Research Article

Analysis Of The M/G/1 Queue With Setup Costs In Fuzzy Environments Using Parametric Nonlinear Programming

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Abstract: Queueing models are used to model various real-life situations and finds various applications in engineering and the sciences. To account for the inherent imprecision in real world data, one resorts to fuzzy set theoretic methods, which are extremely versatile. In this paper, we analyze the M/G/1 queueing system with setup costs in fuzzy environments. A solution procedure is proposed to achieve the primary goal of determining the fuzzy performance measures of the system. The problem is reduced effectively to the problem of determining the solutions to a pair of parametric nonlinear programs. The graded mean integration scheme is also used for defuzzification of the fuzzy characteristics. An example is presented to illustrate the proposed solution procedure.

Keywords: M/G/1 queues, parametric nonlinear programming, fuzzy sets

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1. Introduction

Zadeh [1] in 1965 introduced the concept of fuzzy sets. Following Zadeh, the theory of fuzzy sets was developed by around 300 researchers – De Kerf [2], Kaufmann [3], Dubois [4], Prade [5], Mizumoto and Tanaka [6] to name a few. These researchers have played crucial roles in shaping the modern theory.

Fuzzy set theory provides a mathematical framework that formalizes the notion of uncertainty and vagueness in data. One can transform real-life situations which involve uncertain or fuzzy data into formal mathematical models. The theory finds many applications in several areas of science and engineering such as statistics, logic, control theory, communication networks, neural networks, operations research etc. Queueing theory is one notable field in which fuzzy set theoretic techniques have been applied and put to great use. Notable researchers in fuzzy queueing theory include Li and Lee [7], Negi and Lee [8], Buckley [9,10] and Chen [11].

Probability theory is central to the subject of classical queueing theory. The parameters of the probability distributions used in classical queueing models are crisp real numbers. Real-life data on the other hand, is inherently fuzzy. Thus, in real-life situations, the parameters involved are often described using linguistic terms, and thus fuzzy set theory comes into picture.

A popular approach to analyzing queueing models in fuzzy environments is parametric nonlinear programming [11,12]. In this paper, we use parametric nonlinear programming to study and analyze the M/G/1 queueing model with setup costs [13] in fuzzy environments. This paper is organized as follows. Sec. 2 and 3 discuss in brief the necessary preliminaries, Sec. 4 describes the queueing model in discussion, Sec. 5 presents the proposed solution procedure, and Sec. 6 illustrates a numerical example. Sec. 7 concludes the study.

2. Preliminaries

2.1 Fuzzy set theoretic definitions

A *fuzzy set* $\tilde{A}$ [14] defined on the *universe* or the *domain of discourse* X is characterized by a function A that maps the universe into the closed interval $[0,1]$. The function A is called the *membership function* associated with the fuzzy set $\tilde{A}$. The value $A(x) \in [0,1]$ of the membership function at a point x of the universe is interpreted as the extent of membership of x in $\tilde{A}$, or its *membership grade*. A *fuzzy subset of* X is a fuzzy set whose universe is X.

A useful family of crisp sets associated with a fuzzy set $\tilde{A}$ are its α-cuts. For $0 \le \alpha \le 1$, the (*weak) α-cut* of $\tilde{A}$, denoted ${}^{\alpha}\tilde{A}$, is defined by

$$ {}^{\alpha}\tilde{A} := \{x \mid x \in X, A(x) \ge \alpha\} \subseteq X \ . \qquad (1)$$

Similarly, the *strong α-cut* of $\tilde{A}$, denoted ${}^{\alpha+}\tilde{A}$, is defined by

$$ {}^{\alpha+}\tilde{A} := \{x \mid x \in X, A(x) > \alpha\} \subseteq X . \quad (2)$$

The α-cuts of a fuzzy set $\tilde{A}$ are nested, *i.e.* for reals α, β, one has

$$0 \le \alpha < \beta \le 1 \Rightarrow {}^{\beta}\tilde{A} \subseteq {}^{\alpha}\tilde{A}. \quad (3)$$

Special α-cuts of $\tilde{A}$ include the *support* and the *core*, denoted $\operatorname{supp}\tilde{A}$ and $\operatorname{core}\tilde{A}$ respectively. These are defined as

$$\operatorname{core}\tilde{A} := {}^{1}\tilde{A}, \quad \text{and} \quad \operatorname{supp}\tilde{A} := {}^{0+}\tilde{A}. \quad (4)$$

A useful quantity associated with a fuzzy set is its height h. It is defined as

$$h\left(\tilde{A}\right) := \sup\left\{A(x) : x \in X\right\}. \quad (5)$$

A fuzzy set is called *normal* iff its height is one.

