Research Article

A Search For Integral Solutions To The Ternary Bi-Quadratic Equation $x^4+x^3y+x^2y^2+xy^3+y^4=(x+y)^2+1+z^2$

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ABSTRACT : This paper deals with the problem of obtaining non-zero distinct integer solutions to the ternary bi-quadratic equation $x^4+x^3y+x^2y^2+xy^3+y^4=(x+y)^2+1+z^2$. A few interesting relations among the solution are presented. Given on integer solution of the equation under consideration, integer solutions for various choices of hyperbola and parabolas are exhibited. The formulation of second order Ramanujan Numbers with base numbers as real integers and Gaussian integers is illustrated and also the sequence of Diophantine 3-tuples are exhibited.

Keywords: Ternary bi-quadratic, integer solutions, parabolas, hyperbolas, Second order Ramanujan numbers, sequence of Diophantine 3-tuples.

INTRODUCTION

In number theory, Diophantine equations play a significant role and have a marvellous effects on credulous people. They occupy a remarkable position due to unquestioned historical importance. The subject of Diophantine equation is quite difficult. Every century has seen the solution of more mathematical problem than the century before and yet many mathematical problem, both major and minor still remains unsolved. It is hard to tell whether a given equation has solution or not and when it does, there may be no method to find all of them. It is difficult to tell which are early solvable and which require advanced techniques. There is no well unified body of knowledge concerning general methods. A Diophantine problem is considered as solved if a method is available to decide whether the problem is solvable or not and in case of its solvability, to exhibit all integers satisfying the requirements set forth in the problem. Many researchers in the subjects of Diophantine equation exhibit great interest in homogeneous and non-homogeneous bi-quadratic Diophantine equations. In this context, are may refer [1-12]. This communication concerns yet another interesting ternary bi-quadratic equation given by $x^4+x^3y+x^2y^2+xy^3+y^4=(x+y)^2+1+z^2$ and is studied for its non-zero distinct integer solution. A few interesting relations among the solution are presented. Given an integer solution of the equation under consideration, integer solutions for various choices of hyperbola and parabolas are exhibited. The formulation of second order Ramanujan Numbers with base numbers as real integers and Gaussian integers is illustrated and also the sequence of Diophantine 3-tuples are exhibited.

METHOD OF ANALYSIS

The ternary bi-quadratic equation under consideration is

$$x^4+x^3y+x^2y^2+xy^3+y^4=(x+y)^2+1+z^2 \quad (1)$$

Introduction of the transformations

$$x=u+v\ ,\ y=u-v\ ,\ z=4uv\ ,\ u\neq v\neq 0 \quad (2)$$

in (1) leads to

$$v^4-6u^2v^2+5u^4-4u^2-1=0 \quad (3)$$

Treating (3) as a quadratic in v^2 and solving for v^2, we've

$$v^2=5u^2+1 \quad (4)$$

which is the well known pellian equation whose general solution given by,

$$v_n=\frac{1}{2}f_n$$
$$u_n=\frac{1}{2\sqrt{5}}g_n \qquad (5)$$

where

$$f_n=(9+4\sqrt{5})^{n+1}+(9-4\sqrt{5})^{n+1}$$
$$g_n=(9+4\sqrt{5})^{n+1}-(9-4\sqrt{5})^{n+1}\ ,\ n=0,1,2,3........$$

In view of (2), the sequence of values of x , y and z satisfying (1) are represented by

$$x_n=u_n+v_n$$
$$=\frac{1}{2\sqrt{5}}g_n+\frac{1}{2}f_n$$
$$\Rightarrow\quad 2\sqrt{5}x_n=g_n+\sqrt{5}f_n \qquad (6)$$
$$y_n=u_n-v_n$$
$$=\frac{1}{2\sqrt{5}}g_n-\frac{1}{2}f_n$$
$$\Rightarrow\quad 2\sqrt{5}y_n=g_n-\sqrt{5}f_n \qquad (7)$$
$$z_n=\frac{f_ng_n}{\sqrt{5}}\ \forall\ n=-1,0,1,2,.....$$
$$\Rightarrow\quad z_n=x_n^2-y_n^2 \qquad (8)$$

