Study of A Class Of Multivalent Functions Defined By Dziak - Srivastava Operat With Application On Fractional Calculus

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Abstract: In the this paper we introduced a class of Multivalent functions with negative coefficients defined by Dziak-Srivastava operator and some application of Fractional Calculus , we obtain some theorems of this class.

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1- Introduction :

Let K denote the class of functions of the form :

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad p \in IN = \{1,2,\ldots\} \tag{1}$$

which are analytic and multivalent in the open unit disk

$U = \{z \in C : |z| < 1\}$. Let K_p^* a subclass of K consisting of function of the form:

$$f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n, \quad a_n \geq 0, \; p \in IN = \{1,2,\ldots\}. \tag{2}$$

For $\{\alpha_1, \alpha_2, \ldots, \alpha_m\} \subseteq C$ and $\{\beta_1, \beta_2, \ldots, \beta_k\} \subseteq C - \{0,-1,-2,\ldots\}$, the generalized hypergeometric function ${}_mF_k\{\alpha_1,\ldots,\alpha_m;\beta_1,\ldots,\beta_k;z\}$ is defined by :

$${}_mF_k\{\alpha_1,\ldots,\alpha_m;\beta_1,\ldots,\beta_k;z\} = \sum_{n=p+1}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_m)_n}{(\beta_1)_n \ldots (\beta_k)_n} z^n \tag{3}$$

$$(m \leq k+1, m,k \in N_0 = \{0,1,2,\ldots\}),$$

where $(x)_n$ is the pochhammer symbol defined by :

$$(x)_n = \begin{cases} 1 & n = 0 \\ x(x+1)\ldots(x+n-1) & n \in IN \end{cases} \tag{4}$$

Dziok and Srivastava consider in [6] a linear operator under the multivalent analytic functions

$$DS_p^{m,k}(\alpha_1,\ldots,\alpha_m;\beta_1,\ldots\beta_k) : K_p^* \to K_p^*$$

defined by the Hadamard product:

$$\begin{aligned} DS_p^{m,k}(f)(z) &= DS_p^{m,k}(\alpha_1,\ldots,\alpha_m;\beta_1,\ldots,\beta_k)f(z) \\ &= h_p(\alpha_1,\ldots,\alpha_m;\beta_1,\ldots\beta_k;z) * f(z), \end{aligned} \tag{5}$$

where

$$h_p(\alpha_1,\ldots,\alpha_m;\beta_1,\ldots,\beta_k;z) = z^p {}_mF_k(\alpha_1,\ldots,\alpha_m;\beta_1,\ldots,\beta_k;z)$$

If $f \in K_p^*$ is given by (2), then we can write the Dziak-Srivastava linear operator (5) as follows:

$$DS_p^{m,k}(f)(z) = z^p - \sum_{n=p+1}^{\infty} \frac{(\alpha_1)_n ... (\alpha_m)_n}{(\beta_1)_n ... (\beta_k)_n n!} a_n z^n. \qquad (6)$$

Definition 1: A function $f(z)$ defined by (2) be in the class

$K(\sigma,\eta,\theta,\ell,A,B,D,p)$ if it satisfies the condition

$$\left| \frac{(1-A)(1-B)\sigma\eta\ell\left(z^2\left(DS_p^{m,k}(f)(z)\right)'' - (p-1)z(DS_p^{m,k}(f)(z))' \right)}{DS_p^{m,k}(f)(z) - \theta(1-D)} \right| \le 1, \qquad (7)$$

where $0 \le \sigma \le 1$, $0 \le \eta \le 1$, $0 \le \ell \le 1$, $0 \le \theta < 1$, $0 \le A,B,D < 1$, $p \in IN$, $z \in U$. $\theta(1-D) < 1$.

2. Main Result

In the next theorem, we obtain a necessary and sufficient condition for functions to be in the class $K(\sigma,\eta,\theta,\ell,A,B,D,p)$.