Fuzzy sets whose α-cuts are all convex in X are called *convex*. A characterization of the convex fuzzy sets is the following: the fuzzy set $\tilde{A}$ is convex iff for each $\beta \in [0,1]$ and for all elements $x_1, x_2 \in X$, we have that

$$A\left(\beta x_1 + (1-\beta)x_2\right) \ge \min\left\{A(x_1), A(x_2)\right\}. \quad (6)$$

2.2 Zadeh's extension principle

Let f be a real-valued function that takes n real arguments, *i.e.* $f : \mathbb{R}^n \to \mathbb{R}$. The function that extends the definition of f so as to admit fuzzy inputs to produce a fuzzy output is called the *fuzzy extension* of f. Such an extension was first provided by Zadeh, through his *extension principle* [14].

Formally, suppose that $\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_n$ are fuzzy subsets of $\mathbb{R}$. Then, $\tilde{B} = f\left(\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_n\right)$ for some fuzzy subset $\tilde{B}$ of $\mathbb{R}$ (the f here is its fuzzy extension – we will not distinguish between the two to avoid possible confusion). The extension principle defines $\tilde{B}$ through the equation in Eq. (7).

$$B(x) = \sup\left\{ \min_{1\le i\le n} A_i(t_i) \;\middle|\; f(t_1, t_2, \cdots, t_n) = x, \text{ where each } t_i \in \mathbb{R} \right\}, \forall x \in \mathbb{R}, \quad (7)$$

along with the convention that $\sup\varnothing = 0$.

2.3 Fuzzy numbers

A fuzzy number [14] is a normal fuzzy subset $\tilde{A}$ of the real numbers $\mathbb{R}$ that satisfies the following properties:

(i) $\operatorname{supp}\tilde{A}$ is bounded, and

(ii) ${}^{\alpha}\tilde{A}$ is a closed interval in $\mathbb{R}$ for $\alpha \in (0,1]$.

Since intervals are convex, it immediately follows that fuzzy numbers are convex.

2.4 Operations on fuzzy numbers

Suppose that the function f is *continuous*, and that the inputs to f are *fuzzy numbers* $\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_n$, with $\tilde{B} = f\left(\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_n\right)$. In principle, $\tilde{B}$ is defined by Zadeh's extension principle (see Eq. (7)), which is extremely difficult to implement and parse. A relatively easier and equivalent approach that uses α-cuts is due to Buckley and Qu [15]. Their result states that $\tilde{B}$ is a fuzzy number defined by

$${}^{\alpha}\tilde{B} = \left\{ x \in \mathbb{R} \;\middle|\; x = f(a_1, a_2, \cdots, a_n) \text{ with } a_i \in {}^{\alpha}\tilde{A}_i \ \forall i \right\}, \text{ for } 0 < \alpha \le 1. \quad (8)$$

This result is very versatile in that it enables one to perform various operations on fuzzy numbers.

3. Trapezoidal Fuzzy Numbers

The fuzzy number $\tilde{B}$ is called *trapezoidal* [16] iff

$$B(x) = \begin{cases} \dfrac{x-\ell_1}{\ell_2-\ell_1} & \text{for } \ell_1 \le x \le \ell_2 \\ 1 & \text{for } \ell_2 \le x \le \ell_3 \\ \dfrac{\ell_4-x}{\ell_4-\ell_3} & \text{for } \ell_3 \le x \le \ell_4 \end{cases} \text{ and zero otherwise,} \quad (9)$$

for some reals $\ell_1, \ell_2, \ell_3, \ell_4 \in \mathbb{R}$ that satisfy $\ell_1 < \ell_2 \le \ell_3 < \ell_4$. We will denote such fuzzy numbers $\tilde{B}$ as $\tilde{B} = (\ell_1, \ell_2, \ell_3, \ell_4)$ for brevity.