Replacing n by $n+1$ in (6) , we get

$$x_{n+1}=\frac{1}{2\sqrt{5}}g_{n+1}+\frac{1}{2}f_{n+1}$$
$$=\frac{1}{2\sqrt{5}}\left(9g_n+4\sqrt{5}f_n\right)+\frac{1}{2}\left(9f_n+4\sqrt{5}g_n\right)$$
$$x_{n+1}=\frac{29}{2\sqrt{5}}g_n+\frac{13}{2}f_n$$
$$2\sqrt{5}x_{n+1}=29g_n+13\sqrt{5}f_n \qquad (9)$$

Replacing n by $n+1$ in (9) , we get

$$x_{n+2}=\frac{29}{2\sqrt{5}}g_{n+1}+\frac{13}{2}f_{n+1}$$
$$=\frac{29}{2\sqrt{5}}\left(9g_n+4\sqrt{5}f_n\right)+\frac{13}{2}\left(9f_n+4\sqrt{5}g_n\right)$$
$$x_{n+2}=\frac{521}{2\sqrt{5}}g_n+\frac{233}{2}f_n$$
$$2\sqrt{5}x_{n+2}=521g_n+233\sqrt{5}f_n \qquad (10)$$

Eliminating f_n and g_n between (6) , (9) and (10) , we have

$$x_n-18x_{n+1}+x_{n+2}=0\ ,\ n=1,2,3,...... \qquad (11)$$

In a similar manner, from (7) one obtains

$$2\sqrt{5}y_{n+1}=-11g_n-5\sqrt{5}f_n \qquad (12)$$
$$2\sqrt{5}y_{n+2}=-199g_n-89\sqrt{5}f_n \qquad (13)$$

Eliminating f_n and g_n between (7) , (11) and (12) , we have

$$y_n - 18y_{n+1} + y_{n+2} = 0 \text{ , } n = 1,2,3,...... \quad (14)$$

Thus (11) and (14) represent recurrence relations satisfied by the values of x and y respectively .

A few numerical examples of x_n , y_n and z_n satisfying (1) are given in the Table 1.1 below

Table: 1.1 Numerical Examples

n	x_n	y_n	z_n
-1	1	-1	0
0	13	-5	144
1	233	-89	46368
2	4181	-1597	14930352
3	75025	-28657	4807526976

From then above table, we observe some interesting relations among the solutions which are presented below:

- Both x_n , y_n values are odd and z_n values are even.
- One can generate second order Ramanujan numbers with base integers as real integers by choosing x , y and z values suitably.