Theorem1: Let the function $f(z) \in K_p^*$ is in class $K(\sigma,\eta,\theta,\ell,A,B,D,p)$ if and only if

$$\sum_{n=p+1}^{\infty} [n(1-A)(1-B)\sigma\eta\ell(n-p)+1] \frac{(\alpha_1)_n ... (\alpha_m)_n}{(\beta_1)_n ... (\beta_k)_n n!} a_n \le 1-\theta(1-D). \qquad (8)$$

The result (8) is sharp

Proof: Let the inequality (8) holds true. Let $|z|=1$. Then

$$\left|(1-A)(1-B)\sigma\eta\ell\left[z^2(DS_p^{m,k}(f)(z))'' - (p-1)z(DS_p^{m,k}(f)(z))'\right]\right| - \left|DS_p^{m,k}(f)(z) - \theta(1-D)\right|$$

$$= \left| -\sum_{n=p+1}^{\infty} [(1-A)(1-B)n\sigma\eta\ell(n-p)] \frac{(\alpha_1)_n ... (\alpha_m)_n}{(\beta_1)_n ... (\beta_k)_n n!} a_n z^n \right| - \left| z^p - \sum_{n=p+1}^{\infty} \frac{(\alpha_1)_n ... (\alpha_m)_n}{(\beta_1)_n ... (\beta_k)_n n!} a_n z^n - \theta(1-D) \right|$$

$$\le \sum_{n=p+1}^{\infty} [(1-A)(1-B)n\sigma\eta\ell(n-p)] \frac{(\alpha_1)_n ... (\alpha_m)_n}{(\beta_1)_n ... (\beta_k)_n n!} a_n |z|^n - |z|^p + \sum_{n=p+1}^{\infty} \frac{(\alpha_1)_n ... (\alpha_m)_n}{(\beta_1)_n ... (\beta_k)_n n!} a_n |z|^n + \theta(1-D)$$

$$\le \sum_{n=p+1}^{\infty} [(1-A)(1-B)n\sigma\eta\ell(n-p)+1] \frac{(\alpha_1)_n ... (\alpha_m)_n}{(\beta_1)_n ... (\beta_k)_n n!} a_n - (1-\theta(1-D)) \le 0.$$

Hence by the principle of maximum modulus $f(z) \in K(\sigma,\eta,\theta,\ell,A,B,D,p)$.

Conversely, assume that

$$\left| \frac{(1-A)(1-B)\sigma\eta\ell\left(z^2\left(DS_p^{m,k}(f)(z)\right)'' - (p-1)z(DS_p^{m,k}(f)(z))' \right)}{DS_p^{m,k}(f)(z) - \theta(1-D)} \right|$$

$$= \left| \frac{-\sum_{n=p+1}^{\infty}(1-A)(1-B)\left[n\sigma\eta\ell(n-p)\right]\frac{(\alpha_1)_n\ldots(\alpha_m)_n}{(\beta_1)_n\ldots(\beta_k)_n n!}a_n z^n}{z^p - \sum_{n=p+1}^{\infty}\frac{(\alpha_1)_n\ldots(\alpha_m)_n}{(\beta_1)_n\ldots(\beta_k)_n n!}a_n z^n - \theta(1-D)} \right| \le 1.$$

Since $|\mathrm{Re}(z)| \le |z|$ for all z, we have

$$\mathrm{Re}\left\{ \frac{\sum_{n=p+1}^{\infty}(1-A)(1-B)\left[n\sigma\eta\ell(n-p)\right]\frac{(\alpha_1)_n\ldots(\alpha_m)_n}{(\beta_1)_n\ldots(\beta_k)_n n!}a_n z^n}{z^p - \sum_{n=p+1}^{\infty}\frac{(\alpha_1)_n\ldots(\alpha_m)_n}{(\beta_1)_n\ldots(\beta_k)_n n!}a_n z^n - \theta(1-D)} \right\} \le 1,$$

we choose the values of z on the real axis and letting $z \to 1^-$, we obtain

$$\sum_{n=p+1}^{\infty}\left[n(1-A)(1-B)\sigma\eta\ell(n-p)+1\right]\frac{(\alpha_1)_n\ldots(\alpha_m)_n}{(\beta_1)_n\ldots(\beta_k)_n n!}a_n \le 1-\theta(1-D).$$

Finally , sharpness follows if we take

$$f(z) = z^p - \frac{1-\theta(1-D)}{\left[(1-A)(1-B)n\sigma\eta\ell(n-p)+1\right]\frac{(\alpha_1)_n\ldots(\alpha_m)_n}{(\beta_1)_n\ldots(\beta_k)_n n!}} z^n. \qquad (9)$$

The proof is complete.