It is easily seen that $\operatorname{supp} \tilde{B} = (\ell_1, \ell_4)$, and that $\operatorname{core} \tilde{B} = [\ell_2, \ell_3]$. Also, the α-cut of $\tilde{B}$ is given by the interval

$$^{\alpha}\tilde{B} = [\ell_1 + (\ell_2 - \ell_1)\alpha,\ \ell_4 - (\ell_4 - \ell_3)\alpha], \text{ for } \alpha \in (0,1]. \tag{10}$$

4. The M/G/1 queue with setup times

4.1 Basic description

Our considerations will be based on the *M/G/1 queueing system*, with *setup costs* (time cost), in fuzzy environments. The M/G/1 queue [17] consists of a single server, whose service time is modelled as a random variable T. The distribution of T is assumed to be a general one, *i.e.* $T \sim G$ for some distribution G. We also assume that the arrivals are *Poisson* distributed with rate parameter λ. There is also a time cost associated with the server. The server is *"switched off"* if there are no customers in the system. The next arrival to the queue *"activates"* the server, but it takes a non-zero amount of time until the service begins. This time is called the *setup time* and is modelled as a random variable S. It is also assumed that the service times and the interarrival times are independent, and that the queueing discipline is *first-come first-served* in nature.

This queueing system has been widely studied, and its performance measures in steady state are already known. One must keep in mind that steady state demands that $\lambda E[T] < 1$.

4.2 Relevant results

We now state the formulae for a few performance measures (in steady state) of the above queueing system [18,19].

1. The average time that a customer spends in the queue, W^q, is given by

$$W^q = \frac{\lambda}{2}\left(\frac{\operatorname{var} T + E[T]^2}{1 - \lambda E[T]}\right) + \frac{\lambda}{2}\left(\frac{\operatorname{var} S - E[S]^2}{1 + \lambda E[S]}\right) + E[S]. \tag{11}$$

2. The average time that a customer spends in the system, W^s, is given by

$$W^s = W^q + E[S]. \tag{12}$$

3. The queue length L^q and the number of customers in the system L^s are given by Little's formulae:

$$L^q = \lambda W^q \text{ and } L^s = \lambda W^s. \tag{13}$$

We shall study the above queueing system in fuzzy environments, wherein all system parameters are intrinsically fuzzy. Our primary goal is to extend the formulae for the above performance measures to admit fuzzy inputs, to produce fuzzy outputs.

5. Solution procedure

The objective is to satisfactorily analyze the system in question in fuzzy environments, in steady state. We shall model any and all fuzziness in the parameters of the system using fuzzy numbers. Henceforth, the Poisson arrival rate shall be denoted $\tilde{\omega} := \lambda$, and the central moments of the service time distribution shall be denoted $\tilde{\tau}_T := E[T]$ and $\tilde{\nu}_T := \operatorname{var} T$. The fuzzy numbers $\tilde{\tau}_S$ and $\tilde{\nu}_S$ are defined similarly. The only (fuzzy) parameters of the system are $\tilde{\omega}$, $\tilde{\tau}_T$, $\tilde{\tau}_S$, $\tilde{\nu}_T$ and $\tilde{\nu}_S$.

It is important to keep in mind that the existence of steady state demands that the traffic intensity ρ ($:= \lambda E[T]$ for a crisp queue) of the system be lesser than one. When the system parameters are fuzzy, this translates to $\sup(\operatorname{supp} \tilde{\omega}) \cdot \sup(\operatorname{supp} \tilde{\tau}_T) < 1$. It should also be noted that if either of the distributions of the random variables T and S is parametrized by a single variable, then the associated mean and variance are not independent fuzzy quantities, and thus the number of parameters of the system falls by one. However, we shall work in a general setting, and thus assume that the mean and variance are independent quantities.

