $n \geq 3$.

We know that $\gamma_p(C_k) = \gamma_p(P_k) = \lceil \frac{k}{3} \rceil$; $k \geq 3$ also $b(C_k) \geq 2$

$\Delta[L(S_m)] = 4$ and $\delta[L(S_m)] = 2$.

The degree sequence $\pi(G)$ of a graph G = $L(S_m)$ the vertex set of G is { $v_1, v_2, \ldots, v_n$ } is the order $\pi = \{$ $deg_1, deg_2, \ldots, deg_n\}$ with $deg_1 \leq deg_2 \leq \ldots \leq deg_n$;where $deg_i = deg_G(v_i)\ \forall\ i = 1,2,\ldots,n$

If $m \equiv 0,2 (\text{mod } 3)$Then the spanning subgraph H obtained by cancelling either four adjacent edges or four non adjacent edges from G. Then consists of three isolated vertex and a path of order $m-3$. Thus,

γ_p (H) = 2 + $\gamma_p\ (P_{m-3})$

$= 2 + \lceil \frac{m-3}{3} \rceil$

$= 2 + \lceil \frac{m}{3} \rceil$ (Since $\gamma_p\ (C_k) = \gamma_p\ (P_k) = \lceil \frac{k}{3} \rceil$; $k \geq 3$ also $b(C_k) > 3$)

$= 2+ \gamma_p(L(S_m))$

hence $b_p(L(S_m)) > 3$ this results shows the bondage number for $L(S_m)= 6$ *since* $b_p\ (C_k) > 3$.

If n ≡ 1(mÒd 3) the cancellation of six edges from $L(S_m)$ produces a graph H consisting of paths P_{mi} where i = 1 to 6, and $\sum_{i=1}^{6} m_i = m$. Then all $m_i \equiv 2(\text{mod } 3)$.

So that,

$\gamma_p\ (H) = \sum_{i=1}^{6} \gamma(P_{mi})$

$= \sum_{i=1}^{6} \lceil \frac{mi}{3} \rceil$

$= \lceil \frac{m1+1}{3} \rceil + \lceil \frac{m2+1}{3} \rceil + \lceil \frac{m3+1}{3} \rceil + \lceil \frac{m4+1}{3} \rceil + \lceil \frac{m5+1}{3} \rceil + \lceil \frac{m6+1}{3} \rceil$

$= \lceil \frac{m+1}{3} \rceil = \lceil \frac{m}{3} \rceil$

Let H be the graph got from the cancellation of six consecutive edges of $L(S_m)$. Then H includes the path of order is n − 5. Thus,

$\gamma_p\ (H) = 6 + \lceil \frac{m-6}{3} \rceil$

$= 6 + \lceil \frac{m-5}{3} \rceil$

$= 6 + \lceil \frac{m}{3} \rceil -2 = 4+ \lceil \frac{m}{3} \rceil$

$= 4+ \gamma_p(L(S_m))$

so that b[$L(S_m)$] ≤ 4. Thus, b($L(S_m)$) = 4. Hence proved the result.

Example 3.1.2: Power bondage number for $G[\ L(S_7)]$ is 6.

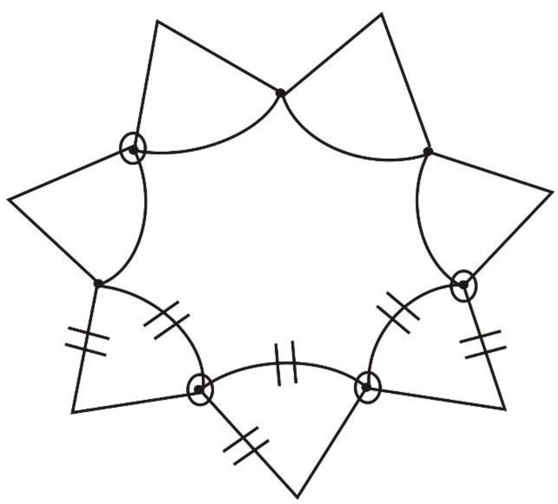

Figure 2: Line graph of 7-Sunlet graph

Section 3.2: Power Bondage number of P-Polygonal snake graphs

Definition 3.2.1: Let us consider the Graph *G(V,E)* = $\mathcal{S}_m(C_n)$ *(n ≥ 3; m≥ 3)* is P-Polygonal Snake. It is formed from the path P_n (m ≥ 2) is changed by cycle C_p. This connected cycle graph form the snake graph and P indicates the number of vertices in cycle.

Theorem 3.2.2: Power bondage number for P-Polygonal snake graph $\mathcal{S}_m(C_n)$ *(n ≥ 3; m≥ 3)* is

$$b_p[(\mathcal{S}_m(C_n))] = \begin{cases} 2; & \text{if } m \equiv 1/2(\text{mÒd } 3); \\ 4; & \text{otherwise;} \end{cases}$$

Proof:

Let us consider $x_1, x_2, \ldots, x_n$ are path vertices and $y_{11}, y_{12}, y_{13}, \ldots, y_{1m}, y_{21}, y_{22}, y_{23}, \ldots, y_{2m}, \ldots \ldots, y_{n1}, y_{n2}, y_{n3}, \ldots, y_{nm}$ are the circuit vertices respectively. By the condition of domination rule the domination number for a circuit with n number of vertices is $\lceil \frac{n}{3} \rceil$ hence the bondage number is less than or equal to 2.