Corollary 1: Let $f(z) \in K(\sigma,\eta,\theta,\ell,A,B,D,p)$. Then

$$a_n \le \frac{1-\theta(1-D)}{\left[(1-A)(1-B)n\sigma\eta\ell(n-p)+1\right]\frac{(\alpha_1)_n\ldots(\alpha_m)_n}{(\beta_1)_n\ldots(\beta_k)_n n!}}. \qquad (10)$$

The equality in (10) is attained for the function *f* given by (9).

3.Application of fractional calculus with Hypergeometric functions

The hypergeometric functions connected with positive and negative coefficients and by applying fractional calculus techniques is of great interest. Such type of study was carried out by various mathematicians, like, Aouf, Shamandy and Yassen [1], Cho and Aouf [5] ,Reddy and padmanabhan [7], Atshan [2], Atshan, Kulkani and Murugusundaramoorthy [4], Atshan and Kulkarni [3] who obtained several growth and distortion properties of functions in the class of operators of fractional integral and fractional derivative.
We need the following definitions given by H.M. Srivastava and S.Owa [8].

Definition 2: The fractional integral of order $\delta(0<\delta)$ is defined by

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)}\int_0^z \frac{f(t)}{(z-t)^{1-\delta}}\,dt, \qquad (11)$$

where $f(z)$ is an analytic function in a simply connected region of z-plane containing the origin and the multiplicity of $(z-t)^{\delta-1}$ is removed by required *log*$(z-t)$ to be real when $(z-t)$>0.

Definition 3: The fractional derivative of order $\delta(0 \le \delta < 1)$ is defined by

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\delta}} dt, \qquad (12)$$

where $f(z)$ is as in Definition 2 and the multiplicity of $(z-t)^{-\delta}$ is removed like Definition 2.
Definition 4: [Under the conditions of Definition 3] the fractional derivative of order $j+\delta(j=0,1,2,\ldots)$ is defined by

$$D_z^{j+\delta} f(z) = \frac{d^j}{dz^j} D_z^{\delta} f(z).$$

From Definition 2 and 3 by applying a simple calculation, we get

$$D_z^{-\delta} f(z) = \frac{\Gamma(p+1)}{\Gamma(p+1+\delta)} z^{\delta+p} - \sum_{n=p+1}^{\infty} \frac{\Gamma(n+p)}{\Gamma(n+p+\delta)} a_n z^{n+\delta}, \qquad (13)$$

$$D_z^{\delta} f(z) = \frac{\Gamma(p+1)}{\Gamma(p+1-\delta)} z^{p-\delta} - \sum_{n=p+1}^{\infty} \frac{\Gamma(n+p)}{\Gamma(n+p-\delta)} a_n z^{n-\delta}. \qquad (14)$$

Now, we introduce a new fractional calculus is fractional calculus (fractional derivative and fractional integral) of order λ.

Definition 5: The fractional derivative of order $\lambda(\lambda=-1,-2,\ldots)$ is defined by

$$D_z^{\lambda} f(z) = \frac{(-1)^{\lambda+1}}{\Gamma(-\lambda)} \frac{d}{dz} \int_0^z (u-z)^{-\lambda-1} f(u)du, \qquad (15)$$

where $f(z)$ is an analytic function in a simply connected region of z-plane containing the origin and the multiplicity of $(u-z)^{-\lambda-1}$ is removed by requiring $log\,(u-z)$ to be real when $(u-z)>0$.

Definition 6: The fractional integral of order $\lambda(\lambda=-1,-2,\ldots)$ is defined by

$$D_z^{-\lambda} f(z) = \frac{(-1)^{\lambda}}{\Gamma(1-\lambda)} \int_0^z (u-z)^{-\lambda} f(u)du, \qquad (16)$$

where $f(z)$ is an analytic function in a simply connected region of z-plane containing the origin and the multiplicity of $(u-z)^{-\lambda}$ is removed by requiring $log\,(u-z)$ to be real when $(u-z)>0$.

