Research Article

Statistical Properties & Different Methods Of Estimation Of A New Extended Weighted Frechet Distribution.

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Abstract: : In this paper, we introduce a new distribution is called the extended weighted Frechet distribution, which we obtain by applying the Azzalini method and deduced some statistical properties such as mean, variance, coefficients of variation, coefficient of skewness, and coefficient of kurtosis. The parameters of the new distribution were estimated by the following estimation methods: Maximum Likelihood Method (MLE) and percentile method. We used the Monte Carlo simulation to compare the performances of the proposed estimators obtained through methods of estimation.

Keywords: Azalini's method, extended weighted Frechet distribution, Percentils method.

1. Introduction

Recently, the known distributions did not represent the data obtained from the real world, which became more complex than before, as the problems associated with that data made it more difficult in terms of representation and modeling. Therefore, the matter required all researchers in the field of statistics to develop and expand these distributions in order to infer through them the results that are unbiased and consistent. As well as to obtain application flexibility for these distributions in data representation and modeling.

The proposed distribution called the expanded weighted frechet distribution that we obtained from applying Azzalini's method is more flexible. Azzalini (1985) introduced the skew-normal distribution when adding a parameter to impart more flexibility to the normal distribution. Also, M. K. Shakhatrah (2012) introduced a new class of distributions called two-parameter exponential distribution (TWE). This new class of distributions generalizes the weighted exponential distribution (WE) proposed by Gupta and Kundu (2009). It turns out that (TWE) better than (WE) application of the two groups from real data. Abbas Mahdavi & Leila Jabari (2017) proposed a new model using Azalini's method and demonstrated through the application that the proposed model is better and more flexible in application to real data. Abdulhakim A. Al-Babtain (2020) proposed a new model for Rayleigh distribution was called type I, the half logistic Rayleigh distribution, and the new model was applied to real data where the data was adopted (63) views from previous research by researchers (Kundu and Raqap, 2009) to clarify the importance and flexibility of the new Rayleigh distribution. In this paper, a new extended weighted frechet distribution has been proposed and studied. Frechet distribution was introduced by Maurice Frechet (French mathematician) (1878-1973). The frechet distribution has many applications in modeling and analyzing events such as earthquakes, floods and wind speeds, diseases, as well as in engineering fields to analyzing the statistical behavior of engineering material properties.

The frechet distribution is inverted Weibull distribution, also called the extreme value distribution. The probability density function (pdf) and the cumulative distribution function (CDF) for frechet distribution are (Kamran Abbas and Tang Yincai, 2012):

$$f(x) = \frac{\lambda}{\theta}\left(\frac{\theta}{x}\right)^{\lambda+1} e^{-\left(\frac{\theta}{x}\right)^{\lambda}} \qquad x > 0 \qquad \dots (1)$$

$$F(x) = e^{-\left(\frac{\theta}{x}\right)^{\lambda}} \qquad x > 0 \qquad \dots.(2)$$

Where: $\lambda > 0$ is the shape parameter of the distribution. , $\theta>0$ is the scale parameter distribution.

2. Azalini's Method

Mathematician Azzalini, A. (1985) introduced this method by inserting an additional parameter into the normal distribution to obtain a new distribution called the skew-normal distribution to achieve more flexibility in the normal distribution function as it is an extension of it. After that, many researchers inserted the shape parameter into non-symmetric distributions such as the T-skew, Skew-Cauchy by (Gupta *et al.*, 2002), and Skew-logistic distribution by (Nadarajah 2009).

The general formula for Azzalini's method is[9]:

$$f(x) = \frac{f(x).\,F(\alpha x)}{P(\alpha X_1 > X_2)} \qquad \dots (3)$$

where:

f(x) is the probability density function of frechet distribution.

F(αx) is the cumulative distribution function after adding α parameter.

X_1, X_2: is independent and identically random variables. Then, the conditional probability density function of $X=X_1$ given $\alpha X_1 > X_2$.

Observe that :

$$P(\alpha X_1 > X_2) = \int_0^\infty \left[\int_0^{\alpha X_1} f_2(X_2)dX_2\right] f_1(X_1)dX_1$$

$$P(\alpha X_1 > X_2) = \int_0^\infty \left[\int_0^{\alpha X_1} \frac{\lambda}{\theta}\left(\frac{\theta}{X_2}\right)^{\lambda+1} e^{-\left(\frac{\theta}{X_2}\right)^\lambda} dX_2\right] \frac{\lambda}{\theta}\left(\frac{\theta}{X_1}\right)^{\lambda+1} e^{-\left(\frac{\theta}{X_1}\right)^\lambda} dX_1$$

Then :

$$P(\alpha X_1 > X_2) = \frac{-1}{\frac{\alpha^\lambda+1}{\alpha^\lambda}} e^{-t^\lambda\left(\frac{\alpha^\lambda+1}{\alpha^\lambda}\right)}\Bigg]_0^\infty$$

$$P(\alpha X_1 > X_2) = \frac{\alpha^\lambda}{\alpha^\lambda+1} \qquad \dots (4)$$

By (4) in the (3), we get the probability density function to the extended weighted frechet distribution new (pdf).

$$f(x,\lambda,\theta,\alpha) = \frac{\lambda}{\theta}\,\frac{\alpha^\lambda+1}{\alpha^\lambda}\,\left(\frac{\theta}{x}\right)^{\lambda+1} e^{-\left(\frac{\theta}{x}\right)^\lambda\left(\frac{\alpha^\lambda+1}{\alpha^\lambda}\right)} \qquad \dots (5)$$

from eq. (5) we found the cumulative function to the new distribution by integration:

$$F(x,\lambda,\theta,\alpha) = \int_0^x \frac{\lambda}{\theta}\,\frac{\alpha^\lambda+1}{\alpha^\lambda}\,\left(\frac{\theta}{x}\right)^{\lambda+1} e^{-\left(\frac{\theta}{x}\right)^\lambda\left(\frac{\alpha^\lambda+1}{\alpha^\lambda}\right)} dx$$

$$F(x,\lambda,\theta,\alpha) = e^{-\left(\frac{\theta}{x}\right)^\lambda\left(\frac{\alpha^\lambda+1}{\alpha^\lambda}\right)} \qquad \dots (6)$$

3. Characteristics of the new extended weighted Frechet Distribution

3.1: mean

The mean can be obtained by finding The r^{th} moment about the origin when the value of r =1:

$$\mu_r = E(x^r) = \int_0^\infty x^r f(x)dx \qquad \dots (7)$$

By (1) in the (7), we get:

$$\mu_r = E(x^r) = \int_0^\infty x^r\,\frac{\lambda}{\theta}\,\frac{\alpha^\lambda+1}{\alpha^\lambda}\,\left(\frac{\theta}{x}\right)^{\lambda+1} e^{-\left(\frac{\theta}{x}\right)^\lambda\left(\frac{\alpha^\lambda+1}{\alpha^\lambda}\right)} dx$$

The r^{th} moment about the origin of extended weighted Frechet distribution is:

$$\mu_r = E(x^r) = \theta^r k^{\frac{r}{\lambda}}\,\Gamma\left(1-\frac{r}{\lambda}\right)$$

At r=1, we obtain the mean :

$$\mu_1 = E(x) = \theta\, k^{\frac{1}{\lambda}}\,\Gamma\left(1-\frac{1}{\lambda}\right) \qquad \dots (8)$$

3.2: Variance

The Variance can be obtained by finding The r^{th} moment about the mean when the value of r=2:

$$E(x-\mu)^r = \int_0^\infty (x-\mu)^r f(x)dx \qquad \dots (9)$$

By (1) in the (9), we get:

$$E(x-\mu)^r = \int_0^\infty (x-\mu)^r\,\frac{\lambda}{\theta}\,\frac{\alpha^\lambda+1}{\alpha^\lambda}\,\left(\frac{\theta}{x}\right)^{\lambda+1} e^{-\left(\frac{\theta}{x}\right)^\lambda\left(\frac{\alpha^\lambda+1}{\alpha^\lambda}\right)} dx$$

The r^{th} moment about the mean of extended weighted Frechet distribution is:

$$E(x-\mu)^r = \sum_{j=0}^{r} \binom{r}{j} \theta^j \left(k^{\frac{j}{\lambda}}\right) (-\mu)^{r-j}\ \Gamma\left(1-\frac{j}{\lambda}\right)$$

At r=2, we obtain the variance:

$$\boxed{E(x-\mu)^2 = \sum_{j=0}^{2} \binom{2}{j} \theta^j \left(k^{\frac{j}{\lambda}}\right) (-\mu)^{2-j}\ \Gamma\left(1-\frac{1}{\lambda}\right) = \text{var}(x)}$$

3.3: Coefficients of Variation

The mathematical formula for the coefficient of variation is:

$$C.V = \frac{\sigma}{\mu} \times 100$$

$$C.V = \frac{\sqrt{\Gamma\left(1-\frac{2}{\lambda}\right) - \left(\Gamma\left(1-\frac{1}{\lambda}\right)\right)^2}}{\Gamma\left(1-\frac{1}{\lambda}\right)} \times 100$$

3.4: Coefficient of Skewness

The mathematical formula for the Coefficient of Skewness is:

$$C.S = \frac{E(x-\mu)^3}{\sigma^3}$$

$$C.S = \frac{-\theta^3 k^{\frac{3}{\lambda}}\,\Gamma\left(1-\frac{1}{\lambda}\right)^3 + 3\theta k^{1/\lambda}\theta k^{\frac{1}{\lambda}}\,\Gamma\left(1-\frac{1}{\lambda}\right)^2 - 3\theta^3 k^{3/\lambda}\,\Gamma\left(1-\frac{1}{\lambda}\right)\Gamma\left(1-\frac{2}{\lambda}\right) + \theta^3 k^{3/\lambda}\Gamma\left(1-\frac{3}{\lambda}\right)}{\theta^3\, k^{3/\lambda}\left(\Gamma\left(1-\frac{2}{\lambda}\right) - \Gamma\left(1-\frac{1}{\lambda}\right)^2\right)^{3/2}}$$