For illustrations, consider

$$z_0 = 144 = 2*72 = 4*36 = 6*24 = 8*18 \quad (*)$$

$$= 37^2 - 35^2 = 20^2 - 16^2 = 15^2 - 9^2 = 13^2 - 5^2$$

Now,

$$37^2 - 35^2 = 20^2 - 16^2 \Rightarrow 37^2 + 16^2 = 20^2 + 35^2 = 1625$$

$$37^2 - 35^2 = 15^2 - 9^2 \Rightarrow 37^2 + 9^2 = 15^2 + 35^2 = 1450$$

$$37^2 - 35^2 = 13^2 - 5^2 \Rightarrow 37^2 + 5^2 = 13^2 + 35^2 = 1394$$

$$20^2 - 16^2 = 15^2 - 9^2 \Rightarrow 20^2 + 9^2 = 15^2 + 16^2 = 481$$

$$20^2 - 16^2 = 13^2 - 5^2 \Rightarrow 20^2 + 5^2 = 13^2 + 16^2 = 425$$

$$15^2 - 9^2 = 13^2 - 5^2 \Rightarrow 15^2 + 5^2 = 13^2 + 9^2 = 250$$

Note: 1

$$2*72 = 4*36$$
$$\rightarrow (2+72)^2 + (4-36)^2 = (2-72)^2 + (4+36)^2$$
$$\rightarrow 74^2 + (-32)^2 = (-70)^2 + 40^2 = 6500$$
$$2*72 = 6*24$$
$$\rightarrow (2+72)^2 + (6-24)^2 = (2-72)^2 + (6+24)^2$$
$$\rightarrow 74^2 + (-18)^2 = (-70)^2 + 30^2 = 5800$$
$$2*72 = 8*18$$
$$\rightarrow (2+72)^2 + (8-18)^2 = (2-72)^2 + (8+18)^2$$
$$\rightarrow 74^2 + (-10)^2 = (-70)^2 + 26^2 = 5576$$
$$4*36 = 6*24$$
$$\rightarrow (4+36)^2 + (6-24)^2 = (4-36)^2 + (6+24)^2$$
$$\rightarrow 40^2 + (-18)^2 = (-32)^2 + 30^2 = 1924$$
$$4*36 = 8*18$$
$$\rightarrow (4+36)^2 + (8-18)^2 = (4-36)^2 + (8+18)^2$$
$$\rightarrow 40^2 + (-10)^2 = (-32)^2 + 26^2 = 1700$$
$$6*24 = 8*18$$
$$\rightarrow (6+24)^2 + (8-18)^2 = (6-24)^2 + (8+18)^2$$
$$\rightarrow 30^2 + (-10)^2 = (-18)^2 + 26^2 = 1000$$

Thus , 1625 , 1450 , 1394 , 481 , 425 , 250 , 6500, 5800, 5576, 1924, 1700, 1000 are second order Ramanujan numbers with base integers as real integers.

- Considering suitable values of x_n and y_n , one generates second order Ramanujan numbers with base integers as Gaussian integers.

For illustrations, consider again z_0 represented by (*)

- $$2*72 = 4*36$$
  $$\rightarrow (2+i72)^2 + (4-i36)^2 = (2-i72)^2 + (4+i36)^2 = -6460$$
  and
  $$2*72 = 4*36$$
  $$\rightarrow (72+i2)^2 + (36-i4)^2 = (72-i2)^2 + (36+i4)^2 = 6460$$
- $$4*36 = 6*24$$
  $$\rightarrow (4+i36)^2 + (6-i24)^2 = (4-i36)^2 + (6+i24)^2 = -1820$$
  and
  $$4*36 = 6*24$$
  $$\rightarrow (36+i4)^2 + (24-i6)^2 = (36-i4)^2 + (24+i6)^2 = 1820$$
- $$6*24 = 8*18$$
  $$\rightarrow (6+i24)^2 + (8-i18)^2 = (6-i24)^2 + (8+i18)^2 = -800$$
  and
  $$6*24 = 8*18$$
  $$\rightarrow (24+i6)^2 + (18-i8)^2 = (24-i6)^2 + (18+i8)^2 = 800$$

Note that -6460, 6460, -1820, 1820, -800, 800 represent second order Ramanujan numbers with base integers as Gaussian integers.

In a similar manner, other second order Ramanujan numbers are obtained.

➢ Formulation of sequence of Diophantine 3-tuples:

Consider the solution to (1) given by

$$y_0 = -5 , x_0 = 13 = c_0 \ (\text{say})$$

It is observed that

$$y_0 x_0 + k^2 + 65 = k^2 \text{ ,a perfect square}$$

The pair (y_0 , x_0) represents Diophantine 2-tuple with property $D(k^2 + 65)$

If c_1 is the 3^{rd} tuple, then it is given by

$$c_1 = y_0 + x_0 + 2k = 2k + 8$$

Note that $(-5 , 13 , 2k + 8)$ represents diophantine 3-tuple with property $D(k^2 + 65)$

The process of obtaining sequence of diophantine 3-tuple with property $D(k^2 + 65)$ is illustrated below:

Let M be a $3*3$ square matrix given by

$$M = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

Now, $(-5 , 13 , 2k + 8)M = (-5 , 2k + 8 , 4k - 7)$

Note that

$$-5*(4k-7) + k^2 + 65 = (k-10)^2 = \text{perfect square}$$

$$-5*(2k+8) + k^2 + 65 = (k-5)^2 = \text{perfect square}$$

$$(2k+8)*(4k-7) + k^2 + 65 = (3k+3)^2 = \text{perfect square}$$

Therefore the triple $(-5 , 2k + 8 , 4k - 7)$ represents diophantine 3-tuple with property $D(k^2 + 65)$. The repetition of the above process leads to sequences of diophantine 3-tuple whose general form $(-5 , c_{s-1} , c_s)$ is given by

$$\left(-5 , -5s^2 + (2k+10)s - 2k + 8 , -5s^2 + 2ks + 13\right), \quad s = 1,2,3.......$$

A few numerical illustrations are given in Table below:

Table: Numerical illustrations

k	$(-5 , c_0 , c_1)$	$(-5 , c_1 , c_2)$	$(-5 , c_2 , c_3)$	$D(k^2 + 65)$
0	$(-5 , 13 , 8)$	$(-5 , 8 , -7)$	$(-5 , -7 , -32)$	$D(65)$
1	$(-5 , 13 , 10)$	$(-5 , 10 , -3)$	$(-5 , -3 , -26)$	$D(66)$
2	$(-5 , 13 , 12)$	$(-5 , 12 , 1)$	$(-5 , 1 , -20)$	$D(69)$

It is note that the triple $(c_{s-1} , c_s - 5 , c_{s+1}), \quad s = 1,2,3..............$ forms an arithmetic progression.

In a similar way one may generate sequences of diophantine 3-tuples with suitable property through the other solutions to (1).

1. Relations among the solutions are given below.

- ❖ $x_{n+2} - 18x_{n+1} + x_n = 0$
- ❖ $8y_n - x_{n+1} + 21x_n = 0$
- ❖ $8y_{n+1} + 3x_{n+1} + x_n = 0$
- ❖ $8y_{n+2} - 55x_{n+1} - 3x_n = 0$
- ❖ $144y_n - x_{n+2} + 377x_n = 0$
- ❖ $48y_{n+1} + x_{n+2} + 7x_n = 0$

- $8y_n - 21x_{n+2} + 377x_{n+1} = 0$
- $8y_{n+1} - x_{n+2} + 21x_{n+1} = 0$
- $8y_{n+2} + 3x_{n+2} + x_{n+1} = 0$
- $21y_{n+1} + 8x_{n+1} - y_n = 0$
- $y_{n+2} + 8x_{n+1} + 3y_{n+1} = 0$
- $377y_{n+1} + 8x_{n+2} - 21y_n = 0$
- $377y_{n+2} + 144x_{n+2} - y_n = 0$
- $21y_{n+2} + 8x_{n+2} - y_{n+1} = 0$
- $y_n - 18y_{n+1} + y_{n+2} = 0$
- $144y_{n+2} + 55x_{n+2} + x_n = 0$
- $y_{n+1} + 8x_n + y_n = 0$
- $y_{n+2} + 144x_n + 5x_n = 0$
- $x_{n+1} - 7y_{n+1} - 48x_n = 0$

2. Each of the following expressions is a nasty number:

Solving (6) and (9) , we get

$$f_n = \frac{1}{8}[29x_n - x_{n+1}] \quad (15)$$

$$g_n = \frac{\sqrt{5}}{8}[x_{n+1} - 13x_n] \quad (16)$$

Replacing n by $2n+1$ in (15) we get

$$f_{2n+1} = \frac{1}{8}[29x_{2n+1} - x_{2n+2}]$$

Now , $f_{2n+1} + 2 = f_n^2$

$$\therefore \quad \frac{3}{4}[29x_{2n+1} - x_{2n+2} + 16] = 6f_n^2 \text{, a nasty number} \quad (16a)$$

For simplicity and clear understanding the other choices of nasty numbers are presented below:

- $\frac{3}{4}[29x_{2n+1} - x_{2n+2} + 16]$
- $\frac{1}{24}[521x_{2n+1} - x_{2n+3} + 288]$
- $6\,[x_{2n+1} - y_{2n+1} + 2]$
- $6\,[22x_{2n+1} + 2y_{2n+2} + 12]$
- $\frac{6}{55}\,[199x_{2n+2} + y_{2n+3} + 110]$
- $\frac{3}{4}[521x_{2n+2} - 29x_{2n+3} + 16]$
- $\frac{2}{7}[x_{2n+2} - 29y_{2n+1} + 4]$
- $6[-11x_{2n+2} - 29y_{2n+1} + 2]$
- $2[199x_{2n+2} + 29y_{2n+3} + 6]$
- $\frac{6}{377}[5x_{2n+3} - 521y_{2n+1} + 754]$

- $\frac{2}{7}[-11x_{2n+3} - 521y_{2n+2} + 42]$
- $6[-199x_{2n+3} - 521y_{2n+3} + 2]$
- $\frac{3}{4}[-11y_{2n+1} - y_{2n+2} + 16]$
- $\frac{1}{24}[-199y_{2n+1} - y_{2n+3} + 288]$
- $\frac{3}{4}[11y_{2n+3} - 199y_{2n+2} + 16]$

3. Each of the following expressions is a cubical integer:

Replacing n by $3n+2$ in (15) we get

$$f_{3n+2} = \frac{1}{8}[29x_{3n+2} - x_{3n+3}]$$

Now, $f_{3n+2} = f_n^3 - 3f_n$

$$f_{3n+2} + 3f_n = f_n^3$$

$$\Rightarrow \frac{1}{8}[29x_{3n+2} - x_{3n+3} + 87x_n - 3x_{n+1}] = f_n^3$$, a cubical integer.

For simplicity and clear understanding the other choices of cubical integers are presented below:

- $\frac{1}{8}[29x_{3n+2} - x_{3n+3} + 87x_n - 3x_{n+1}]$
- $\frac{1}{144}[521x_{3n+2} - x_{3n+4} + 1563x_n - 3x_{n+2}]$
- $[x_{3n+2} - y_{3n+2} + 3x_n - 3y_n]$
- $\frac{1}{3}[11x_{3n+2} + y_{3n+3} + 33x_n + 3y_{n+2}]$
- $\frac{1}{55}[199x_{3n+2} + y_{3n+4} + 597x_n + 3y_{n+1}]$
- $\frac{1}{8}[521x_{3n+3} - 29x_{3n+4} + 1563x_{n+1} - 87x_{n+2}]$
- $\frac{1}{21}[x_{3n+3} - 29x_{3n+2} + 3x_{n+1} - 87y_n]$
- $[-11x_{3n+3} - 29y_{3n+3} - 33x_{n+1} - 87y_{n+1}]$
- $\frac{1}{3}[199x_{3n+3} + 29y_{3n+4} + 597x_{n+1} + 87y_{n+2}]$
- $\frac{1}{377}[x_{3n+4} - 521y_{3n+2} + 3x_{n+2} - 1563y_n]$
- $\frac{1}{21}[-11x_{3n+4} - 521y_{3n+3} - 33x_{n+2} - 1563y_{n+1}]$
- $[-199x_{3n+4} - 521y_{3n+4} - 597x_{n+2} - 1563y_{n+2}]$
- $\frac{1}{8}[-11y_{3n+2} - y_{3n+2} - 33y_n - 3y_{n+1}]$
- $\frac{1}{144}[-199y_{3n+2} - y_{3n+4} - 597y_n - 3y_{n+2}]$

- $\frac{1}{8}[11y_{3n+4}-199y_{3n+3}+33y_{n+2}-597y_{n+1}]$

4. Each of the following expressions is a bi-quadratic integer.

Replacing n by $4n+3$ in (15) we get

$$f_{4n+3}=\frac{1}{8}[29x_{4n+3}-x_{4n+4}]$$

Now, $f_{4n+3}+4f_n^2-2=f_n^4$

$$\Rightarrow\frac{1}{8}[29x_{4n+3}-x_{4n+4}+116x_{2n+1}-4x_{2n+2}+48]=f_n^4,$$

a bi-quadratic integer.