Lemma 1: Let $f(z) \in K_p^*$. Then

$$G(z) = \frac{\Gamma(p-\lambda)}{\Gamma(p+1)} z^{\lambda+1} D_z^{\lambda} f(z) = z^p - \sum_{n=p+1}^{\infty} \frac{n!\,\Gamma(p-\lambda)}{\Gamma(p+1)\Gamma(n-\lambda)} a_n z^n. \qquad (17)$$

Proof:

$$D_z^{\lambda} f(z) = \frac{(-1)^{\lambda+1}}{\Gamma(-\lambda)} \frac{d}{dz} \left[\int_0^z (u-z)^{-\lambda-1} \left[u^p - \sum_{n=p+1}^{\infty} a_n u^n \right] du \right]$$

$$=\frac{(-1)^{\lambda+1}}{\Gamma(-\lambda)}\frac{d}{dz}\left[\int_0^z (-1)^{-\lambda-1}(z-u)^{-\lambda-1}u^p du-\sum_{n=p+1}^{\infty}a_n(-1)^{-\lambda-1}\int_o^z (z-u)^{-\lambda-1}u^n du\right]$$

$$=z^{p-\lambda-1}\frac{\Gamma(p+1)}{\Gamma(p-\lambda)}-\sum_{n=p+1}^{\infty}\frac{n!}{\Gamma(n-\lambda)}a_n z^{n-\lambda-1}.$$

Hence

$$G(z)=z^p-\sum_{n=p+1}^{\infty}\frac{n!\Gamma(p-\lambda)}{\Gamma(p+1)\Gamma(n-\lambda)}a_n z^n.$$

In particular, when p=1, then

$$G(z)=z-\sum_{n=2}^{\infty}\frac{n!\Gamma(1-\lambda)}{\Gamma(n-\lambda)}a_n z^n.$$

Lemma 2: Let $f(z)\in K_p^*$. Then

$$F(z)=\frac{\Gamma(2+p-\lambda)}{\Gamma(p+1)}z^{\lambda-1}D_z^{-\lambda}f(z)=z^p-\sum_{n=p+1}^{\infty}\frac{n!\Gamma(2+p-\lambda)}{\Gamma(p+1)\Gamma(n-\lambda+2)}a_n z^n . \qquad (18)$$

Proof of Lemma 2 is similar to that of Lemma1 and hence details are omitted.

In the next, we obtain distortion theorems for the class $K(\sigma,\eta,\theta,\ell,A,B,D,p)$.

Theorem 2: Let $f(z)$ defined by (2) be in the class $K(\sigma,\eta,\theta,\ell,A,B,D,p)$.
Then

$$\left|\frac{\Gamma(p+1-\delta)}{\Gamma(p+1)}z^{\delta}D_z^{\delta}f(z)\right|\le|z|^p+\frac{2\Gamma(2p)\Gamma(p+1-\delta)\left[(1-\theta(1-D))(p+1)!\right]}{\Gamma(2p+1-\delta)\Gamma(p)\left[(1-A)(1-B)\sigma\eta\ell(p+1)+1\right]\frac{(\alpha_1)_{p+1}...(\alpha_m)_{p+1}}{(\beta_1)_{p+1}...(\beta_k)_{p+1}}}|z|^{p+1} \qquad (19)$$

and

$$\left|\frac{\Gamma(p+1-\delta)}{\Gamma(p+1)}z^{\delta}D_z^{\delta}f(z)\right|\ge|z|^p-\frac{2\Gamma(2p)\Gamma(p+1-\delta)\left[(1-\theta(1-D))(p+1)!\right]}{\Gamma(2p+1-\delta)\Gamma(p)\left[(1-A)(1-B)\sigma\eta\ell(p+1)+1\right]\frac{(\alpha_1)_{p+1}...(\alpha_m)_{p+1}}{(\beta_1)_{p+1}...(\beta_k)_{p+1}}}|z|^{p+1}. \qquad (20)$$

The inequalities in (19) and (20) are attained for the function

$$f(z)=z^p-\frac{(1-\theta(1-D))(p+1)!}{\left[(1-A)(1-B)\sigma\eta\ell(p+1)+1\right]\frac{(\alpha_1)_{p+1}...(\alpha_m)_{p+1}}{(\beta_1)_{p+1}...(\beta_k)_{p+1}}}z^{p+1} . \qquad (21)$$