3.5: Coefficient of Kurtosis

The mathematical formula for the Coefficient of Kurtosis is:

$$K.S = \frac{E(x-\mu)^4}{\sigma^4}$$

$$K.S = \frac{\left(-3\,\Gamma\left(1-\frac{1}{\lambda}\right)^4 + 6\,\Gamma\left(1-\frac{1}{\lambda}\right)^2\Gamma\left(1-\frac{2}{\lambda}\right) - 4\,\Gamma\left(1-\frac{1}{\lambda}\right)\Gamma\left(1-\frac{3}{\lambda}\right) + \Gamma\left(1-\frac{4}{\lambda}\right)\right)}{\left(\Gamma\left(1-\frac{2}{\lambda}\right)^2 - 2\Gamma\left(1-\frac{2}{\lambda}\right)\Gamma\left(1-\frac{1}{\lambda}\right)^2 + \Gamma\left(1-\frac{1}{\lambda}\right)^4\right)}$$

4. Estimation

4.1: Maximum Likelihood Method (MLE)

If $x_1, x_2 \ldots x_n$ are random variables distributed in the extended weighted Freight distribution, then:

$$f(x,\lambda,\theta,\alpha) = \frac{\lambda}{\theta}\left(1+\alpha^{-\lambda}\right)\left(\frac{\theta}{x}\right)^{\lambda+1} e^{-\left(\frac{\theta}{x}\right)^{\lambda}\left(1+\alpha^{-\lambda}\right)}$$

$$L(x_1, x_2 \ldots\ldots\ldots x_n, \lambda, \theta, \alpha) = \prod_{i=1}^{n} f(x_i, \lambda, \theta, \alpha)$$

$$L(x,\lambda,\theta,\alpha) = \frac{\lambda^n}{\theta}\left(1+\alpha^{-\lambda}\right)^n \prod_{i=1}^{n}\left(\frac{\theta}{x}\right)^{\lambda+1} e^{-\left(1+\alpha^{-\lambda}\right)\sum_{i=1}^{n}\left(\frac{\theta}{x}\right)^{\lambda}}$$

$$\ln L = n\ln\lambda - n\ln\theta + n\ln\left(1+\alpha^{-\lambda}\right) + (\lambda+1)\sum \ln\left(\frac{\theta}{x_i}\right) - \left(1+\alpha^{-\lambda}\right)\sum\left(\frac{\theta}{x_i}\right)^{\lambda}$$

$$\frac{\partial \ln L}{\partial \lambda} = \frac{n}{\lambda} - \frac{n\alpha^{-\lambda}\ln(\alpha)}{(1+\alpha^{-\lambda})} + \sum \ln\left(\frac{\theta}{x_i}\right) - \sum\left(\frac{\theta}{x_i}\right)^{\lambda} \ln\left(\frac{\theta}{x_i}\right) - \alpha^{-\lambda}\sum\left(\frac{\theta}{x_i}\right)^{\lambda}\ln\left(\frac{\theta}{x_i}\right) + \sum\left(\frac{\theta}{x_i}\right)^{\lambda}\alpha^{-\lambda}\ln\alpha \quad \ldots(10)$$

To estimate the parameter (θ):

$$\frac{\partial \ln L}{\partial \theta} = -\frac{n}{\theta} + \frac{\lambda\sum\frac{1}{x_i}}{\sum\left(\frac{\theta}{x_i}\right)} + \frac{\sum\frac{1}{x_i}}{\sum\left(\frac{\theta}{x_i}\right)} - \alpha\sum\left(\frac{\theta}{x_i}\right)^{\lambda-1}\left(\frac{1}{x_i}\right) - \lambda\alpha^{-\lambda}\sum\left(\frac{\theta}{x_i}\right)^{\lambda-1}\left(\frac{1}{x_i}\right) \quad \ldots(11)$$

To estimate the parameter (α):

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n(-\lambda)\alpha^{-(\lambda+1)}}{(1+\alpha^{-\lambda})} + \lambda\alpha^{-(\lambda+1)} \sum \left(\frac{\theta}{x_i}\right)^{\lambda} \qquad \ldots(12)$$

The MLE $\hat{\lambda}, \hat{\theta}$ and $\hat{\alpha}$ Can be obtained by solving the likelihood eqs.

$$\left.\frac{\partial lnL}{\partial \lambda}\right|_{\lambda=\hat{\lambda}} = 0 \quad , \quad \left.\frac{\partial lnL}{\partial \theta}\right|_{\theta=\hat{\theta}} = 0 \quad \text{and} \quad \left.\frac{\partial lnL}{\partial \theta}\right|_{\alpha=\hat{\alpha}} = 0$$

Clearly, it is difficult to solve the equations (10) , (11), and (12) therefore applying Newton-Raphson's method.

4.2: Maximum product of spacing estimator method

$$F(x) = e^{-\left(\frac{\theta}{x}\right)^{\lambda}\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)}$$

$$F(x) = e^{-k\left(\frac{\theta}{x}\right)^{\lambda}}$$

$$G = \left(\prod_{i=1}^{n+1} D_i\right)^{\frac{1}{n+1}}$$

$$D_i = \begin{cases} D_1 = F(x_1) \\ D_i = F(x_i) - F(x_{i-1}) = F\big(x_{(2:m)}\big); \quad i = 2, \ldots., m \\ D_{m+1} = 1 - F(x_m) \end{cases}$$

$$\ln G = \frac{1}{n+1}\left[\left(-\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\frac{\theta}{x_1}\right)^{\lambda}\right) + \sum_{i=2}^{n} \ln\left(e^{-\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\frac{\theta}{x_i}\right)^{\lambda}} - e^{-\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\frac{\theta}{x_{i-1}}\right)^{\lambda}}\right) + ln\left(1 - e^{-\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\frac{\theta}{x_n}\right)^{\lambda}}\right)\right]$$

$$\frac{\partial \ln(G)}{\partial \alpha} = \left(\frac{1}{n+1}\right)\left[\frac{-\left(\frac{\lambda}{\alpha} - \frac{\lambda(\alpha^{\lambda}+1)}{\alpha^{\lambda+1}}\right)\left(\frac{\theta}{x_1}\right)^{\lambda} + \sum_{i=2}^{n}\left(-\left(\left(\frac{\lambda}{\alpha} - \frac{\lambda(\alpha^{\lambda}+1)}{\alpha^{\lambda+1}}\left(\frac{\theta}{x_i}\right)^{\lambda}\right)e^{-\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\frac{\theta}{x_i}\right)^{\lambda}}\right)\right) + \left(\left(\frac{\lambda}{\alpha} - \frac{\lambda(\alpha^{\lambda}+1)}{\alpha^{\lambda+1}}\left(\frac{\theta}{x_{i-1}}\right)^{\lambda}\right)e^{-\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\frac{\theta}{x_{i-1}}\right)^{\lambda}}\right)}{\sum_{i=2}^{n}\left(e^{-\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\frac{\theta}{x_i}\right)^{\lambda}} - e^{-\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\frac{\theta}{x_{i-1}}\right)^{\lambda}}\right) + \sum_{i=2}^{n}\frac{\left(\left(\frac{\lambda}{\alpha} - \frac{\lambda(\alpha^{\lambda}+1)}{\alpha^{\lambda+1}}\right)\left(\frac{\theta}{x_n}\right)^{\lambda}\right)e^{-\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\frac{\theta}{x_n}\right)^{\lambda}}}{1 - e^{-\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\frac{\theta}{x_n}\right)^{\lambda}}}}\right] \qquad \ldots(13)$$

$$\frac{\partial \ln(G)}{\partial \theta} = \left(\frac{1}{n+1}\right)\left[\frac{-\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\frac{\lambda\left(\frac{\theta}{x_1}\right)^{\lambda}}{\theta}\right) + \sum_{i=2}^{n}\left(-\left(\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\frac{\lambda\left(\frac{\theta}{x_i}\right)^{\lambda}}{\theta}\right)e^{-\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\frac{\theta}{x_i}\right)^{\lambda}}\right)\right) + \left(\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\frac{\lambda\left(\frac{\theta}{x_{i-1}}\right)^{\lambda}}{\theta}\right)e^{-\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\frac{\theta}{x_{i-1}}\right)^{\lambda}}\right)}{\sum_{i=2}^{n}\left(e^{-\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\frac{\theta}{x_i}\right)^{\lambda}} - e^{-\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\frac{\theta}{x_{i-1}}\right)^{\lambda}}\right) + \sum_{i=2}^{n}\frac{\left(\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\frac{\lambda\left(\frac{\theta}{x_n}\right)^{\lambda}}{\theta}\right)\right)e^{-\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\frac{\theta}{x_n}\right)^{\lambda}}}{1 - e^{-\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\frac{\theta}{x_n}\right)^{\lambda}}}}\right] \qquad \ldots(14)$$

$$\frac{\partial \ln(G)}{\partial \lambda} = \left(\frac{1}{n+1}\right)\left[\frac{-\left(\ln(\alpha) - \frac{(\alpha^{\lambda}+1)\ln(\alpha)}{\alpha^{\lambda}}\right)\left(\frac{\theta}{x_1}\right)^{\lambda} - \left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\left(\frac{\theta}{x_1}\right)^{\lambda}\ln\left(\frac{\theta}{x_1}\right)\right) + \sum_{i=2}^{n}\left(-\left(\ln(\alpha) - \frac{(\alpha^{\lambda}+1)\ln(\alpha)}{\alpha l^{\lambda}}\right)\left(\frac{\theta}{x_i}\right)^{\lambda} - \left(\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\frac{\theta}{x_i}\right)^{\lambda} e^{-\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\frac{\theta}{x_i}\right)^{\lambda}}\right)\right) - \left(-\left(\ln(\alpha) - \frac{(\alpha^{\lambda}+1)\ln(\alpha)}{\alpha^{\lambda}}\right)\left(\frac{\theta}{x_{i-1}}\right)^{\lambda} - \left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\left(\frac{\theta}{x_{i-1}}\right)^{\lambda}\right.\right.}{\sum_{i=2}^{n}\left(\left(\frac{\theta}{x_i}\right)^{\lambda} e^{-\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\frac{\theta}{x_i}\right)^{\lambda}}\right) - \left(\left(\frac{\theta}{x_{i-1}}\right)^{\lambda} e^{-\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\frac{\theta}{x_{i-1}}\right)^{\lambda}}\right) - \sum_{i=2}^{n}\frac{\left(-\left(\ln(\alpha) - \frac{(\alpha^{\lambda}+1)\ln(\alpha)}{\alpha l^{\lambda}}\right)\left(\frac{\theta}{x_i}\right)^{\lambda} - \left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\left(\frac{\theta}{x_n}\right)^{\lambda}\ln\left(\frac{\theta}{x_n}\right)\right)e^{-\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\frac{\theta}{x_n}\right)^{\lambda}}\right)}{1 - e^{-\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\frac{\theta}{x_n}\right)^{\lambda}}}}\right.$$

The MLE $\hat{\lambda}, \hat{\theta}$ and $\hat{\alpha}$ can be obtained by solving the likelihood eqs.

$$\left.\frac{\partial lnG}{\partial \lambda}\right|_{\lambda=\hat{\lambda}} = 0 \quad , \quad \left.\frac{\partial lnG}{\partial \theta}\right|_{\theta=\hat{\theta}} = 0 \quad \text{and} \quad \left.\frac{\partial lnG}{\partial \theta}\right|_{\alpha=\hat{\alpha}} = 0$$

Clearly, it is difficult to solve the equations (13) , (14), and (15) therefore applying Newton-Raphson's method.

