*Research Article*

# Cyclic Group Of Rational Functions With Coefficients As Fibonacci Numbers

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**Article History:** Received: 10 January 2021; Revised: 12 February 2021; Accepted: 27 March 2021; Published online: 16 April 2021

**Abstract :** Group theory is main topic modern algebra and group is also useful in many other fields. In this paper we show a special type relationship between group theory and number theory. In this paper will give type group of rational functions with coefficients as Fibonacci number with respect composition of mapping operation. We also show that is not only a group but also a cyclic group. In this represent a special relation between properties of group and rational functions with Fibonacci numbers coefficients. In this paper we gave a special type of recurrence relation sequence of rational functions with coefficients as Fibonacci numbers and also we proved the collection of all such rational function form a cyclic group with respect to the composition of function operation. In this represent a special relation between properties of group and rational functions with Fibonacci numbers coefficients. In this paper we gave a special type of recurrence relation sequence of rational functions with coefficients as Fibonacci numbers and also we proved the collection of all such rational function form a cyclic group with respect to the composition of function operation.

**Keywords:** Recurrence relation, group, cyclic.

## 1. Introduction

### 1.1. GROUP

In modern algebra a group is a set which satisfied four properties with respect to the given operation. Four axioms are namely

Closure: - let **H** be any set and $*$ be any operation on **H** if $a * b \in \mathbf{H}\ for\ all\ a, b \in \mathbf{H}$ then called **H** satisfied closure property.

Associative: - if $(a * b) * c = a * (b * c)\ for\ all\ a, b, c \in \mathbf{H}$ then called **H** satisfied the associative property.

Existence of Identity: - If there exist an element $e\ in\ \mathbf{H}$ such that $a * e = e * a = a\ \ for\ all\ a \in \mathbf{H}$ then called identity is exist.

Existence of Inverse: - if $for\ all\ \ a \in \mathbf{H}\ there\ exist\ b \in \mathbf{H}\ such\ that\ \ a * b = b * a = e$ then called inverse exist.

### 1.2. Cyclic

If in a group every elements of a group can be generates by single element of group the group called the cyclic group. For example set of integer is a cyclic group with respect to addition.

### 1.3. Fibonacci numbers

In Number Theory there are many special types of numbers according to their special properties. Fibonacci numbers are special type of numbers obtained from recurrence relation with given initial terms. Recurrence relation is an equation that defines a sequence based on a method that gives the next term as relation of the previous terms [2, 3, 4, 7]. Recurrence relations are used in various fields of mathematics.

In Number Theory there are many special types of Sequences of numbers. Fibonacci numbers sequence and Luca numbers Sequence both are special type of recurrence relation numbers with given initial terms. Italian Mathematician Leonardo of Pisa who is also known as by his nickname Fibonacci (1170-1240) he wrote (Book of the Abacus) in 1202. He was 1st European mathematician which work on Indian and Arabian mathematics. He gave a special type sequence

$$F_n = F_{n-1} + F_{n-2} n \geq 2 \qquad (1.1)$$

with initial Term $F_0 = 0$ and $F_1 = 1$

Edouard Lucas dominated the field recursive series during the period 1878-1891 he was 1st mathematician who applied Fibonacci's name for sequence (1.1) and it has been known as Fibonacci sequence since then. Lucas sequence defined by the recurrence relation [5, 6, 8, 11, 17]

$L_n = L_{n-1} + L_{n-2} n \geq 2$ (1.2)
With initial term, $L_0 = 2 \qquad L_1 = 1$

Terms of the Lucas sequence are called Lucas numbers. Binet forms of $nth$ Fibonacci and $nth$Lucas numbers were given by Bernoulli (1724) and Euler (1726) respectively [9,10,12,13]

**1.4. Rational function**

$$f(x) = \frac{p(x)}{q(x)}$$

$where\ p(x)\ and\ \ q(x)\ are\ polynomials\ then\ f(x) is\ called\ the\ rational\ function.$

In this paper we will show that a relationship between group theory and number theory. We will represent a special type sequence of rational functions with coefficients as Fibonacci numbers. We will prove collection of all rational functions defined by us form a cyclic group.