Now, the primary goal is to extend the formulae for the performance measures in the previous section to hold in fuzzy environments, so that they admit fuzzy numbers as input and produce fuzzy outputs. More precisely, if $\tilde{z}$ denotes a fuzzified performance measure of interest, given the membership functions ω, τ_T, τ_S, ν_T and ν_S, and the function $f : \mathbb{R}^5 \to \mathbb{R}$ that relates the crisp parameters of the system to the crisp performance measure of interest, we are to construct the membership function z of the fuzzified performance measure $\tilde{z}$.

Theoretically, we can achieve this by appealing to Zadeh's extension principle, defined by Eq. (7). We have

$$z(y) = \sup\left\{\min\left\{\omega(x_1), \tau_T(x_2), \tau_S(x_3), \nu_T(x_4), \nu_S(x_5)\right\} : \underline{x} \in \mathbb{R}^5 \text{ with } f(\underline{x}) = y\right\} \tag{14}$$

for each $y \in \mathbb{R}$, where $\underline{x} = (x_1, x_2, x_3, x_4, x_5)$. We also use the convention that $\sup \varnothing = 0$. But this approach is extremely difficult to implement and use. Thus, we appeal to the result due to Buckley and Qu in Eq. (8), which uses α-cuts rather than defining the membership function of the output pointwise. Applying this result yields

$$^{\alpha}\tilde{z} = \left\{y = f(x_1, x_2, x_3, x_4, x_5) : x_1 \in {^{\alpha}\tilde{\omega}},\ x_2 \in {^{\alpha}\tilde{\tau}_T},\ x_3 \in {^{\alpha}\tilde{\tau}_S},\ x_4 \in {^{\alpha}\tilde{\nu}_T} \text{ and } x_5 \in {^{\alpha}\tilde{\nu}_S}\right\} =: S_{\alpha} \tag{15}$$

for $0 < \alpha \le 1$, where the fuzzy output $\tilde{z}$ is a fuzzy number. This equation is simpler in form and is easier to interpret. Henceforth, it will be assumed that $0 < \alpha \le 1$ unless stated otherwise. Since $\tilde{z}$ is a fuzzy number, its α-cuts are closed intervals in $\mathbb{R}$. Therefore, ${}^{\alpha}\tilde{z} = \left[z_\alpha^L, z_\alpha^U \right]$ with

$$z_\alpha^L = \min {}^{\alpha}\tilde{z} = \min\left\{ y \in \mathbb{R} \mid z(y) \ge \alpha \right\} \text{ and } z_\alpha^U = \max {}^{\alpha}\tilde{z} = \max\left\{ y \in \mathbb{R} \mid z(y) \ge \alpha \right\} \tag{16}$$

We can also write similar equations for the fuzzy rates. Combining Eqs. (15) and (16) for the interval ${}^{\alpha}\tilde{z}$, namely that ${}^{\alpha}\tilde{z} = [z_\alpha^L, z_\alpha^U]$ and ${}^{\alpha}\tilde{z} = S_\alpha$, we get

$$\begin{aligned} z_\alpha^L &= \min S_\alpha \\ &= \min f(x_1, x_2, x_3, x_4, x_5) \\ &\quad \text{subject to } x_1 \in {}^{\alpha}\tilde{\omega},\ x_2 \in {}^{\alpha}\tilde{\tau}_T,\ x_3 \in {}^{\alpha}\tilde{\tau}_S,\ x_4 \in {}^{\alpha}\tilde{\nu}_T \text{ and } x_5 \in {}^{\alpha}\tilde{\nu}_S \end{aligned} \tag{17}$$

$$\begin{aligned} z_\alpha^U &= \max S_\alpha \\ &= \max f(x_1, x_2, x_3, x_4, x_5) \\ &\quad \text{subject to } x_1 \in {}^{\alpha}\tilde{\omega},\ x_2 \in {}^{\alpha}\tilde{\tau}_T,\ x_3 \in {}^{\alpha}\tilde{\tau}_S,\ x_4 \in {}^{\alpha}\tilde{\nu}_T \text{ and } x_5 \in {}^{\alpha}\tilde{\nu}_S \end{aligned} \tag{18}$$