4.3:Method of Cramer-Von Mises Minimum

The CVME, as a type of minimum distance estimator, has less bias than the other minimum distance estimators

$$C(\lambda,\alpha,\theta)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F(x_i,\lambda,\alpha,\theta)-\frac{2i-1}{2n}\right]^2$$

$$C(\lambda,\alpha,\theta)=\frac{1}{12n}+\sum_{i=1}^{n}\left[e^{-\left(\frac{\theta}{x}\right)^{\lambda}\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)}-\frac{2i-1}{2n}\right]^2$$

$$\frac{\partial C(\lambda,\alpha,\theta)}{\partial\lambda}=2\sum_{i=1}^{n}\left[e^{-\left(\frac{\theta}{x}\right)^{\lambda}\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)}-\frac{2i-1}{2n}\right]\left[e^{-\left(\frac{\theta}{x}\right)^{\lambda}\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)}\left(-\left(\left(\frac{\theta}{x}\right)^{\lambda}\left(\frac{\alpha^{\lambda}\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)ln\alpha}{\alpha^{2\lambda}}\right)-\left(\alpha^{\lambda}+1\right)\alpha^{\lambda}ln(\alpha)\right)+\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\frac{\theta}{x}\right)^{\lambda}ln\left(\frac{\theta}{x}\right)\right)\right]$$

And

$$\frac{\partial C(\lambda,\alpha,\theta)}{\partial\theta}=2\sum_{i=1}^{n}\left[e^{-\left(\frac{\theta}{x}\right)^{\lambda}\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)}-\frac{2i-1}{2n}\right]\left[e^{-\left(\frac{\theta}{x}\right)^{\lambda}\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)}\lambda\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)\left(\frac{\theta}{x}\right)^{\lambda-1}\left(\frac{1}{x}\right)\right]=0$$

And

$$\frac{\partial C(\lambda,\alpha,\theta)}{\partial\alpha}==-2\sum_{i=1}^{n}\left[e^{-\left(\frac{\theta}{x}\right)^{\lambda}\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)}-\frac{2i-1}{2n}\right]\left[e^{-\left(\frac{\theta}{x}\right)^{\lambda}\left(\frac{\alpha^{\lambda}+1}{\alpha^{\lambda}}\right)}\left(\left(\frac{\theta}{x}\right)^{\lambda}\left(\frac{\lambda}{\alpha}-\frac{(\alpha^{\lambda}+1)\lambda}{\alpha^{\lambda}\alpha}\right)\right)\right]$$

The MLE $\hat{\lambda},\hat{\theta}$ *and* $\hat{\alpha}$ can be obtained by solving the likelihood eqs.

$$\left.\frac{\partial lnC}{\partial\lambda}\right|_{\lambda=\hat{\lambda}}=0\quad,\quad\left.\frac{\partial lnC}{\partial\theta}\right|_{\theta=\hat{\theta}}=0\quad\text{and}\quad\left.\frac{\partial lnC}{\partial\theta}\right|_{\alpha=\hat{\alpha}}=0$$

Clearly, it is difficult to solve the equations () , (), therefore applying Newton-Raphson's method.

5. Simulation Study

We ran a simulation to compare the behavior of the estimates with respect to mean squares of error (MSEs) using the rank method, as this method is based on selecting the lowest order for the sum of the total and partial ranks of all estimation methods for a set of imposed parameter values $(\hat{\lambda},\hat{\theta},\hat{\alpha})$. The results were as shown in Table (1).

Table (1): Simulation results for $(\lambda=2,\ \theta=4,\alpha=4)$

Estimated parameters			$\hat{\lambda}$ / $\hat{\theta}$ / $\hat{\alpha}$		$\hat{\lambda}$ / $\hat{\theta}$ / $\hat{\alpha}$		$\hat{\lambda}$ / $\hat{\theta}$ / $\hat{\alpha}$	
			1.71 / 2.77 / 7.29		2.27 / 3.91 / 3.52		2.00 / 4.00 / 4.00	
N	**ti**	**Real**	**Mle**	**MSE**	**MPS**	**MSE**	**CVMD**	**MSE**
35	13.2984	0.1985	0.2578	0.0302 {3}	0.2267	0.0008{2}	0.1986	0.000003 {1}
	12.8853	0.1984	0.2577	0.0281 {3}	0.2267	0.0008{2}	0.1986	0.000003 {1}
	12.5260	0.1884	0.2565	0.0228 {3}	0.2086	0.0006{2}	0.1885	0.000003 {1}
	12.4007	0.1844	0.2406	0.0074 {3}	0.2083	0.0005{2}	0.1845	0.000003 {1}
	12.3796	0.1833	0.2404	0.0073 {3}	0.2024	0.0004{2}	0.1834	0.000002 {1}
	11.140	0.1787	0.2291	0.0032	0.199	0.0003{2	0.17	0.000002

	8			{3}	9	}	89	{1}
	10.3747	0.1768	0.2233	$0.0027^{\{3\}}$	0.1938	$0.0003^{\{2\}}$	0.1769	$0.000002^{\{1\}}$
	10.2858	0.1721	0.2172	$0.0025^{\{3\}}$	0.1840	$0.0002^{\{2\}}$	0.1723	$0.000002^{\{1\}}$
	10.0135	0.1603	0.2119	$0.0025^{\{3\}}$	0.1786	$0.0002^{\{2\}}$	0.1603	$0.000001^{\{1\}}$
	8.4484	0.1550	0.1731	$0.0023^{\{3\}}$	0.1593	$0.0002^{\{2\}}$	0.1552	$0.000001^{\{1\}}$
$\sum Rank$			$30^{\{3\}}$		$20^{\{2\}}$		$10^{\{1\}}$	
Estimated parameters			$\hat{\lambda}$ $\hat{\theta}$ $\hat{\alpha}$		$\hat{\lambda}$ $\hat{\theta}$ $\hat{\alpha}$		$\hat{\lambda}$ $\hat{\theta}$ $\hat{\alpha}$	
			1.83 2.71 6.83		4.12 3.28 3.35		2.05 3.94 3.93	
N	**ti**	**Real**	**Mle**	**MSE**	**MPS**	**MSE**	**CVMD**	**MSE**
75	13.2984	0.1988	0.2779	$0.0375^{\{2\}}$	0.4713	$0.0780^{\{3\}}$	0.2062	$0.0001^{\{1\}}$
	12.8853	0.1988	0.2779	$0.0359^{\{2\}}$	0.4704	$0.0746^{\{3\}}$	0.2062	$0.0001^{\{1\}}$
	12.5421	0.1988	0.2776	$0.0352^{\{2\}}$	0.4614	$0.0746^{\{3\}}$	0.2062	$0.0001^{\{1\}}$
	12.5260	0.1986	0.2769	$0.0294^{\{2\}}$	0.4596	$0.0712^{\{3\}}$	0.2061	$0.0001^{\{1\}}$
	12.5131	0.1985	0.2591	$0.0109^{\{2\}}$	0.4528	$0.0688^{\{3\}}$	0.2056	$0.0001^{\{1\}}$
	12.4007	0.1985	0.2589	$0.0108^{\{2\}}$	0.4515	$0.0682^{\{3\}}$	0.2056	$0.0001^{\{1\}}$
	12.3796	0.1984	0.2495	$0.0066^{\{2\}}$	0.4508	$0.0640^{\{3\}}$	0.2055	$0.0001^{\{1\}}$
	11.9692	0.1974	0.2459	$0.0054^{\{2\}}$	0.4488	$0.0638^{\{3\}}$	0.2051	$0.0001^{\{1\}}$
	11.1408	0.1955	0.2407	$0.0040^{\{2\}}$	0.4467	$0.0635^{\{3\}}$	0.2019	$0.0001^{\{1\}}$
	10.9277	0.1925	0.2392	$0.0037^{\{2\}}$	0.4346	$0.0625^{\{3\}}$	0.1999	$0.0001^{\{1\}}$
$\sum Rank$			$20^{\{2\}}$		$30^{\{3\}}$		$10^{\{1\}}$	
Estimated parameters			$\hat{\lambda}$ $\hat{\theta}$ $\hat{\alpha}$		$\hat{\lambda}$ $\hat{\theta}$ $\hat{\alpha}$		$\hat{\lambda}$ $\hat{\theta}$ $\hat{\alpha}$	
			1.86 2.62 7.26		3.53 3.76 3.99		2.12 3.85 3.85	
n	**ti**	**Real**	**Mle**	**MSE**	**MPS**	**MSE**	**CVMD**	**MSE**
100	13.2984	0.1988	0.2914	$0.0569^{\{3\}}$	0.3576	$0.0255^{\{2\}}$	0.2176	$0.0004^{\{1\}}$
	12.8853	0.1988	0.2910	$0.0430^{\{3\}}$	0.3570	$0.0253^{\{2\}}$	0.2175	$0.0004^{\{1\}}$
	12.5421	0.1988	0.2908	$0.0411^{\{3\}}$	0.3568	$0.0251^{\{2\}}$	0.2175	$0.0004^{\{1\}}$
	12.5260	0.1988	0.2880	$0.0402^{\{3\}}$	0.3566	$0.0251^{\{2\}}$	0.2174	$0.0004^{\{1\}}$