Proof : By using Theorem 1, we have

$$\sum_{n=p+1}^{\infty} a_n \leq \frac{(1-\theta(1-D))(p+1)!}{\left[(1-A)(1-B)\sigma\eta\ell(p+1)+1\right]\dfrac{(\alpha_1)_{p+1}\ldots(\alpha_m)_{p+1}}{(\beta_1)_{p+1}\ldots(\beta_k)_{p+1}}}. \tag{22}$$

By Definition 4, we have

$$D_z^{\delta} f(z) = \frac{\Gamma(p+1)}{\Gamma(p+1-\delta)} z^{p-\delta} - \sum_{n=p+1}^{\infty} \frac{\Gamma(n+p)}{\Gamma(n+p-\delta)} a_n z^{n-\delta},$$

and

$$\frac{\Gamma(p+1-\delta)}{\Gamma(p+1)} z^{\delta} D_z^{\delta} f(z) = z^p - \sum_{n=p+1}^{\infty} \frac{\Gamma(n+p)\Gamma(p+1-\delta)}{\Gamma(n+p-\delta)\Gamma(p+1)} a_n z^n = z^p - \sum_{n=p+1}^{\infty} \Im(n,p,\delta) a_n z^n, \tag{23}$$

where $\Im(n,p,\delta) = \dfrac{\Gamma(n+p)\Gamma(p+1-\delta)}{\Gamma(n+p-\delta)\Gamma(p+1)}$.

We know that $\Im(n,p,\delta)$ is a decreasing function of *n* and

$$0 < \Im(n,p,\delta) \leq \Im(p+1,p,\delta) = \frac{2\Gamma(2p)\Gamma(p+1-\delta)}{\Gamma(2p+1-\delta)\Gamma(p)}.$$

So by using (22) and (23), we have

$$\left|\frac{\Gamma(p+1-\delta)}{\Gamma(p+1)} z^{\delta} D_z^{\delta} f(z)\right| \leq |z|^p + \Im(p+1,p,\delta)|z|^{p+1} \sum_{n=p+1}^{\infty} a_n$$

$$\leq |z|^p + \frac{2\Gamma(2p)\Gamma(p+1-\delta)\left[(1-\theta(1-D))(p+1)!\right]}{\Gamma(2p+1-\delta)\Gamma(p)\left[(1-A)(1-B)\sigma\eta\ell(p+1)+1\right]\dfrac{(\alpha_1)_{p+1}\ldots(\alpha_m)_{p+1}}{(\beta_1)_{p+1}\ldots(\beta_k)_{p+1}}} |z|^{p+1},$$

which gives (19); we also have

$$\left|\frac{\Gamma(p+1-\delta)}{\Gamma(p+1)} z^{\delta} D_z^{\delta} f(z)\right| \geq |z|^p - \Im(p+1,p,\delta)|z|^{p+1} \sum_{n=p+1}^{\infty} a_n$$

$$\geq |z|^p - \frac{2\Gamma(2p)\Gamma(p+1-\delta)\left[(1-\theta(1-D))(p+1)!\right]}{\Gamma(2p+1-\delta)\Gamma(p)\left[(1-A)(1-B)\sigma\eta\ell(p+1)+1\right]\dfrac{(\alpha_1)_{p+1}\ldots(\alpha_m)_{p+1}}{(\beta_1)_{p+1}\ldots(\beta_k)_{p+1}}} |z|^{p+1},$$

which gives (20).

Theorem 3: Let $f(z)$ defined by (2) be in the class $K(\sigma,\eta,\theta,\ell,A,B,D,p)$. Then

$$\left|\frac{\Gamma(p+1+\delta)}{\Gamma(p+1)} z^{-\delta} D_z^{-\delta} f(z)\right| \leq |z|^p + \frac{2\Gamma(2p)\Gamma(p+1+\delta)\left[(1-\theta(1-D))(p+1)!\right]}{\Gamma(2p+1+\delta)\Gamma(p)\left[(1-A)(1-B)\sigma\eta\ell(p+1)+1\right]\dfrac{(\alpha_1)_{p+1}\ldots(\alpha_m)_{p+1}}{(\beta_1)_{p+1}\ldots(\beta_k)_{p+1}}} |z|^{p+1}, \tag{24}$$

and

$$\left|\frac{\Gamma(p+1+\delta)}{\Gamma(p+1)} z^{-\delta} D_z^{-\delta} f(z)\right| \geq |z|^p - \frac{2\Gamma(2p)\Gamma(p+1+\delta)\left[(1-\theta(1-D))(p+1)!\right]}{\Gamma(2p+1+\delta)\Gamma(p)\left[(1-A)(1-B)\sigma\eta\ell(p+1)+1\right]\dfrac{(\alpha_1)_{p+1}...(\alpha_m)_{p+1}}{(\beta_1)_{p+1}...(\beta_k)_{p+1}}} |z|^{p+1}. (25)$$

The inequalities in (24) and (25) are attained for the function $f(z)$ given by (21).