$\Delta[\mathcal{S}_m(C_n)] = 4$ and $\delta[\mathcal{S}_m(C_n)] = 2$.

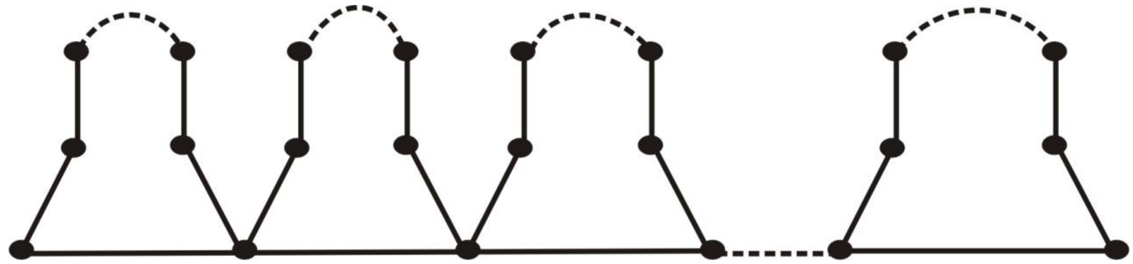

Figure 3: Polygonal snake graph $\mathcal{S}_m(C_n)$

The spanning sub graph H of G is getting from the deletion of some edges from $\mathcal{S}_m(C_n)$
If $m \equiv 1,2$(mÒd 3)Then the spanning subgraph H obtained by cancelling either two adjacent edges or two non adjacent edges from G. Then G consists of two isolated vertex and a path of order $m - 2$. Thus,
γ_p (H) = 1 + γ_p (P_{m-2})
= $1 + \lceil \frac{m-2}{3} \rceil$
= $1 + \lceil \frac{m}{3} \rceil$ (Since $\gamma(C_n) \geq 2$; $n \geq 3$ also $b(C_n) \leq 2$
=$1+\gamma(L(S_m))$
hence $b_p(\mathcal{S}_m(C_n)) \leq 2$ this results shows the bondage number for $L(S_m) = 2$

If $n \equiv 0$(mÒd 3) the cancellation of four edges from $L(S_m)$ produces a graph H consisting of paths P_{mi} where i = 1 to 4, and $\sum_{i=1}^{4} m_i = m$. Then all $m_i \equiv 0 \pmod 3$.
So that,
$\gamma(H) = \sum_{i=1}^{4} \gamma(P_{mi})$
$= \sum_{i=1}^{4} \lceil \frac{mi}{3} \rceil$
$= \lceil \frac{m1+1}{3} \rceil + \lceil \frac{m2+1}{3} \rceil + \lceil \frac{m3+1}{3} \rceil + \lceil \frac{m4+1}{3} \rceil$
$= \lceil \frac{m+1}{3} \rceil = \lceil \frac{m}{3} \rceil$

Let H be the graph got from the cancellation of four consecutive edges of $\mathcal{S}_m(C_n)$. Then H includes order of the path is $n - 3$. Thus,
γ_p (H) = $4 + \lceil \frac{m-3}{3} \rceil$
= $4 + \lceil \frac{m-2}{3} \rceil$
= $4 + \lceil \frac{m}{3} \rceil = 4 + \lceil \frac{m}{3} \rceil$
= $4 + \gamma_p[\mathcal{S}_m(C_n)]$
so that $b_p[\mathcal{S}_m(C_n)] \leq 4$. Thus, b_p ($L(S_m)$) = 4

Example 3.2.3: Power bondage number for Polygonal snake graph $\mathcal{S}_6(C_3)$ is *four*.

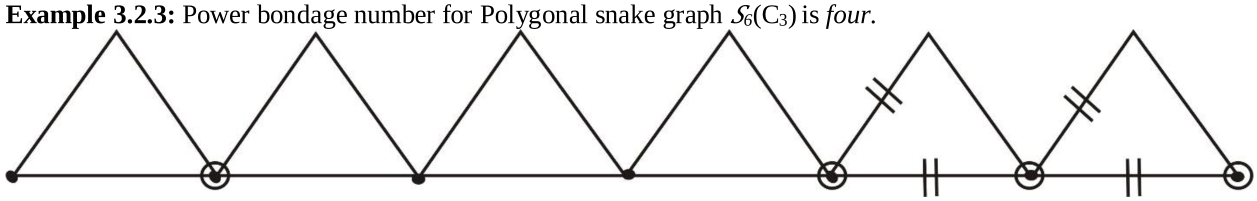

Figure 4: Polygonal snake graph $\mathcal{S}_6(C_3)$

Conclusion:

If the power domination number increases in a graph then the power bondage number of the system is in stable stage. So that the accurate value of power bondage number for a graphs are totally depends upon its power domination number. The power bondage number as a parameter for measuring the problem of the interconnection network under circuit failure problems. we can find the power bondage number for different kinds of graph family.

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