<table>
<tr><td></td><td>12.5131</td><td>0.1986</td><td>0.2851</td><td>0.0334$^{\{3\}}$</td><td>0.3544</td><td>0.0250$^{\{2\}}$</td><td>0.2174</td><td>0.0004$^{\{1\}}$</td></tr>
<tr><td></td><td>12.4007</td><td>0.1985</td><td>0.2813</td><td>0.0289$^{\{3\}}$</td><td>0.3518</td><td>0.0240$^{\{2\}}$</td><td>0.2170</td><td>0.0004$^{\{1\}}$</td></tr>
<tr><td></td><td>12.3796</td><td>0.1985</td><td>0.2751</td><td>0.0188$^{\{2\}}$</td><td>0.3508</td><td>0.0238$^{\{3\}}$</td><td>0.2163</td><td>0.0004$^{\{1\}}$</td></tr>
<tr><td></td><td>11.9692</td><td>0.1984</td><td>0.2644</td><td>0.0120$^{\{2\}}$</td><td>0.3505</td><td>0.0234$^{\{3\}}$</td><td>0.2162</td><td>0.0004$^{\{1\}}$</td></tr>
<tr><td></td><td>11.1416</td><td>0.1981</td><td>0.2642</td><td>0.0119$^{\{2\}}$</td><td>0.3491</td><td>0.0231$^{\{3\}}$</td><td>0.2160</td><td>0.0004$^{\{1\}}$</td></tr>
<tr><td></td><td>11.1408</td><td>0.1974</td><td>0.2531</td><td>0.0072$^{\{2\}}$</td><td>0.3480</td><td>0.0230$^{\{3\}}$</td><td>0.2154</td><td>0.0004$^{\{1\}}$</td></tr>
<tr><td colspan="3">$\sum Rank$</td><td colspan="2">26$^{\{3\}}$</td><td colspan="2">24$^{\{2\}}$</td><td colspan="2">10$^{\{1\}}$</td></tr>
<tr><td colspan="3">Over ranks</td><td colspan="2">11</td><td colspan="2">10</td><td colspan="2">3</td></tr>
</table>

Table (2): Simulation results for $(\lambda = 3.5,\ \theta = 4,\alpha = 3.5)$

<table>
<tr><th colspan="3">Estimated parameters</th><th>$\hat{\lambda}$</th><th>$\hat{\theta}$</th><th>$\hat{\alpha}$</th><th>$\hat{\lambda}$</th><th>$\hat{\theta}$</th><th>$\hat{\alpha}$</th><th>$\hat{\lambda}$</th><th>$\hat{\theta}$</th><th>$\hat{\alpha}$</th></tr>
<tr><th rowspan="2">N</th><th rowspan="2">Ti</th><th rowspan="2">Real</th><td>2.19</td><td>2.65</td><td>6.28</td><td>3.68</td><td>5.30</td><td>1.52</td><td>3.11</td><td>4.15</td><td>3.83</td></tr>
<tr><th>Mle</th><th colspan="2">MSE</th><th>MPS</th><th colspan="2">MSE</th><th>CV MD</th><th colspan="2">MSE</th></tr>
<tr><td rowspan="10">35</td><td>9.8791</td><td>0.3329</td><td>0.2503</td><td colspan="2">0.0865$^{\{3\}}$</td><td>0.2860</td><td colspan="2">0.0296$^{\{2\}}$</td><td>0.2860</td><td colspan="2">0.0022$^{\{1\}}$</td></tr>
<tr><td>9.5171</td><td>0.3329</td><td>0.2494</td><td colspan="2">0.0836$^{\{3\}}$</td><td>0.2826</td><td colspan="2">0.0291$^{\{2\}}$</td><td>0.2859</td><td colspan="2">0.0022$^{\{1\}}$</td></tr>
<tr><td>9.2221</td><td>0.3168</td><td>0.2483</td><td colspan="2">0.0803$^{\{3\}}$</td><td>0.2743</td><td colspan="2">0.0261$^{\{2\}}$</td><td>0.2807</td><td colspan="2">0.0022$^{\{1\}}$</td></tr>
<tr><td>9.1230</td><td>0.3078</td><td>0.2482</td><td colspan="2">0.0799$^{\{3\}}$</td><td>0.2436</td><td colspan="2">0.0246$^{\{2\}}$</td><td>0.2754</td><td colspan="2">0.0021$^{\{1\}}$</td></tr>
<tr><td>9.1065</td><td>0.3069</td><td>0.2438</td><td colspan="2">0.0792$^{\{3\}}$</td><td>0.2434</td><td colspan="2">0.0244$^{\{2\}}$</td><td>0.2600</td><td colspan="2">0.0019$^{\{1\}}$</td></tr>
<tr><td>8.2114</td><td>0.2993</td><td>0.2419</td><td colspan="2">0.0739$^{\{3\}}$</td><td>0.2296</td><td colspan="2">0.0238$^{\{2\}}$</td><td>0.2595</td><td colspan="2">0.0017$^{\{1\}}$</td></tr>
<tr><td>7.7136</td><td>0.2888</td><td>0.2369</td><td colspan="2">0.0600$^{\{3\}}$</td><td>0.2233</td><td colspan="2">0.0217$^{\{2\}}$</td><td>0.2535</td><td colspan="2">0.0015$^{\{1\}}$</td></tr>
<tr><td>7.6579</td><td>0.2834</td><td>0.2202</td><td colspan="2">0.0598$^{\{3\}}$</td><td>0.2170</td><td colspan="2">0.0199$^{\{2\}}$</td><td>0.2449</td><td colspan="2">0.0013$^{\{1\}}$</td></tr>
<tr><td>7.4895</td><td>0.2770</td><td>0.2154</td><td colspan="2">0.0408$^{\{3\}}$</td><td>0.2117</td><td colspan="2">0.0146$^{\{2\}}$</td><td>0.2392</td><td colspan="2">0.0012$^{\{1\}}$</td></tr>
<tr><td>6.5717</td><td>0.2547</td><td>0.2104</td><td colspan="2">0.0302$^{\{3\}}$</td><td>0.1766</td><td colspan="2">0.0127$^{\{2\}}$</td><td>0.2354</td><td colspan="2">0.0010$^{\{1\}}$</td></tr>
<tr><td colspan="3">$\sum Rank$</td><td colspan="3">30$^{\{3\}}$</td><td colspan="3">20$^{\{2\}}$</td><td colspan="3">10$^{\{1\}}$</td></tr>
<tr><th colspan="3">Estimated parameters</th><th>$\hat{\lambda}$</th><th>$\hat{\theta}$</th><th>$\hat{\alpha}$</th><th>$\hat{\lambda}$</th><th>$\hat{\theta}$</th><th>$\hat{\alpha}$</th><th>$\hat{\lambda}$</th><th>$\hat{\theta}$</th><th>$\hat{\alpha}$</th></tr>
<tr><th rowspan="2">N</th><th rowspan="2">Ti</th><th rowspan="2">Real</th><td>2.33</td><td>2.60</td><td>6.27</td><td>3.54</td><td>4.17</td><td>3.71</td><td>3.22</td><td>4.10</td><td>3.74</td></tr>
<tr><th>Mle</th><th colspan="2">MSE</th><th>MPS</th><th colspan="2">MSE</th><th>CV MD</th><th colspan="2">MSE</th></tr>
<tr><td>75</td><td>9.8791</td><td>0.3329</td><td>0.3016</td><td colspan="2">0.0299$^{\{3\}}$</td><td>0.3223</td><td colspan="2">0.0012$^{\{1\}}$</td><td>0.3001</td><td colspan="2">0.0027$^{\{2\}}$</td></tr>
</table>

n	Ti	Real	Mle	MSE	MPS	MSE	CVMD	MSE
	9.5171	0.3329	0.2989	0.0299$^{\{3\}}$	0.3214	0.0012$^{\{1\}}$	0.2997	0.0027$^{\{2\}}$
	9.2350	0.3329	0.2976	0.0296$^{\{3\}}$	0.3197	0.0012$^{\{1\}}$	0.2996	0.0026$^{\{2\}}$
	9.2221	0.3317	0.2880	0.0291$^{\{3\}}$	0.3183	0.0012$^{\{1\}}$	0.2995	0.0024$^{\{2\}}$
	9.2119	0.3315	0.2528	0.0289$^{\{3\}}$	0.3160	0.0012$^{\{1\}}$	0.2986	0.0023$^{\{2\}}$
	9.1230	0.3314	0.2526	0.0289$^{\{3\}}$	0.3157	0.0011$^{\{1\}}$	0.2978	0.0023$^{\{2\}}$
	9.1065	0.3313	0.2410	0.0288$^{\{3\}}$	0.3155	0.0011$^{\{1\}}$	0.2975	0.0023$^{\{2\}}$
	8.7947	0.3300	0.2370	0.0275$^{\{3\}}$	0.3129	0.0011$^{\{1\}}$	0.2974	0.0022$^{\{2\}}$
	8.2114	0.3283	0.2314	0.0269$^{\{3\}}$	0.3108	0.0011$^{\{1\}}$	0.2958	0.0021$^{\{2\}}$
	8.0695	0.3263	0.2299	0.0266$^{\{3\}}$	0.3100	0.0011$^{\{1\}}$	0.2933	0.0020$^{\{2\}}$
$\sum Rank$			30$^{\{3\}}$		10$^{\{1\}}$		20$^{\{2\}}$	