Equivalently, we can write

$$\begin{aligned} z_\alpha^L &= \min f(x_1, x_2, x_3, x_4, x_5) \\ &\quad \text{subject to } \omega_\alpha^L \le x_1 \le \omega_\alpha^U \\ &\qquad (\tau_T)_\alpha^L \le x_2 \le (\tau_T)_\alpha^U \\ &\qquad (\tau_S)_\alpha^L \le x_3 \le (\tau_S)_\alpha^U \\ &\qquad (\nu_T)_\alpha^L \le x_4 \le (\nu_T)_\alpha^U \\ &\qquad (\nu_S)_\alpha^L \le x_5 \le (\nu_S)_\alpha^U \end{aligned} \tag{19}$$

$$\begin{aligned} z_\alpha^U &= \max f(x_1, x_2, x_3, x_4, x_5) \\ &\quad \text{subject to } \omega_\alpha^L \le x_1 \le \omega_\alpha^U \\ &\qquad (\tau_T)_\alpha^L \le x_2 \le (\tau_T)_\alpha^U \\ &\qquad (\tau_S)_\alpha^L \le x_3 \le (\tau_S)_\alpha^U \\ &\qquad (\nu_T)_\alpha^L \le x_4 \le (\nu_T)_\alpha^U \\ &\qquad (\nu_S)_\alpha^L \le x_5 \le (\nu_S)_\alpha^U \end{aligned} \tag{20}$$

Thus, to determine the function z, it suffices to solve the optimization problems in Eqs. (19) and (20). These problems constitute a pair of parametric nonlinear programs – parametric because the feasible region is parametrized by a confidence level $\alpha \in (0,1]$, and nonlinear since the function f is nonlinear in general. The function f is continuous on its domain (in particular, it is continuous on the feasible region), and the feasible region is closed and bounded in $\mathbb{R}^5$. Therefore, the extreme value theorem guarantees the solvability of the programs in Eqs. (19) and (20).

We also define the quantities z_0^L and z_0^U as the numbers that one obtains when zero is substituted for α and simplified in the expressions for the solutions z_α^L and z_α^U to the programs in Eqs. (19) and (20). Due to continuity, it is easily seen that these numbers are the endpoints of the support of $\tilde{z}$.

It remains to construct the map z. We now exploit the nested structure of the α-cuts of the fuzzy number $\tilde{z}$, as in Eq. (3). For real numbers α, β with $0 \le \alpha < \beta \le 1$, we have

$${}^{\beta}\tilde{z} = \left[z_\beta^L, z_\beta^U \right] \subset \left[z_\alpha^L, z_\alpha^U \right] = {}^{\alpha}\tilde{z}. \tag{21}$$

Notice how the inclusion is strict – this is not necessarily the case for general fuzzy sets but holds necessarily for fuzzy numbers. Now, we turn our attention to the maps $\alpha \mapsto z_\alpha^L$ and $\alpha \mapsto z_\alpha^U$. The above inclusion immediately implies that these maps are strictly increasing and strictly decreasing, respectively. Therefore, these functions are injections, and thus are invertible on their respective ranges. Denote the inverses by $L : \left[z_0^L, z_1^L \right] \to [0,1]$ and

$R:\left[z_1^U, z_0^U\right] \to [0,1]$ respectively. Then, clearly, by definition of the α-cut, it follows that the membership function z of $\tilde{z}$ is expressible as

$$z(y)=\begin{cases} L(y) & z_0^L \le y \le z_1^L \\ 1 & z_1^L \le y \le z_1^U \\ R(y) & z_1^U \le y \le z_0^U \end{cases}, \text{ and zero otherwise.} \tag{22}$$

It is not easy to obtain simple closed-form expressions for $L(y)$ and $R(y)$, and thus one resorts to numerical approximations. The collection of intervals (the α-cuts)

$$\left\{ \left[z_\alpha^L, z_\alpha^U\right] \;\middle|\; \alpha \in [0,1] \right\} \tag{23}$$

can be used to arrive at approximate plot of the map z by performing interpolation on a finite subset of the collection.