**1.5 Generalized Fibonacci sequences**

Generalized Fibonacci sequence [13,16], is defined as
$F_k = pF_{k-1} + qF_{k-2}\ , k \geq 2\ with\ F_0 = a, F_1 = b$
where p, q, a & b are positive integers [1,14,15,]
For different values of p, q, a & b many sequences can be determined.
We will focus on two cases of sequences $\{V_k\}_{k\geq 0}$ and $\{U_k\}_{k\geq 0}$ which generated in
If p = 1 , q = a = b = 2 , then we get

$$V_k = V_{k-1} + 2V_{k-2}\ , k \geq 2\ with\ v_0 = 2, V_1 = 2$$

The first few terms of $\{V_k\}_{k\geq 0}$ are 2, 2, 6, 10, 22, 42 and so on.
If p = 1, q = a = 2 , b = 0 , then we get

$$U_k = U_{k-1} + 2U_{k-2}\ , for\ \ k \geq 2\ with\ U_0 = 2, U_1 = 0$$

The first few terms of $\{U_k\}_{k\geq 0}$ are 2, 0, 4, 4, 12, 20 and so on.

**2. Main result of paper**

Consider a real valued function $u: (0, \infty) \to (0,1)$ given by

$$u(x) = \frac{1}{1+x}$$

This function is clearly continuous on its domain. Clearly $codmain\ of\ u\ subset\ of\ domain\ of\ u$ so consider function

$$u\ o\ u = \frac{1}{1+\frac{1}{1+x}}$$

Now we define $z_k(x) = (uououou \ldots ou)(x), where\ there\ are\ (k)\ compositions.$
Now we define a recurrence relation sequence of rational function

$$z_1(x) = u(x) = \frac{1}{1+x} \quad \text{and}$$

$$z_k(z) = \frac{1}{1+z_{k-1}(x)} \quad \text{for all } k \geq 2$$

Now we shall show that every member of this family has Fibonacci coefficients. For this purpose we define the Fibonacci sequence starting 0,1,1,2,3,5,8,13 ...
Where $f_0 = 0\ \ and\ f_1 = 1\ and\ \ f_n = f_{n-1} + f_{n-2}\ \ for\ all\ n \geq 2$
Now we have $z_n(x) = \frac{f_{n-1}\,x+f_n}{f_n\,x+f_{n+1}}$ [1], where
$f_i\ is\ (i)th\ Fibonacci\ number\ and\ \ \ z_n(x)\ nth\ term\ of\ above\ defined\ sequence\ of\ rational\ functions$
For any $n \in N\ , the\ codomain\ of\ z_n(x)\ is$
$A_n = \left(\min\left\{\frac{f_{n-1}}{f_n}, \frac{f_n}{f_{n+1}}\right\}, \max\left\{\frac{f_{n-1}}{f_n}, \frac{f_n}{f_{n+1}}\right\}\right)$ [1]
For example we can say that

$$codomain\ of\ z_1(x)\ is\ \ A_1 = (0,1)$$

$$codomain\ of\ z_2(x)\ is\ \ A_2 = \left(\frac{1}{2}, 1\right)$$

$$codomain\ of\ z_3(x)\ is\ \ A_3 = \left(\frac{1}{2}, \frac{2}{3}\right)$$

And so on we can find co-domain of all functions.
In particular if k is odd
$A_n = \left(\frac{f_{n-1}}{f_n} \frac{f_n}{f_{n+1}}\right)$