Estimated parameters	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$
	2.70	2.50	5.11	3.25	4.05	3.50	3.75	3.91	3.29

n	Ti	Real	Mle	MSE	MPS	MSE	CV MD	MSE
100	11.5386	0.3329	0.3514	0.0640$^{\{3\}}$	0.3312	0.0001$^{\{1.5\}}$	0.3408	0.0001$^{\{1.5\}}$
	10.9245	0.3328	0.3452	0.0550$^{\{3\}}$	0.3310	0.00005$^{\{1\}}$	0.3406	0.0001$^{\{2\}}$
	9.0953	0.3326	0.3287	0.0343$^{\{3\}}$	0.3307	0.00005$^{\{1\}}$	0.3404	0.0001$^{\{2\}}$
	8.9342	0.3318	0.3039	0.0297$^{\{3\}}$	0.3307	0.00004$^{\{1\}}$	0.3401	0.0001$^{\{2\}}$
	8.8230	0.3318	0.3001	0.0297$^{\{3\}}$	0.3307	0.00004$^{\{1\}}$	0.3401	0.0001$^{\{2\}}$
	8.2663	0.3315	0.2917	0.0297$^{\{3\}}$	0.3306	0.00004$^{\{1\}}$	0.3399	0.0001$^{\{2\}}$
	8.1689	0.3313	0.2732	0.0297$^{\{3\}}$	0.3290	0.00002$^{\{1\}}$	0.3387	0.0001$^{\{2\}}$
	7.9143	0.3313	0.2681	0.0297$^{\{3\}}$	0.3290	0.00002$^{\{1\}}$	0.3387	0.0001$^{\{2\}}$
	7.8467	0.3303	0.2589	0.0297$^{\{3\}}$	0.3290	0.00002$^{\{1\}}$	0.3381	0.0001$^{\{2\}}$
	7.2624	0.3296	0.2560	0.0296$^{\{3\}}$	0.3280	0.00001$^{\{1\}}$	0.3376	0.0001$^{\{2\}}$
$\sum Rank$			30$^{\{3\}}$		10.5$^{\{1\}}$		19.5$^{\{2\}}$	

Table (3): Simulation results for $(\lambda = 3.5\,,\ \theta = 5\,, \alpha = 2.5)$

Estimated parameters			$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$
N	Ti	Real	2.69	3.11	5.81	3.07	5.69	2.71	3.96	4.80	2.34

			Mle	MSE	MPS	MSE	CV MD	MSE
35	11.4077	0.2642	0.2755	0.0473{3}	0.2644	0.000001{1}	0.3307	0.0064{2}
	11.3483	0.2629	0.2313	0.0230{3}	0.2632	0.000001{1}	0.3272	0.0064{2}
	9.9721	0.2627	0.2251	0.0230{3}	0.2628	0.000001{1}	0.3271	0.0062{2}
	9.6911	0.2589	0.2094	0.0229{3}	0.2589	0.000001{1}	0.3189	0.0061{2}
	9.1054	0.2559	0.1965	0.0229{3}	0.2558	0.000008{1}	0.3161	0.0053{2}
	7.6634	0.2544	0.1833	0.0229{3}	0.2543	0.000007{1}	0.3146	0.0046{2}
	7.5934	0.2507	0.1691	0.0229{3}	0.2506	0.000006{1}	0.3027	0.0042{2}
	7.3583	0.2505	0.1667	0.0228{3}	0.2504	0.000005{1}	0.2951	0.0040{2}
	6.7842	0.2494	0.1578	0.0225{3}	0.2492	0.000004{1}	0.2946	0.0027{2}
	6.2422	0.2485	0.1355	0.0224{3}	0.2491	0.000001{1}	0.2915	0.0027{2}
$\sum Rank$			30{3}		10{1}		20{2}	
Estimated parameters			$\hat{\lambda}$, $\hat{\theta}$, $\hat{\alpha}$		$\hat{\lambda}$, $\hat{\theta}$, $\hat{\alpha}$		$\hat{\lambda}$, $\hat{\theta}$, $\hat{\alpha}$	
N	Ti	Real	2.59, 3.10, 5.82		3.21, 5.14, 2.48		3.85, 4.82, 2.37	
			Mle	MSE	MPS	MSE	CV MD	MSE
75	12.5256	0.2643	0.2746	0.0496{3}	0.1854	0.0062{2}	0.3279	0.0056{1}
	11.6799	0.2643	0.2559	0.0302{3}	0.1854	0.0062{2}	0.3252	0.0056{1}
	11.4277	0.2643	0.2491	0.0239{3}	0.1854	0.0062{2}	0.3246	0.0056{1}
	9.5865	0.2643	0.2290	0.0226{3}	0.1853	0.0062{2}	0.3243	0.0056{1}
	9.4554	0.2641	0.2277	0.0225{3}	0.1853	0.0062{2}	0.3229	0.0056{1}
	9.3709	0.2634	0.2273	0.0223{3}	0.1847	0.0062{2}	0.3221	0.0054{1}
	8.9241	0.2632	0.2254	0.0223{3}	0.1846	0.0062{2}	0.3218	0.0053{1}
	8.7807	0.2630	0.2205	0.0219{3}	0.1845	0.0062{2}	0.3218	0.0053{1}
	8.6519	0.2620	0.2150	0.0218{3}	0.1842	0.0061{2}	0.3205	0.0053{1}
	8.6278	0.2606	0.2019	0.0218{3}	0.1839	0.0060{2}	0.3180	0.0052{1}
$\sum Rank$			30{3}		20{2}		10{1}	

Estimated parameters			$\hat{\lambda}$ $\hat{\theta}$ $\hat{\alpha}$		$\hat{\lambda}$ $\hat{\theta}$ $\hat{\alpha}$		$\hat{\lambda}$ $\hat{\theta}$ $\hat{\alpha}$	
			2.61 3.08 6.11		3.35 5.10 2.50		3.91 4.81 2.35	
n	**ti**	**Real**	**Mle**	**MSE**	**MPS**	**MSE**	**CVMD**	**MSE**
100	11.6483	0.2643	0.2798	0.0435{3}	0.2573	0.0003{1}	0.2885	0.0007{2}
	11.3357	0.2642	0.2745	0.0377{3}	0.2572	0.0003{1}	0.2884	0.0007{2}
	10.1570	0.2641	0.2608	0.0240{3}	0.2572	0.0003{1}	0.2881	0.0007{2}
	10.0341	0.2640	0.2403	0.0203{3}	0.2572	0.0003{1}	0.2875	0.0007{2}
	9.9475	0.2640	0.2372	0.0203{3}	0.2567	0.0003{1}	0.2875	0.0007{2}
	9.4914	0.2637	0.2304	0.0203{3}	0.2565	0.0003{1}	0.2873	0.0007{2}
	9.4078	0.2631	0.2154	0.0203{3}	0.2561	0.0003{1}	0.2868	0.0007{2}
	9.1838	0.2617	0.2114	0.0203{3}	0.2561	0.0003{1}	0.2868	0.0007{2}
	9.1231	0.2617	0.2040	0.0202{3}	0.2558	0.0003{1}	0.2859	0.0007{2}
	8.5764	0.2617	0.2017	0.0202{3}	0.2541	0.0002{1}	0.2856	0.0007{2}
$\sum Rank$			30{3}		10{1}		20{2}	

Table (4): Simulation results for $(\lambda = 4,\ \theta = 6.5,\alpha = 3)$

Estimated parameters			$\hat{\lambda}$ $\hat{\theta}$ $\hat{\alpha}$		$\hat{\lambda}$ $\hat{\theta}$ $\hat{\alpha}$		$\hat{\lambda}$ $\hat{\theta}$ $\hat{\alpha}$	
			2.64 4.12 6.32		7.04 7.25 3.06		4.31 6.36 2.84	
N	**Ti**	**Real**	**Mle**	**MSE**	**MPS**	**MSE**	**CVMD**	**MSE**
35	15.8429	0.2323	0.1976	0.0206{3}	0.2332	0.000003{1}	0.2752	0.0026{2}
	15.7023	0.2316	0.1682	0.0196{3}	0.2326	0.000003{1}	0.2730	0.0026{2}
	13.0341	0.2302	0.1641	0.0196{3}	0.2309	0.000003{1}	0.2703	0.0025{2}
	12.5816	0.2264	0.1538	0.0195{3}	0.2269	0.000003{1}	0.2639	0.0024{2}
	11.6990	0.2234	0.1453	0.0193{3}	0.2238	0.000003{1}	0.2631	0.0023{2}
	9.7573	0.2220	0.1367	0.0193{3}	0.2223	0.000002{1}	0.2607	0.0019{2}
	9.6687	0.2203	0.1273	0.0192{3}	0.2217	0.000002{1}	0.2549	0.0017{2}
	9.3736	0.2185	0.1257	0.0191{3}	0.2188	0.000002{1}	0.2490	0.0016{2}