Proof: By Definition 4, we have

$$D_z^{-\delta} f(z) = \frac{\Gamma(p+1)}{\Gamma(p+1+\delta)} z^{\delta+p} - \sum_{n=p+1}^{\infty} \frac{\Gamma(n+p)}{\Gamma(n+p+\delta)} a_n z^{n+\delta},$$

and

$$\frac{\Gamma(p+1+\delta)}{\Gamma(p+1)} z^{-\delta} D_z^{-\delta} f(z) = z^p - \sum_{n=p+1}^{\infty} \frac{\Gamma(n+p)\Gamma(p+1+\delta)}{\Gamma(n+p+\delta)\Gamma(p+1)} a_n z^n = z^p - \sum_{n=p+1}^{\infty} \Phi(n,p,\delta) a_n z^n, (26)$$

where $\Phi(n,p,\delta) = \dfrac{\Gamma(n+p)\Gamma(p+1+\delta)}{\Gamma(n+p+\delta)\Gamma(p+1)}$.

We know that $\Phi(n,p,\delta)$ is a decreasing function of n and

$$0 < \Phi(n,p,\delta) \leq \Phi(p+1,p,\delta) = \frac{2\Gamma(2p)\Gamma(p+1+\delta)}{\Gamma(2p+1+\delta)\Gamma(p)}.$$

Hence by using (22) and (26), we have

$$\left|\frac{\Gamma(p+1+\delta)}{\Gamma(p+1)} z^{-\delta} D_z^{-\delta} f(z)\right| \leq |z|^p + \Phi(p+1,p,\delta)|z|^{p+1} \sum_{n=p+1}^{\infty} a_n$$

$$\leq |z|^p + \frac{2\Gamma(2p)\Gamma(p+1+\delta)\left[(1-\theta(1-D))(p+1)!\right]}{\Gamma(2p+1+\delta)\Gamma(p)\left[(1-A)(1-B)\sigma\eta\ell(p+1)+1\right]\dfrac{(\alpha_1)_{p+1}...(\alpha_m)_{p+1}}{(\beta_1)_{p+1}...(\beta_k)_{p+1}}} |z|^{p+1},$$

which gives (24) ; we also have

$$\left|\frac{\Gamma(p+1+\delta)}{\Gamma(p+1)} z^{-\delta} D_z^{-\delta} f(z)\right| \geq |z|^p - \Phi(p+1,p,\delta)|z|^{p+1} \sum_{n=p+1}^{\infty} a_n$$

$$\geq |z|^p - \frac{2\Gamma(2p)\Gamma(p+1+\delta)\left[(1-\theta(1-D))(p+1)!\right]}{\Gamma(2p+1+\delta)\Gamma(p)\left[(1-A)(1-B)\sigma\eta\ell(p+1)+1\right]\dfrac{(\alpha_1)_{p+1}...(\alpha_m)_{p+1}}{(\beta_1)_{p+1}...(\beta_k)_{p+1}}} |z|^{p+1},$$

which gives (25).

Theorem 4: Let $f(z)$ defined by (2) be in the class $K(\sigma,\eta,\theta,\ell,A,B,D,p)$ Then

$$|G(z)| \leq |z|^p + \frac{(p+1)(p+1)!(1-\theta(1-D))}{(p-\lambda)\left[(1-A)(1-B)\sigma\eta\ell(p+1)+1\right]\dfrac{(\alpha_1)_{p+1}...(\alpha_m)_{p+1}}{(\beta_1)_{p+1}...(\beta_k)_{p+1}}} |z|^{p+1}, \qquad (27)$$

and

$$|G(z)| \geq |z|^p - \frac{(p+1)(p+1)!(1-\theta(1-D))}{(p-\lambda)\left[(1-A)(1-B)\sigma\eta\ell(p+1)+1\right]\frac{(\alpha_1)_{p+1}\dots(\alpha_m)_{p+1}}{(\beta_1)_{p+1}\dots(\beta_k)_{p+1}}}|z|^{p+1}. \qquad (28)$$

The inequalities in (27) and (28) are attained for the function $f(z)$ given by (21).