For simplicity and clear understanding the other choices of bi-quadratic integers are presented below:

- $\frac{1}{8}[29x_{4n+3}-x_{4n+4}+116x_{2n+1}-4x_{2n+2}+48]$
- $\frac{1}{144}[521x_{4n+3}-x_{4n+5}+2084x_{2n+1}-4x_{2n+3}+864]$
- $[x_{4n+3}-y_{4n+3}+4x_{2n+1}-4y_{2n+1}+6]$
- $\frac{1}{3}[11x_{4n+3}+y_{4n+4}+44x_{2n+1}+4x_{2n+2}+18]$
- $\frac{1}{55}[199x_{4n+3}+y_{2n+3}+796x_{2n+1}+4y_{2n+3}+330]$
- $\frac{1}{8}[521x_{4n+4}-29x_{4n+5}+2084x_{2n+2}-116x_{2n+3}+48]$
- $\frac{1}{21}[x_{4n+4}-29y_{4n+3}+4x_{2n+2}-116y_{2n+1}+126]$
- $[-11x_{4n+4}-29y_{4n+4}-44x_{2n+2}-116y_{2n+2}+6]$
- $\frac{1}{3}[199x_{4n+4}+29y_{4n+4}+796x_{2n+2}+116y_{2n+3}+18]$
- $\frac{1}{377}[x_{4n+5}-521y_{4n+4}+4x_{2n+3}-2084y_{2n+1}+2262]$
- $\frac{1}{21}[-11x_{4n+5}-521y_{4n+4}-44x_{2n+3}-2084y_{2n+2}+126]$
- $[-199x_{4n+5}-521y_{4n+5}-796x_{2n+3}-2084y_{2n+3}+6]$
- $\frac{1}{8}[-11y_{4n+3}-y_{4n+4}-44y_{2n+1}-4y_{2n+2}+48]$
- $\frac{1}{144}[-199y_{4n+3}-y_{4n+5}-796y_{2n+1}-4y_{2n+3}+864]$
- $\frac{1}{8}[11y_{4n+5}-199y_{4n+4}+44y_{2n+3}-796y_{2n+2}+48]$

5. Each of the following expressions is a quintic integer:

Replacing n by $5n+4$ in (15) we get

$$f_{5n+4}=\frac{1}{8}[29x_{5n+4}-x_{5n+5}]$$

Now, $f_{5n+4}=f_n^5-5f_n^3+5f_n$

$$f_n^5 = f_{5n+4} + 5f_n^3 - 5f_n$$

$\Rightarrow \frac{1}{8}[29x_{5n+4} - x_{5n+5} + 145x_{3n+2} - 5x_{3n+3} + 290x_n - 10x_{n+1}] = f_n^5$, a quintic integer.

For simplicity and clear understanding the other choices of quintic integers are presented below:

- $[x_{5n+4} - y_{5n+4} + 5x_{3n+2} - 5y_{3n+2} + 10x_n - 10y_n]$
- $\frac{1}{21}[x_{5n+5} - 29y_{5n+4} + 5x_{3n+3} - 145y_{3n+2} + 10x_n - 290y_n]$
- $\frac{1}{377}[x_{5n+6} - 521y_{5n+4} + 5x_{3n+4} - 2605y_{3n+2} + 10x_{n+1} - 5210y_n]$
- $\frac{1}{8}[-11y_{5n+4} - y_{5n+5} - 55y_{3n+2} - 5y_{3n+3} - 110y_n - 10y_{n+1}]$
- $\frac{1}{144}[521x_{5n+4} - x_{5n+6} + 2605x_{3n+2} - 5x_{3n+4} + 5210x_n - 10x_{n+2}]$
- $[199x_{5n+6} - 521y_{5n+6} - 995x_{3n+4} - 2605y_{3n+4} - 1990x_{n+2} - 5210y_{n+2}]$
- $\frac{1}{377}[x_{5n+6} - 521y_{5n+4} + 5x_{3n+4} - 2605y_{3n+2} + 10x_n - 5210y_{n+1}]$
- $\frac{1}{21}[-11x_{5n+6} - 521y_{5n+5} - 55x_{3n+4} - 2605y_{3n+3} - 110x_{n+2} - 5210y_{n+1}]$
- $\frac{1}{144}[-199y_{5n+4} - y_{5n+4} - 995y_{3n+2} - 5y_{3n+4} - 1990y_n - 10y_{n+2}]$
- $\frac{1}{8}[11y_{5n+6} - 199y_{5n+5} + 55y_{3n+4} - 99y_{3n+3} + 110y_{n+2} - 1990y_{n+1}]$
- $\frac{1}{3}[11x_{5n+4} + y_{5n+5} + 55x_{3n+2} + 5y_{3n+3} + 110x_n + 10y_{n+1}]$
- $\frac{1}{55}[199x_{5n+4} + y_{5n+6} + 995x_{3n+2} + 5y_{3n+4} + 1990x_n + 10y_{n+2}]$
- $\frac{1}{8}[521x_{5n+5} - 29x_{5n+6} + 2605x_{3n+3} - 145x_{3n+4} + 5210x_{n+1} - 290x_{n+2}]$
- $\frac{1}{21}[x_{5n+5} - 29y_{5n+4} + 5x_{3n+3} - 145y_{3n+2} + 10x_n - 290y_n]$
- $[-11x_{5n+5} - 29y_{5n+5} - 55x_{3n+3} - 145y_{3n+3} - 110x_{n+1} - 290y_{n+1}]$
- $\frac{1}{3}[199x_{5n+5} + 29y_{5n+6} + 995x_{3n+3} + 145y_{3n+4} + 1990x_{n+1} + 290y_{n+2}]$

REMARKABLE OBSERVATIONS

I. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in the Table 2 below:

Illustration

Let

$$X_n = x_{n+1} - 13x_n$$

$$Y_n = 29x_n - x_{n+1}$$

$$f_n = \frac{Y_n}{8} \quad (17)$$

$$g_n = \frac{\sqrt{5}}{8}X_n \quad (18)$$

W.K.T

$$f_n^2 - g_n^2 = 4 \quad (19)$$

Substituting (17) and (18) in (19) we have

$$\frac{1}{64}Y_n^2 - \frac{5}{64}X_n^2 = 4$$

$$Y_n^2 - 5X_n^2 = 256$$

which represents a hyperbola.

For simplicity and clear understanding, the other choices of hyperbola are presented in the table 1.2 below:

Table:1. 2 Hyperbola

S. NO	Hyperbola	(X,Y)
1	$Y^2 - 5X^2 = 256$	$(x_{n+1} - 13x_n, 29x_n - x_{n+1})$
2	$Y^2 - 5X^2 = 82944$	$(x_{n+2} - 233x_n, 521x_n - x_{n+2})$
3	$Y^2 - 5X^2 = 16$	$(2x_n + 2y_n, 2x_n - 2y_n)$
4	$Y^2 - 5X^2 = 144$	$(2y_{n+1} + 10x_n, 22x_n + 2y_n)$
5	$Y^2 - 5X^2 = 48400$	$(2y_{n+2} + 178x_n, 398x_{n+2} + 2y_{n+2})$
6	$Y^2 - 5X^2 = 256$	$(1x_{n+2} - 233x_{n+1}, 521x_{n+1} - 29x_{n+2})$
7	$Y^2 - 5X^2 = 1764$	$(x_{n+1} + 13y_n, x_{n+1} - 29y_n)$
8	$Y^2 - 5X^2 = 16$	$(10x_{n+1} + 26y_{n+1}, -22x_{n+1} - 58y_{n+1})$
9	$Y^2 - 5X^2 = 324$	$(-89x_{n+1} - 13y_{n+2}, 199x_{n+1} + 29y_{n+2})$
10	$Y^2 - 5X^2 = 568516$	$(x_{n+2} + 233y_n, 521y_n - x_{n+2})$
11	$Y^2 - 5X^2 = 1764$	$(5x_{n+2} - 233y_{n+1}, -11x_n - 521y_{n+1})$
12	$Y^2 - 5X^2 = 16$	$(178x_{n+2} + 466y_{n+2}, -398x_{n+2} - 1042y_{n+2})$
13	$Y^2 - 5X^2 = 256$	$(5y_n - y_{n+1}, -11y_n - y_{n+1})$
14	$Y^2 - 5X^2 = 82944$	$(89y_n - y_{n+2}, -199y_n -_{n+2})$
15	$Y^2 - 5X^2 = 256$	$(89y_{n+1} - 5y_{n+2}, -11y_{n+2} - 199y_{n+2})$

II. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in Table1. 3 below:

Illustration

Let

$$Y_n = 29x_{2n+1} - x_{2n+2} + 16$$

From (16a),

$$f_n^2 = \frac{1}{8}Y_n$$

In view of (19), one has

$$\frac{1}{8}Y_n - \frac{5}{64}X_n^2 = 4$$

$$8Y_n - 5X_n^2 = 256$$

which represents a parabola.

For simplicity and clear understanding the other choices of parabola are presented below in Table 1.3

Table:1. 3 parabola

S. NO	Parabola	(X,Y)

1	$8Y-5X^2=256$	$\left(x_{n+1}-13x_n,\ 29x_{2n+1}-x_{2n+2}+16\right)$
2	$144Y-5X^2=82944$	$\left(x_{n+2}-233x_n,\ 521x_{2n+1}-x_{2n+3}+288\right)$
3	$2Y-5X^2=16$	$\left(2x_n+2y_n,\ 2x_{2n+1}-2y_{2n+1}+4\right)$
4	$6Y-5X^2=144$	$\left(2y_{n+1}+10x_n,\ 22x_{2n+1}+2y_{2n+2}+12\right)$
5	$110Y-5X^2=48400$	$\left(2y_{n+2}+178x_n,\ 398x_{2n+1}+2y_{2n+3}+220\right)$
6	$8Y-5X^2=256$	$\left(13x_{n+2}-233x_{n+1},\ 521x_{2n+2}-29x_{2n+3}+16\right)$
7	$21Y-5X^2=1764$	$\left(x_{n+1}+13y_n,\ x_{2n+2}-29y_{2n+1}+42\right)$
8	$2Y-5X^2=16$	$\left(10x_{n+1}+26y_{n+1},\ -22x_{2n+2}-58y_{2n+2}+4\right)$
9	$3Y-5X^2=36$	$\left(-89x_{n+1}-13y_{n+2},\ 199x_{2n+2}+29y_{2n+3}+6\right)$
10	$377Y-5X^2=568516$	$\left(x_{n+2}+233y_n,\ x_{2n=3}-521y_{2n+1}+754\right)$
11	$21Y-5X^2=1764$	$\left(5x_{n+2}-233y_{n+1},\ -11x_{2n+3}-521y_{2n+2}+42\right)$
12	$2Y-5X^2=16$	$\left(178x_{n+2}+466y_{n+2},\ -398x_{2n+3}-1042y_{2n+3}+4\right)$
13	$8Y-5X^2=256$	$\left(5y_n-y_{n+1},\ -11y_{2n+1}-y_{2n+2}+16\right)$
14	$144Y-5X^2=82944$	$\left(89y_n-y_{n+2},\ -199y_{2n+1}-y_{2n+3}+288\right)$
15	$8Y-5X^2=256$	$\left(89y_{n+1}-5y_{n+2},\ -11y_{2n+3}-199y_{2n+2}+16\right)$

CONCLUSION

In this paper an attempt has been made to obtained integer solutions to the ternary bi-quadratic equations. Since these equations are rich in verity, one may search for integer solutions to other choices of bi-quadratic equations with multiple variables.

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