N	Ti	Real	Mle	MSE	MPS	MSE	CV MD	MSE
	8.6676	0.2183	0.1197	0.0191$^{\{3\}}$	0.2186	0.000001$^{\{1\}}$	0.2462	0.0011$^{\{2\}}$
	8.0150	0.2172	0.1047	0.0190$^{\{3\}}$	0.2175	0.000001$^{\{1\}}$	0.2427	0.0011$^{\{2\}}$
$\sum Rank$			30$^{\{3\}}$		10$^{\{1\}}$		20$^{\{2\}}$	
Estimated parameters			$\hat{\lambda}$	$\hat{\theta}$ $\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$ $\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$ $\hat{\alpha}$
N	Ti	Real	2.59	4.10 6.27	3.91	6.81 3.02	4.22	6.39 2.88
			Mle	MSE	MPS	MSE	CV MD	MSE
75	19.3327	0.2323	0.1962	0.0218$^{\{3\}}$	0.2321	0.0035$^{\{2\}}$	0.2716	0.0021$^{\{1\}}$
	16.5281	0.2323	0.1831	0.0199$^{\{3\}}$	0.2320	0.0034$^{\{2\}}$	0.2700	0.0021$^{\{1\}}$
	15.8907	0.2323	0.1784	0.0198$^{\{3\}}$	0.2313	0.0034$^{\{2\}}$	0.2691	0.0021$^{\{1\}}$
	12.4183	0.2323	0.1649	0.0197$^{\{3\}}$	0.2312	0.0033$^{\{2\}}$	0.2686	0.0021$^{\{1\}}$
	12.2176	0.2321	0.1640	0.0197$^{\{3\}}$	0.2301	0.0033$^{\{2\}}$	0.2686	0.0021$^{\{1\}}$
	12.0900	0.2309	0.1637	0.0194$^{\{3\}}$	0.2298	0.0033$^{\{2\}}$	0.2684	0.0021$^{\{1\}}$
	11.4394	0.2308	0.1625	0.0194$^{\{3\}}$	0.2295	0.0033$^{\{2\}}$	0.2680	0.0020$^{\{1\}}$
	11.2379	0.2306	0.1592	0.0193$^{\{3\}}$	0.2283	0.0032$^{\{2\}}$	0.2675	0.0020$^{\{1\}}$
	11.0596	0.2294	0.1555	0.0191$^{\{3\}}$	0.2277	0.0031$^{\{2\}}$	0.2654	0.0020$^{\{1\}}$
	11.0265	0.2291	0.1468	0.0190$^{\{3\}}$	0.2259	0.0028$^{\{2\}}$	0.2632	0.0020$^{\{1\}}$
$\sum Rank$			30$^{\{3\}}$		20$^{\{2\}}$		10$^{\{1\}}$	
Estimated parameters			$\hat{\lambda}$	$\hat{\theta}$ $\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$ $\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$ $\hat{\alpha}$
n	ti	Real	2.61	4.08 6.20	3.60	6.59 3.07	4.28	6.37 2.84
			Mle	MSE	MPS	MSE	CV MD	MSE
100	16.4447	0.2323	0.2008	0.0182$^{\{3\}}$	0.2323	0.0000002$^{\{1\}}$	0.2377	0.00004$^{\{2\}}$
	15.6729	0.2323	0.1970	0.0182$^{\{3\}}$	0.2323	0.0000002$^{\{1\}}$	0.2377	0.00003$^{\{2\}}$
	13.3447	0.2323	0.1873	0.0182$^{\{3\}}$	0.2323	0.0000002$^{\{1\}}$	0.2376	0.00003$^{\{2\}}$
	13.1371	0.2318	0.1734	0.0181$^{\{3\}}$	0.2318	0.0000002$^{\{1\}}$	0.2369	0.00003$^{\{2\}}$
	12.9936	0.2318	0.1713	0.0181$^{\{3\}}$	0.2318	0.0000002$^{\{1\}}$	0.2369	0.00003$^{\{2\}}$
	12.272	0.231	0.166	0.0181{	0.231	0.000000	0.23	0.00003{

	2	3	7	3}	3	$1^{\{1\}}$	65	2}
	12.1455	0.2307	0.1566	$0.0181^{\{3\}}$	0.2307	$0.0000001^{\{1\}}$	0.2365	$0.00003^{\{2\}}$
	11.8131	0.2307	0.1539	$0.0180^{\{3\}}$	0.2307	$0.0000001^{\{1\}}$	0.2363	$0.00003^{\{2\}}$
	11.7247	0.2306	0.1490	$0.0180^{\{3\}}$	0.2306	$0.0000001^{\{1\}}$	0.2363	$0.00003^{\{2\}}$
	10.9562	0.2305	0.1475	$0.0180^{\{3\}}$	0.2305	$0.0000001^{\{1\}}$	0.2355	$0.00003^{\{2\}}$
$\sum Rank$			$30^{\{3\}}$		$10^{\{1\}}$		$20^{\{2\}}$	

Table (5): Simulation results for $(\lambda = 3,\ \theta = 6.5\ , \alpha = 4)$

Estimated parameters			$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$
n	Ti	Real	2.33	4.08	6.67	2.78	7.16	3.19	3.25	6.30	3.74
			Mle	MSE		MPS	MSE		CV MD	MSE	
35	20.5665	0.1776	0.2035	$0.0251^{\{3\}}$		0.1100	$0.0051^{\{2\}}$		0.2123	$0.0018^{\{1\}}$	
	20.3480	0.1773	0.1743	$0.0078^{\{3\}}$		0.1099	$0.0050^{\{2\}}$		0.2122	$0.0018^{\{1\}}$	
	16.1594	0.1740	0.1699	$0.0078^{\{3\}}$		0.1096	$0.0050^{\{2\}}$		0.2095	$0.0018^{\{1\}}$	
	15.4495	0.1716	0.1586	$0.0078^{\{3\}}$		0.1096	$0.0048^{\{2\}}$		0.2079	$0.0017^{\{1\}}$	
	14.0734	0.1699	0.1491	$0.0078^{\{3\}}$		0.1085	$0.0046^{\{2\}}$		0.2063	$0.0016^{\{1\}}$	
	11.1116	0.1671	0.1393	$0.0078^{\{3\}}$		0.1071	$0.0045^{\{2\}}$		0.2003	$0.0015^{\{1\}}$	
	10.9794	0.1664	0.1285	$0.0078^{\{3\}}$		0.1068	$0.0043^{\{2\}}$		0.1940	$0.0012^{\{1\}}$	
	10.5413	0.1657	0.1267	$0.0078^{\{3\}}$		0.1060	$0.0043^{\{2\}}$		0.1920	$0.0011^{\{1\}}$	
	9.5079	0.1648	0.1198	$0.0078^{\{3\}}$		0.1038	$0.0042^{\{2\}}$		0.1868	$0.0009^{\{1\}}$	
	8.5734	0.1626	0.1026	$0.0076^{\{3\}}$		0.1031	$0.0041^{\{2\}}$		0.1844	$0.0007^{\{1\}}$	
$\sum Rank$			$30^{\{3\}}$			$20^{\{2\}}$			$10^{\{1\}}$		
Estimated parameters			$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$
n	Ti	Real	2.30	4.06	6.56	2.69	6.81	3.20	3.18	6.34	3.78
			Mle	MSE		MPS	MSE		CV MD	MSE	
75	25.7884	0.1774	0.2046	$0.0269^{\{3\}}$		0.1705	$0.0001^{\{1\}}$		0.2109	$0.0014^{\{2\}}$	
	21.6257	0.1772	0.1925	$0.0168^{\{3\}}$		0.1703	$0.00005^{\{1\}}$		0.2102	$0.0014^{\{2\}}$	
	20.6409	0.1770	0.1879	$0.0135^{\{3\}}$		0.1701	$0.00005^{\{1\}}$		0.2099	$0.0014^{\{2\}}$	

n	Ti	Real	Mle	MSE	MPS	MSE	CVMD	MSE
	15.1940	0.1769	0.1737	0.0078{3}	0.1701	0.00005{1}	0.2085	0.0014{2}
	14.8803	0.1767	0.1727	0.0078{3}	0.1698	0.00005{1}	0.2074	0.0014{2}
	14.6814	0.1766	0.1725	0.0077{3}	0.1696	0.00005{1}	0.2065	0.0014{2}
	13.6714	0.1749	0.1711	0.0076{3}	0.1683	0.00005{1}	0.2064	0.0014{2}
	13.3604	0.1747	0.1675	0.0076{3}	0.1682	0.00005{1}	0.2057	0.0014{2}
	13.0861	0.1744	0.1634	0.0075{3}	0.1679	0.00005{1}	0.2055	0.0013{2}
	13.0352	0.1738	0.1536	0.0074{3}	0.1669	0.00004{1}	0.2053	0.0013{2}
$\sum Rank$			30{3}		10{1}		20{2}	

Estimated parameters			$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$
			2.32	4.04	6.56	2.98	6.57	3.67	3.23	6.32	3.73

n	Ti	Real	Mle	MSE	MPS	MSE	CV MD	MSE
100	21.4974	0.1774	0.2073	0.0233{3}	0.1274	0.0026{2}	0.1833	0.00004{1}
	20.3022	0.1774	0.2042	0.0204{3}	0.1274	0.0026{2}	0.1833	0.00004{1}
	16.6476	0.1774	0.1956	0.0135{3}	0.1274	0.0026{2}	0.1833	0.00004{1}
	16.3213	0.1773	0.1820	0.0066{3}	0.1274	0.0025{2}	0.1829	0.00004{1}
	16.0959	0.1772	0.1798	0.0066{3}	0.1272	0.0025{2}	0.1828	0.00004{1}
	14.9657	0.1770	0.1750	0.0066{3}	0.1268	0.0025{2}	0.1827	0.00004{1}
	14.7678	0.1766	0.1642	0.0066{3}	0.1267	0.0025{2}	0.1825	0.00004{1}
	14.2504	0.1760	0.1612	0.0066{3}	0.1266	0.0025{2}	0.1818	0.00004{1}
	14.1132	0.1759	0.1557	0.0066{3}	0.1264	0.0025{2}	0.1812	0.00004{1}
	12.9274	0.1756	0.1540	0.0066{3}	0.1257	0.0025{2}	0.1812	0.00004{1}
$\sum Rank$			30{3}		20{2}		10{1}	

Table (6): Simulation results for $(\lambda = 2,\ \theta = 4,\ \alpha = 2)$

Estimated parameters			$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$
			2.39	2.19	5.47	1.89	3.57	1.64	2.65	3.40	1.92

N	Ti	Real	Mle	MSE	MPS	MSE	CV MD	MSE
35	8.1785	0.1833	0.409	0.1486{	0.179	0.00005	0.33	0.0346{2}