If n is even then
$A_n = \left(\frac{f_n}{f_{n+1}}, \frac{f_{n-1}}{f_n}\right)$
Let $I\colon (0,\infty) \to (0,\infty)\ such\ that\ I(x) = x$
Let ***G*** be set of all $z_n(x)\ for\ all\ n$ and including **I** function defined above. Now will prove ***G*** is cyclic group with respect composition operation.
**Closure: -** let $z_n\ and\ z_m$ any two function in ***G*** then we according to definition

$$z_n(x) = (uououou \ldots ou)(x), where\ there\ are\ (n)\ compositions.$$
$$z_m(x) = (uououou \ldots ou)(x), where\ there\ are\ (m)\ compositions.$$
$$(z_n o z_m)\ (x) = (uououou \ldots ou)(x), where\ there\ are\ (m+n)\ compositions.$$
$$(z_n o z_m)\ (x) = z_{m+n}(x) \in \boldsymbol{G}$$

So closure property is satisfied.
**Associative: -** all compositions are of u so associative property is clearly satisfied.
**Existence of Identity: -** since ***G*** is including **I** so clearly identity is exist.

**Inverse: -** First we will prove all functions are one-one onto.
$z_n(x) = \frac{f_{n-1}\,x + f_n}{f_n\,x + f_{n+1}}$ , $z_n(y) = \frac{f_{n-1}\,y + f_n}{f_n\,y + f_{n+1}}$
$z_n(x) = z_n(y)$
$\frac{f_{n-1}\,x + f_n}{f_n\,x + f_{n+1}} = \frac{f_{n-1}\,y + f_n}{f_n\,y + f_{n+1}}$ After solving this we have $x = y$ so we can say that all function all one-one.
**Let** $\frac{f_{n-1}\,x + f_n}{f_n\,x + f_{n+1}} = y$ solving this we have $x = \frac{f_{n+1}\,y - f_n}{f_{n-1} - f_n y}$
Let if possible $f_{n-1} - f_n y = 0\ this\ implies\ y = \frac{f_{n-1}}{f_n} \notin A_n\ , clesrly\ x > 0\ for\ all\ y\ in\ A_n.$

$$so\ we\ can\ say\ that\ every\ elements\ of\ A_n\ have\ pre-image\ under\ \ z_n$$

So we can say that $z_n\ is\ onto\ for\ all\ n$
So we can say that every member of ***G*** is one-one and onto. So we can say that every member of ***G*** is invertible.
Cyclic Property:- every member can be generates by $z_1(x) = u(x) = \frac{1}{1+x}$ so we can say that ***G*** is a cyclic group under the composition operation

## 1. 2nd Main result of paper

Consider a real valued function $u\colon (0,\infty) \to (0,1)$ given by

$$u(x) = \frac{1}{q+x}$$

This function is clearly continuous on its domain. Clearly $codmain\ of\ u\ subset\ of\ domain\ of\ u$ so consider function

$$u\ o\ u = \frac{1}{1+\frac{1}{q+x}}$$

Now we define $z_k(x) = (uououou \ldots ou)(x), where\ there\ are\ (k)\ compositions.$
Now we define a recurrence relation sequence of rational function

$$z_1(x) = u(x) = \frac{1}{q+x} \quad \text{and}$$
$$z_k(x) = \frac{1}{q+z_{k-1}(x)} \quad \text{For all } k \geq 2$$

Now we shall show that every member of this family has Generalized Fibonacci coefficients. For this purpose we define the Generalized Fibonacci sequence starting
With $f_0 = 0\ \ and\ f_1 = 1\ and\ \ f_n = qf_{n-1} + f_{n-2}\ \ for\ all\ n \geq 2$ ,where q is any positive integer.
Now we will prove $z_n(x) = \frac{f_{n-1}\,x + f_n}{f_n\,x + f_{n+1}}$ , where (1)

$$f_i\ is\ (i)th\ Generalized\ Fibonacci\ number$$

$$z_n(x)\ nth\ term\ of\ above\ defined\ sequence\ of\ rational\ functions$$

We will prove result (1) by principal mathematical induction (PMI).