Finally, we *defuzzify* the fuzzy output, namely the performance measure of interest, into a crisp value for practical use. All input fuzziness is encoded in the defuzzified value. There are several defuzzification techniques available in the literature. We shall use the *graded mean integration scheme*, which defuzzifies a given fuzzy number $\tilde{z}$ into a crisp quantity by means of the formula

$$\phi(\tilde{z})=\frac{\int_0^1 \frac{\alpha}{2}\left(z_\alpha^L+z_\alpha^U\right) d\alpha}{\int_0^1 \alpha \, d\alpha}=\int_0^1 \alpha\cdot\left(z_\alpha^L+z_\alpha^U\right) d\alpha \tag{24}$$

where $\phi(\tilde{z})$ is the defuzzified value.

6. Numerical example

We now present a numerical example that illustrates the proposed solution procedure. We assume that the arrival rate is a trapezoidal fuzzy number given by

$$\lambda=\tilde{\omega}=\left(\frac{2}{60},\frac{3}{60},\frac{3}{60},\frac{4}{60}\right) \text{min}^{-1}. \tag{25}$$

We also assume that the setup times are exponentially distributed with expected value $E[S] = \tilde{\tau}_S = (10,12,13,15)$ min . Note that the exponential distribution is a single parameter distribution, and thus its variance and mean are dependent. Indeed, we have

$$\operatorname{var} S = E[S]^2. \tag{26}$$

Further, we assume that the service times are distributed so that $E[T]=\tilde{\tau}_T=(9,10,11,12)$ min and $\operatorname{var} T=\tilde{v}_T=(1,2,2,3)$ min^2 . We shall construct the fuzzified queue length $\tilde{L}$ and the fuzzified waiting time $\tilde{W}$ (in the queue) in steady state. To this end, we first write down the α-cuts of the system parameters. Henceforth, we shall assume $0<\alpha\le 1$. We have

$${}^\alpha\tilde{\omega}=\left[\frac{2+\alpha}{60},\frac{4-\alpha}{60}\right],\ {}^\alpha\tilde{\tau}_T=[9+\alpha,12-\alpha],\ {}^\alpha\tilde{\tau}_S=[10+2\alpha,15-2\alpha],\ {}^\alpha\tilde{v}_T=[1+\alpha,3-\alpha] \tag{27}$$

Now, suppose that f and g are the functions that relate the crisp system parameters, namely the arrival rate, the mean service time, the mean setup time and the service time variance in that order with the crisp queue length and the crisp queue waiting time. Then, using Eq. (26) and Eqs. (11), (12) and (13), we see that

$$\begin{aligned} f(x_1,x_2,x_3,x_4)&=\frac{x_1^2}{2}\left(\frac{x_2^2+x_4}{1-x_1x_2}\right)+x_1x_3 \\ g(x_1,x_2,x_3,x_4)&=\frac{x_1}{2}\left(\frac{x_2^2+x_4}{1-x_1x_2}\right)+x_3. \end{aligned} \tag{28}$$

Our considerations in the previous section applied to the present case (cf. Eqs. (19) and (20)) yield the following pairs of parametric nonlinear programs for the α-cuts of the two performance measures:

$$L_\alpha^L = \min\ f(x_1,x_2,x_3,x_4) = \frac{x_1^2}{2}\left(\frac{x_2^2+x_4}{1-x_1x_2}\right)+x_1x_3$$
$$\text{subject to } \frac{2+\alpha}{60} \le x_1 \le \frac{4-\alpha}{60}$$
$$9+\alpha \le x_2 \le 12-\alpha \tag{29}$$
$$10+2\alpha \le x_3 \le 15-2\alpha$$
$$1+\alpha \le x_4 \le 3-\alpha$$

$$L_\alpha^U = \max\ f(x_1,x_2,x_3,x_4) = \frac{x_1^2}{2}\left(\frac{x_2^2+x_4}{1-x_1x_2}\right)+x_1x_3$$
$$\text{subject to } \frac{2+\alpha}{60} \le x_1 \le \frac{4-\alpha}{60}$$
$$9+\alpha \le x_2 \le 12-\alpha \tag{30}$$
$$10+2\alpha \le x_3 \le 15-2\alpha$$
$$1+\alpha \le x_4 \le 3-\alpha$$