			3	3}	8	{1}	71	
	8.1609	0.1832	0.3554	0.0831{3}	0.1798	0.00005{1}	0.3323	0.0344{2}
	7.6595	0.1830	0.3453	0.0732{3}	0.1796	0.00005{1}	0.3302	0.0336{2}
	7.5309	0.1824	0.3182	0.0500{3}	0.1794	0.00005{1}	0.3282	0.0327{2}
	7.2291	0.1823	0.2947	0.0338{3}	0.1793	0.00004{1}	0.3205	0.0297{2}
	6.2637	0.1822	0.2704	0.0204{2}	0.1791	0.00004{1}	0.3108	0.0239{3}
	6.2082	0.1819	0.2437	0.0099{2}	0.1787	0.00003{1}	0.3001	0.0236{3}
	6.0160	0.1814	0.2393	0.0085{2}	0.1787	0.00003{1}	0.2835	0.0203{3}
	5.5093	0.1813	0.2225	0.0076{2}	0.1781	0.00003{1}	0.2834	0.0158{3}
	4.9867	0.1796	0.1819	0.0074{2}	0.1773	0.00003{1}	0.2737	0.0139{3}
$\sum Rank$			25{2.5}		10{1}		25{2.5}	

Estimated parameters	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$
	2.15	2.20	5.92	1.93	3.84	1.92	2.64	3.34	1.94

N	Ti	Real	Mle	MSE	MPS	MSE	CVMD	MSE
75	8.4581	0.1831	0.3850	0.1328{3}	0.1488	0.0023{1}	0.3344	0.0332{2}
	8.2550	0.1830	0.3694	0.1107{3}	0.1487	0.0021{1}	0.3344	0.0331{2}
	8.1843	0.1827	0.3616	0.1011{3}	0.1487	0.0020{1}	0.3338	0.0331{2}
	7.4805	0.1824	0.3346	0.0718{3}	0.1487	0.0018{1}	0.3319	0.0330{2}
	7.4152	0.1822	0.3326	0.0699{3}	0.1485	0.0017{1}	0.3314	0.0328{2}
	7.3718	0.1820	0.3320	0.0693{3}	0.1482	0.0017{1}	0.3306	0.0328{2}
	7.1257	0.1818	0.3292	0.0666{3}	0.1478	0.0017{1}	0.3304	0.0327{2}
	7.0404	0.1807	0.3217	0.0596{3}	0.1478	0.0017{1}	0.3303	0.0317{2}
	6.9611	0.1803	0.3130	0.0522{3}	0.1477	0.0017{1}	0.3299	0.0311{2}
	6.9460	0.1798	0.2916	0.0359{3}	0.1471	0.0017{1}	0.3191	0.0305{2}
$\sum Rank$			10{3}		10{1}		20{2}	

Estimated parameters			$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$
n	Ti	Real	2.16	2.19	5.76	1.91	3.85	1.88	2.66	3.34	1.93

			Mle	MSE	MPS	MSE	CV MD	MSE
100	8.2464	0.1833	0.3892	0.1286$^{\{3\}}$	0.1492	0.0022$^{\{1\}}$	0.2541	0.0066$^{\{2\}}$
	8.1572	0.1832	0.3870	0.1235$^{\{3\}}$	0.1491	0.0020$^{\{1\}}$	0.2541	0.0066$^{\{2\}}$
	7.7388	0.1831	0.3769	0.1067$^{\{3\}}$	0.1491	0.0019$^{\{1\}}$	0.2541	0.0065$^{\{2\}}$
	7.6866	0.1830	0.3534	0.0779$^{\{3\}}$	0.1491	0.0017$^{\{1\}}$	0.2534	0.0064$^{\{2\}}$
	7.6486	0.1830	0.3492	0.0734$^{\{3\}}$	0.1491	0.0016$^{\{1\}}$	0.2534	0.0063$^{\{2\}}$
	7.4333	0.1829	0.3394	0.0639$^{\{3\}}$	0.1490	0.0016$^{\{1\}}$	0.2528	0.0063$^{\{2\}}$
	7.3909	0.1829	0.3164	0.0444$^{\{3\}}$	0.1487	0.0016$^{\{1\}}$	0.2519	0.0063$^{\{2\}}$
	7.2723	0.1829	0.3098	0.0396$^{\{3\}}$	0.1486	0.0016$^{\{1\}}$	0.2519	0.0062$^{\{2\}}$
	7.2389	0.1824	0.2975	0.0315$^{\{3\}}$	0.1486	0.0016$^{\{1\}}$	0.2519	0.0062$^{\{2\}}$
	6.9134	0.1822	0.2936	0.0292$^{\{3\}}$	0.1483	0.0016$^{\{1\}}$	0.2516	0.0060$^{\{2\}}$
$\sum Rank$			$30^{\{3\}}$		$10^{\{1\}}$		$20^{\{2\}}$	

Table (7): Simulation results for $(\lambda = 4.5,\ \theta = 5, \alpha = 3)$

Estimated parameters			$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$
n	Ti	Real	3.34	3.23	4.93	4.04	5.56	3.24	4.80	4.92	2.87
			Mle	MSE		MPS	MSE		CV MD	MSE	
35	11.5433	0.3382	0.2646	0.0462$^{\{3\}}$		0.3417	0.0001$^{\{1\}}$		0.3960	0.0046$^{\{2\}}$	
	11.4325	0.3368	0.2242	0.0461$^{\{3\}}$		0.3412	0.0001$^{\{1\}}$		0.3935	0.0046$^{\{2\}}$	
	9.4620	0.3359	0.2187	0.0460$^{\{3\}}$		0.3377	0.0001$^{\{1\}}$		0.3867	0.0045$^{\{2\}}$	
	9.1451	0.3307	0.2048	0.0455$^{\{3\}}$		0.3315	0.0001$^{\{1\}}$		0.3819	0.0043$^{\{2\}}$	
	8.5355	0.3267	0.1935	0.0454$^{\{3\}}$		0.3269	0.0001$^{\{1\}}$		0.3756	0.0037$^{\{2\}}$	
	7.2169	0.3246	0.1820	0.0453$^{\{3\}}$		0.3267	0.0001$^{\{1\}}$		0.3719	0.0035$^{\{2\}}$	
	7.1571	0.3198	0.1696	0.0452$^{\{3\}}$		0.3246	0.0001$^{\{1\}}$		0.3698	0.0031$^{\{2\}}$	
	6.9579	0.3196	0.1675	0.0448$^{\{3\}}$		0.3194	0.0001$^{\{1\}}$		0.3618	0.0029$^{\{2\}}$	
	6.4812	0.3193	0.1595	0.0448$^{\{3\}}$		0.3191	0.0001$^{\{1\}}$		0.3579	0.0021$^{\{2\}}$	
	6.0396	0.3181	0.1398	0.0435$^{\{3\}}$		0.3174	0.00002$^{\{1\}}$		0.3493	0.0018$^{\{2\}}$	

$\sum Rank$			$30^{\{3\}}$		$10^{\{1\}}$		$20^{\{2\}}$	
Estimated parameters			$\hat{\lambda}$ $\hat{\theta}$ $\hat{\alpha}$		$\hat{\lambda}$ $\hat{\theta}$ $\hat{\alpha}$		$\hat{\lambda}$ $\hat{\theta}$ $\hat{\alpha}$	
			2.87 3.17 5.54		3.81 5.24 3.06		4.71 4.94 2.89	
N	**Ti**	**Real**	**Mle**	**MSE**	**MPS**	**MSE**	**CVMD**	**MSE**
75	14.7829	0.3383	0.2549	$0.0477^{\{3\}}$	0.1942	$0.0208^{\{2\}}$	0.3898	$0.0038^{\{1\}}$
	12.0986	0.3383	0.2373	$0.0475^{\{3\}}$	0.1942	$0.0208^{\{2\}}$	0.3889	$0.0038^{\{1\}}$
	11.5813	0.3383	0.2311	$0.0474^{\{3\}}$	0.1941	$0.0208^{\{2\}}$	0.3879	$0.0038^{\{1\}}$
	9.0316	0.3382	0.2134	$0.0470^{\{3\}}$	0.1941	$0.0208^{\{2\}}$	0.3872	$0.0038^{\{1\}}$
	8.8925	0.3381	0.2123	$0.0469^{\{3\}}$	0.1940	$0.0207^{\{2\}}$	0.3870	$0.0037^{\{1\}}$
	8.8044	0.3368	0.2119	$0.0468^{\{3\}}$	0.1936	$0.0205^{\{2\}}$	0.3858	$0.0037^{\{1\}}$
	8.3578	0.3366	0.2103	$0.0467^{\{3\}}$	0.1934	$0.0205^{\{2\}}$	0.3846	$0.0036^{\{1\}}$
	8.2203	0.3363	0.2060	$0.0454^{\{3\}}$	0.1934	$0.0205^{\{2\}}$	0.3836	$0.0036^{\{1\}}$
	8.0988	0.3348	0.2013	$0.0452^{\{3\}}$	0.1933	$0.0202^{\{2\}}$	0.3804	$0.0035^{\{1\}}$
	8.0763	0.3330	0.1901	$0.0452^{\{3\}}$	0.1928	$0.0198^{\{2\}}$	0.3798	$0.0035^{\{1\}}$
$\sum Rank$			$30^{\{3\}}$		$20^{\{2\}}$		$10^{\{1\}}$	
Estimated parameters			$\hat{\lambda}$ $\hat{\theta}$ $\hat{\alpha}$		$\hat{\lambda}$ $\hat{\theta}$ $\hat{\alpha}$		$\hat{\lambda}$ $\hat{\theta}$ $\hat{\alpha}$	
			2.95 3.16 5.29		4.29 5.15 3.28		4.78 4.92 2.86	
n	**Ti**	**Real**	**Mle**	**MSE**	**MPS**	**MSE**	**CVMD**	**MSE**
100	12.0296	0.3383	0.2699	$0.0433^{\{3\}}$	0.3203	$0.0064^{\{2\}}$	0.3417	$0.00001^{\{1\}}$
	11.4094	0.3382	0.2645	$0.0433^{\{3\}}$	0.3203	$0.0064^{\{2\}}$	0.3417	$0.00001^{\{1\}}$
	9.6816	0.3381	0.2509	$0.0433^{\{3\}}$	0.3202	$0.0063^{\{2\}}$	0.3417	$0.00001^{\{1\}}$
	9.5347	0.3378	0.2316	$0.0432^{\{3\}}$	0.3202	$0.0062^{\{2\}}$	0.3411	$0.00001^{\{1\}}$
	9.4336	0.3378	0.2288	$0.0432^{\{3\}}$	0.3201	$0.0060^{\{2\}}$	0.3411	$0.00001^{\{1\}}$
	8.9303	0.3372	0.2225	$0.0432^{\{3\}}$	0.3201	$0.0060^{\{2\}}$	0.3405	$0.00001^{\{1\}}$
	8.8427	0.3363	0.2089	$0.0430^{\{3\}}$	0.3193	$0.0059^{\{2\}}$	0.3395	$0.00001^{\{1\}}$
	8.6137	0.3354	0.2052	$0.0429^{\{3\}}$	0.3192	$0.0051^{\{2\}}$	0.3391	$0.00001^{\{1\}}$
	8.5531	0.3354	0.1986	$0.0427^{\{3\}}$	0.3190	$0.0050^{\{2\}}$	0.3391	$0.00001^{\{1\}}$