Proof: From (17) , we have $G(z) = z^p - \sum_{n=p+1}^{\infty} \prod(n,p,\lambda)a_n z^n$,

where $\prod(n,p,\lambda) = \frac{n!\Gamma(p-\lambda)}{\Gamma(p+1)\Gamma(n-\lambda)}$, we know that

$$0 < \prod(n,p,\lambda) \leq \prod(p+1,p,\lambda) = \frac{(p+1)}{(p-\lambda)}.$$

So by using (22) and (17), we get

$$|G(z)| \leq |z|^p + \frac{(p+1)(p+1)!(1-\theta(1-D))}{(p-\lambda)\left[(1-A)(1-B)\sigma\eta\ell(p+1)+1\right]\frac{(\alpha_1)_{p+1}\dots(\alpha_m)_{p+1}}{(\beta_1)_{p+1}\dots(\beta_k)_{p+1}}}|z|^{p+1}.$$

Similarity, we get

$$|G(z)| \geq |z|^p - \frac{(p+1)(p+1)!(1-\theta(1-D))}{(p-\lambda)\left[(1-A)(1-B)\sigma\eta\ell(p+1)+1\right]\frac{(\alpha_1)_{p+1}\dots(\alpha_m)_{p+1}}{(\beta_1)_{p+1}\dots(\beta_k)_{p+1}}}|z|^{p+1}.$$

Theorem 5: Let $f(z)$ defined by (2) be in the class $K(\sigma,\eta,\theta,\ell,A,B,D,p)$ Then

$$|F(z)| \leq |z|^p + \frac{(p+1)(p+1)!(1-\theta(1-D))}{(p+2-\lambda)\left[(1-A)(1-B)\sigma\eta\ell(p+1)+1\right]\frac{(\alpha_1)_{p+1}\dots(\alpha_m)_{p+1}}{(\beta_1)_{p+1}\dots(\beta_k)_{p+1}}}|z|^{p+1}, \qquad (29)$$

and

$$|F(z)| \geq |z|^p - \frac{(p+1)(p+1)!(1-\theta(1-D))}{(p+2-\lambda)\left[(1-A)(1-B)\sigma\eta\ell(p+1)+1\right]\frac{(\alpha_1)_{p+1}\dots(\alpha_m)_{p+1}}{(\beta_1)_{p+1}\dots(\beta_k)_{p+1}}}|z|^{p+1}. \qquad (30)$$

The inequalities in (29) and (30) are attained for the function *f*(z) given by (21).

Proof: From (18), we have $F(z) = z^p - \sum_{n=p+1}^{\infty} \gamma(n,p,\lambda)a_n z^n$, where $\gamma(n,p,\lambda)=\dfrac{n!\Gamma(2+p-\lambda)}{\Gamma(p+1)\Gamma(n-\lambda+2)}$, we know that

$$0<\gamma(n,p,\lambda)\le\gamma(p+1,p,\lambda)=\frac{(p+1)}{(p+2-\lambda)}.$$

So by using (22) and (18), we have

$$|F(z)|\le|z|^p+\frac{(p+1)(p+1)!(1-\theta(1-D))}{(p+2-\lambda)[1-A)(1-B)\sigma\eta\ell(p+1)+1]\dfrac{(\alpha_1)_{p+1}\ldots(\alpha_m)_{p+1}}{(\beta_1)_{p+1}\ldots(\beta_k)_{p+1}}}|z|^{p+1}.$$

Similarity , we get

$$|F(z)|\ge|z|^p-\frac{(p+1)(p+1)!(1-\theta(1-D))}{(p+2-\lambda)[1-A)(1-B)\sigma\eta\ell(p+1)+1]\dfrac{(\alpha_1)_{p+1}\ldots(\alpha_m)_{p+1}}{(\beta_1)_{p+1}\ldots(\beta_k)_{p+1}}}|z|^{p+1}.$$

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