$$W_\alpha^L = \min\ g(x_1,x_2,x_3,x_4) = \frac{x_1}{2}\left(\frac{x_2^2+x_4}{1-x_1x_2}\right)+x_3$$
$$\text{subject to } \frac{2+\alpha}{60} \le x_1 \le \frac{4-\alpha}{60}$$
$$9+\alpha \le x_2 \le 12-\alpha \tag{31}$$
$$10+2\alpha \le x_3 \le 15-2\alpha$$
$$1+\alpha \le x_4 \le 3-\alpha$$

$$W_\alpha^U = \max\ g(x_1,x_2,x_3,x_4) = \frac{x_1}{2}\left(\frac{x_2^2+x_4}{1-x_1x_2}\right)+x_3$$
$$\text{subject to } \frac{2+\alpha}{60} \le x_1 \le \frac{4-\alpha}{60}$$
$$9+\alpha \le x_2 \le 12-\alpha \tag{32}$$
$$10+2\alpha \le x_3 \le 15-2\alpha$$
$$1+\alpha \le x_4 \le 3-\alpha$$

The feasible region as parametrized by α is given by

$$F(\alpha)=\left[\frac{2+\alpha}{60},\frac{4-\alpha}{60}\right]\times[9+\alpha,12-\alpha]\times[10+2\alpha,15-2\alpha]\times[1+\alpha,3-\alpha]. \tag{33}$$

Also observe that

$$\Omega := \left[\frac{2}{60},\frac{4}{60}\right]\times[9,12]\times[10,15]\times[1,3]\supset F(\alpha). \tag{34}$$

We also observe that these nonlinear programs are essentially global optimization problems in 11119four variables, and thus techniques of multivariable calculus can be used. The use of a computing utility like MATLAB R2020b reveals the following information about the partial derivatives of f and g:

1. All the four first partial derivatives of f are positive on Ω, and
2. All the four first partial derivatives of g are positive on Ω.

Therefore, both f and g increase with respect to all their arguments on Ω (and hence on $F(\alpha)$) and thus f and g both attain their maximum and minimum on $F(\alpha)$ at the points

$$\left(\frac{2+\alpha}{60},9+\alpha,10+2\alpha,1+\alpha\right) \text{ and } \left(\frac{4-\alpha}{60},12-\alpha,15-2\alpha,3-\alpha\right) \tag{35}$$

Therefore, the α-cuts of the fuzzy performance measures $\tilde{L}$ and $\tilde{W}$ are given by

$$^{\alpha}\tilde{L}=\left[f\left(\frac{2+\alpha}{60},9+\alpha,10+2\alpha,\ 1+\alpha\right),\ f\left(\frac{4-\alpha}{60},12-\alpha,\ 15-2\alpha,3-\alpha\right)\right] \tag{36}$$

$$^{\alpha}\tilde{W}=\left[g\left(\frac{2+\alpha}{60},9+\alpha,10+2\alpha,\ 1+\alpha\right),\ g\left(\frac{4-\alpha}{60},12-\alpha,\ 15-2\alpha,3-\alpha\right)\right]$$

We now construct the membership functions of the fuzzy performance measures $\tilde{L}$ and $\tilde{W}$. Towards this, we determine the intervals

$$\left\{ {}^{\alpha}\tilde{z}=\left[z_{\alpha}^{L},z_{\alpha}^{U}\right] \ \middle|\ \alpha\in\{0.0,0.1,\cdots,1.0\} \right\} \tag{37}$$

where $z=L,W$, using the above expressions for the α-cuts. These intervals are tabulated in Table 1.