	8.0285	0.3350	0.1966	$0.0427^{\{3\}}$	0.3187	$0.0048^{\{2\}}$	0.3388	$0.00001^{\{1\}}$
$\sum Rank$			$30^{\{3\}}$		$20^{\{2\}}$		$10^{\{1\}}$	

Table (8): Simulation results for $(\lambda = 2.5\,,\ \theta = 5\,, \alpha = 3.5)$

Estimated parameters			$\hat{\lambda}$ $\hat{\theta}$ $\hat{\alpha}$		$\hat{\lambda}$ $\hat{\theta}$ $\hat{\alpha}$		$\hat{\lambda}$ $\hat{\theta}$ $\hat{\alpha}$	
			2.21 3.08 6.53		3.30 5.09 3.01		2.84 4.72 3.20	
N	**Ti**	**Real**	**Mle**	**MSE**	**MPS**	**MSE**	**CVMD**	**MSE**
35	15.5345	0.1941	0.2768	$0.0522^{\{3\}}$	0.3519	$0.0249^{\{2\}}$	0.2483	$0.0039^{\{1\}}$
	15.4271	0.1938	0.2412	$0.0180^{\{2\}}$	0.3460	$0.0232^{\{3\}}$	0.2451	$0.0039^{\{1\}}$
	12.9692	0.1901	0.2352	$0.0143^{\{2\}}$	0.3372	$0.0224^{\{3\}}$	0.2443	$0.0039^{\{1\}}$
	12.4761	0.1876	0.2193	$0.0070^{\{2\}}$	0.3233	$0.0181^{\{3\}}$	0.2427	$0.0037^{\{1\}}$
	11.4603	0.1857	0.2056	$0.0069^{\{2\}}$	0.3164	$0.0177^{\{3\}}$	0.2408	$0.0035^{\{1\}}$
	9.0421	0.1825	0.1912	$0.0069^{\{2\}}$	0.3096	$0.0168^{\{3\}}$	0.2294	$0.0028^{\{1\}}$
	8.9282	0.1818	0.1752	$0.0069^{\{2\}}$	0.3018	$0.0135^{\{3\}}$	0.2282	$0.0026^{\{1\}}$
	8.5483	0.1810	0.1725	$0.0069^{\{2\}}$	0.2883	$0.0112^{\{3\}}$	0.2174	$0.002^{\{1\}}$
	7.6380	0.1800	0.1622	$0.0069^{\{2\}}$	0.2821	$0.0102^{\{3\}}$	0.2116	$0.0019^{\{1\}}$
	6.8036	0.1776	0.1365	$0.0068^{\{2\}}$	0.2686	$0.0087^{\{3\}}$	0.2107	$0.0016^{\{1\}}$
$\sum Rank$			$21^{\{2\}}$		$29^{\{3\}}$		$10^{\{1\}}$	
Estimated parameters			$\hat{\lambda}$ $\hat{\theta}$ $\hat{\alpha}$		$\hat{\lambda}$ $\hat{\theta}$ $\hat{\alpha}$		$\hat{\lambda}$ $\hat{\theta}$ $\hat{\alpha}$	
			2.17 3.07 6.63		2.22 5.04 2.85		2.76 4.74 3.26	
n	**Ti**	**Real**	**Mle**	**MSE**	**MPS**	**MSE**	**CVMD**	**MSE**
75	17.5673	0.1939	0.2734	$0.0530^{\{3\}}$	0.1894	$0.00002^{\{1\}}$	0.2457	$0.0034^{\{2\}}$
	16.0273	0.1936	0.2601	$0.0371^{\{3\}}$	0.1890	$0.00002^{\{1\}}$	0.2455	$0.0034^{\{2\}}$
	15.5705	0.1934	0.2544	$0.0314^{\{3\}}$	0.1890	$0.00002^{\{1\}}$	0.2452	$0.0034^{\{2\}}$
	12.2934	0.1934	0.2361	$0.0169^{\{3\}}$	0.1888	$0.00002^{\{1\}}$	0.2445	$0.0034^{\{2\}}$
	12.0653	0.1931	0.2348	$0.0160^{\{3\}}$	0.1887	$0.00002^{\{1\}}$	0.2427	$0.0034^{\{2\}}$
	11.9186	0.1930	0.2344	$0.0158^{\{3\}}$	0.1884	$0.00002^{\{1\}}$	0.2415	$0.0034^{\{2\}}$
	11.149	0.191	0.232	$0.0146^{\{3\}}$	0.1864	0.00002{	0.241	0.0034{

	4	1	5			1}	4	2}
	10.9047	0.1908	0.2276	0.0119$^{\{3\}}$	0.1864	0.00002$^{\{1\}}$	0.2413	0.0033$^{\{2\}}$
	10.6861	0.1905	0.2220	0.0091$^{\{3\}}$	0.1862	0.00002$^{\{1\}}$	0.2410	0.0032$^{\{2\}}$
	10.6452	0.1900	0.2083	0.0071$^{\{3\}}$	0.1859	0.00002$^{\{1\}}$	0.2390	0.0031$^{\{2\}}$
$\sum Rank$			30$^{\{3\}}$		10$^{\{1\}}$		20$^{\{2\}}$	

Estimated parameters			$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$
n	**Ti**	**Real**	2.18	3.04	6.68	2.42	4.88	3.24	2.80	4.73	3.22

n	**Ti**	**Real**	**Mle**	**MSE**	**MPS**	**MSE**	**CVMD**	**MSE**
100	15.9700	0.1939	0.2749	0.0467$^{\{3\}}$	0.1886	0.0001$^{\{1\}}$	0.2146	0.0005$^{\{2\}}$
	15.4043	0.1939	0.2722	0.0425$^{\{3\}}$	0.1886	0.0001$^{\{1\}}$	0.2146	0.0005$^{\{2\}}$
	13.2955	0.1938	0.2631	0.0314$^{\{3\}}$	0.1884	0.0001$^{\{1\}}$	0.2146	0.0005$^{\{2\}}$
	13.0786	0.1938	0.2465	0.0173$^{\{3\}}$	0.1879	0.0001$^{\{1\}}$	0.2143	0.0005$^{\{2\}}$
	12.9259	0.1937	0.2437	0.0154$^{\{3\}}$	0.1878	0.0001$^{\{1\}}$	0.2136	0.0005$^{\{2\}}$
	12.1278	0.1935	0.2374	0.0118$^{\{3\}}$	0.1877	0.0001$^{\{1\}}$	0.2131	0.0005$^{\{2\}}$
	11.9826	0.1930	0.2226	0.0058$^{\{3\}}$	0.1877	0.0001$^{\{1\}}$	0.2128	0.0005$^{\{2\}}$
	11.5952	0.1923	0.2185	0.0058$^{\{3\}}$	0.1877	0.0001$^{\{1\}}$	0.2123	0.0005$^{\{2\}}$
	11.4907	0.1923	0.2108	0.0058$^{\{3\}}$	0.1872	0.0001$^{\{1\}}$	0.2119	0.0005$^{\{2\}}$
	10.5584	0.1920	0.2084	0.0057$^{\{3\}}$	0.1865	0.0001$^{\{1\}}$	0.2101	0.0005$^{\{2\}}$
$\sum Rank$			30$^{\{3\}}$		10$^{\{1\}}$		20$^{\{2\}}$	

Table (9): Partial and overall ranks of all estimation methods for various combinations of $(\hat{\lambda}, \hat{\theta}, \hat{\alpha})$

Paramerters case	N	Method		
		MLE	PSM	CVM
$(\hat{\lambda} = 2, \hat{\theta} = 4, \hat{\alpha} = 4)$	35	3	2	1
	75	2	3	1
	100	3	2	1
$\lambda = 3.5,\ \theta = 4, \alpha = 3.5$	35	3	2	1
	75	3	1	2
	100	3	1	2
$\lambda = 3.5, \theta = 5, \alpha = 2.5$	35	3	1	2
	75	3	2	1
	100	3	1	2

$\lambda = 4, \theta = 6.5, \alpha = 3$	35	3	1	2
	75	3	2	1
	100	3	1	2
$\lambda = 3, \theta = 6.5, \alpha = 4$	35	3	2	1
	75	3	1	2
	100	3	2	1
$\lambda = 2, \theta = 4, \alpha = 2$	35	2.5	1	2.5
	75	3	1	2
	100	3	1	2
$\lambda = 4.5, \theta = 5, \alpha = 3$	35	3	1	2
	75	3	2	1
	100	3	2	1
$\lambda = 2.5, \theta = 5, \alpha = 3.5$	35	2	3	1
	75	3	1	2
	100	3	1	2
∑Ranks		69.5	37 37 37.5	37.5
Overall rank		3	1	2
Percentages		0.483	0.257	0.260

From table (10), we can observe that:

- The maximum product spacing of the estimator method is the best method over the rest of the estimation methods.
- The sample size (n=100) is the best over the rest of the sample sizes.
- And the values imposed for the parameters when ($\lambda = 4$, $\theta = 6.5$, $\alpha = 3$).
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Conclusion

In this paper, we proposed a new distribution derived from the Frechet distribution called the new extended weighted Frechet distribution. The properties of the new distribution were derived through mathematical and statistical operations. We also estimated the parameters of the new distribution by estimation methods Max. Production spacing of estimator method, and estimation of failure function by simulation.

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