For $n = 1$ $z_1(x) = u(x) = \frac{1}{q+x}$ and $f_0 = 0$ , $f_1 = 1$ $and$ $f_2 = q$ so we can say that (1) result is true for $n = 1$

Now suppose that result is true for $n = k$ so let $z_k(x) = \frac{f_{k-1}x+f_k}{f_k x+f_{k+1}}$

Now we will show that result is also true for $n = k + 1$

Consider $z_{k+1}(x) = \frac{1}{1+z_k(x)}$ now put the value of $z_k(x) = \frac{f_{k-1}x+f_k}{f_k x+f_{k+1}}$

We have $z_{k+1}(x) = \frac{1}{q+\frac{f_{k-1}x+f_k}{f_k x+f_{k+1}}} = \frac{f_k x + f_{k+1}}{(qf_k + f_{k-1})x + (qf_{k+1} + f_k)} = \frac{f_k x+f_{k+1}}{f_{k+1}x+f_{k+2}}$

So result is also true for $n = k + 1$ so we can say that result is true for all positive integer k by PMI.

**Theorem**: - prove that $z_k(x)$ are monotonic functions. Particularly if $k$ is odd then $z_k(x)$ is monotonically decreasing and if k is even then $z_k(x)$ is monotonically increasing.

Proof: - clearly $z_k(x)$ all are differentiable on given domain. So this theorem we will prove by First derivative test.

We have $\frac{dz_1}{dx} = \frac{-1}{(q+x)^2} < 0$ , so $z_1(x)$ is clearly monotonically decreasing function by First derivative test.

Now by recurrence relation of functions we have

$$\frac{dz_k}{dx} = \frac{-1}{(q+z_{k-1})^2}\frac{dz_{k-1}}{dx}$$

So we have $$sgn\left[\frac{dz_k}{dx}\right] = -sgn\left[\frac{dz_{k-1}}{dx}\right]$$

So we can say that if $k$ is odd then $\frac{dz_k}{dx} < o$

If $k$ is odd then $\frac{dz_k}{dx} > 0$

So finally we can say that if $k$ is odd then $z_k(x)$ is monotonically decreasing and if k is even then $z_k(x)$ is monotonically increasing.

**Corollary**: - For any $k \in N$ , $the\ range\ set\ A_k\ of\ z_k(x)\ is$

If k is odd $A_k = \left(\frac{f_{k-1}}{f_k}\frac{f_k}{f_{k+1}}\right)$ , if n is even then $A_k = \left(\frac{f_k}{f_{k+1}}, \frac{f_{k-1}}{f_k}\right)$

Proof: - let $k$ is odd then $z_k(x)$ is monotonically decreasing and we have

$$z_k(x) = \frac{f_{k-1}x+f_k}{f_k x+f_{k+1}}$$

So $z_k(x)$ approach to its maximum value as $x \to 0$ so we can say that maximum value of $z_k(x) \to \frac{f_k}{f_{k+1}}$ and $z_k(x)$ approach to its minimum value as $x \to \infty$ so we can say that minimum value of $z_k(x) \to \frac{f_{k-1}}{f_k}$ , so finally we can say that $the\ range\ set\ A_k\ of\ z_k(x)\ is$

$A_k = \left(\frac{f_{k-1}}{f_k}\frac{f_k}{f_{k+1}}\right)$ if k is odd.

Let $k$ is even then $z_k(x)$ is monotonically increasing and we have

$$z_k(x) = \frac{f_{k-1}x+f_k}{f_k x+f_{k+1}}$$

So $z_k(x)$ approach to its minimum value as $x \to 0$ so we can say that minimum value of $z_k(x) \to \frac{f_k}{f_{k+1}}$ and $z_k(x)$ approach to its maximum value as $x \to \infty$ so we can say that maximum value of $z_k(x) \to \frac{f_{k-1}}{f_k}$ , so finally we can say that $the\ range\ set\ A_k\ of\ z_k(x)\ is$

In particular if k is odd

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## 2. Conclusion

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