	${}^{\alpha}\tilde{L}=[L_{\alpha}^{L},L_{\alpha}^{U}]$	${}^{\alpha}\tilde{W}=[W_{\alpha}^{L},W_{\alpha}^{U}]$
$\alpha=0.0$	[0.3984, 2.6333]	[11.9524, 39.5000]
$\alpha=0.1$	[0.4324, 2.3098]	[12.3547, 35.5354]
$\alpha=0.2$	[0.4684, 2.0521]	[12.7748, 32.4018]
$\alpha=0.3$	[0.5066, 1.8410]	[13.2148, 29.8543]
$\alpha=0.4$	[0.5471, 1.6641]	[13.6769, 27.7355]
$\alpha=0.5$	[0.5902, 1.5132]	[14.1638, 25.9399]
$\alpha=0.6$	[0.6361, 1.3823]	[14.6785, 24.3938]
$\alpha=0.7$	[0.6851, 1.2674]	[15.2248, 23.0445]
$\alpha=0.8$	[0.7377, 1.1655]	[15.8069, 21.8530]
$\alpha=0.9$	[0.7941, 1.0742]	[16.4299, 20.7901]
$\alpha=1.0$	[0.8550, 0.9917]	[17.1000, 19.8333]

Table 1

Now, we perform linear interpolation on the data in Table 1 to arrive at the plots of the required membership functions. We have used MATLAB R2020b for this purpose, and the results are depicted in Figures 1 and 2.

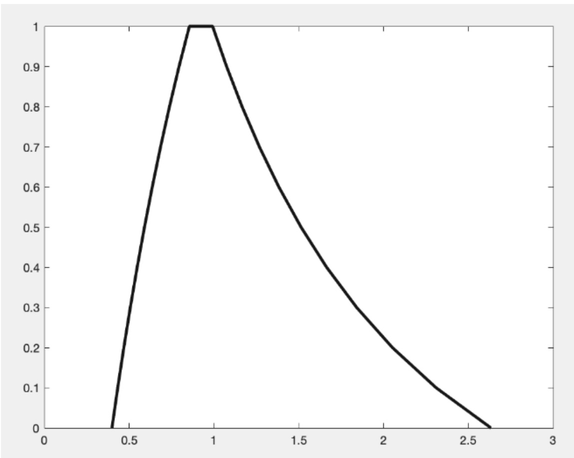

Fig. 1: Plot of the membership function L

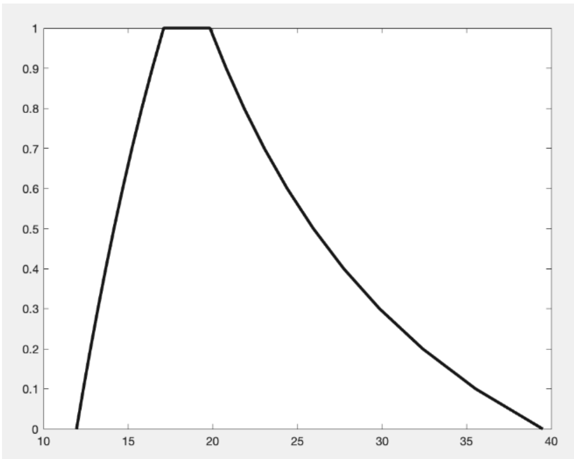

Fig. 2: Plot of the membership function W

It remains to defuzzify the two fuzzy outputs. We use the graded mean integration scheme, which uses the following formula for defuzzification of a fuzzy number $\tilde{z}$ (cf. Eq. (24))

$$\phi(\tilde{z}) = \frac{\int_0^1 \frac{\alpha}{2} \cdot \left(z_\alpha^L + z_\alpha^U \right) \, d\alpha}{\int_0^1 \alpha \, d\alpha} = \int_0^1 \alpha \cdot \left(z_\alpha^L + z_\alpha^U \right) \, d\alpha \,, \tag{38}$$

where ${}^{\alpha}\tilde{z} = \left[z_\alpha^L, z_\alpha^U \right]$. We use MATLAB R2020b to evaluate the integrals for $z = L, W$. The defuzzified values are

$$\phi(\tilde{L}) = 1.0135 \text{ and } \phi(\tilde{W}) = 19.5825. \tag{39}$$

Practitioners will find these values extremely useful.

7. Conclusion

The queueing model discussed in this paper has a wide range of applications – particularly in systems where idle servers are deactivated to conserve power. Incorporating fuzziness in the analysis of such queueing systems makes the model a better approximation to what happens in reality. The proposed solution procedure reduces the problem of determining the performance measures to pairs of optimization problems. The procedure is very generic and applies to a wide range of queueing models. Practitioners will find the data obtained through this analysis helpful in the design of efficient systems.

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