

Generating Matrices and Fibonacci-Like Numbers

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Abstract: Among several classes of interesting numbers, Fibonacci numbers plays a significant role and these numbers very often turns up in several branches of Science, Engineering and Technology. Considering two by two square matrices whose entries are Fibonacci and Fibonacci like sequence numbers, we investigate the integral powers of such matrices and prove interesting results concerning them. Finally, the limiting cases of entries of n th power of such matrices are derived.

Keywords: Fibonacci Numbers, Fibonacci – Like Numbers, Generating Matrices, Cassini’s Identity, Golden Ratio, Limiting Matrices.

1. Introduction

The concept of Fibonacci Numbers was first introduced to Europe through the memorable book “Liber Abaci” written by Italian mathematician Leonardo Fibonacci. Ever since its publication in 1202 CE, this classic book was responsible for two important mathematical achievements. The first being spreading the modern Hindu – Arabic numeral system throughout the globe thereby replacing the then existing Roman numeral system. Second, in one of many amusing problems present in the book, Fibonacci posed a problem about the growth of immortal rabbits whose solution turns to be Fibonacci Numbers. So much about Fibonacci numbers has been written and investigated since that time that a separate journal in the name “Fibonacci Quarterly” was devoted to study their properties. This journal beginning its journey from 1963 continues to this day to publish new results about these ever fascinating numbers. In this paper, we shall discuss novel approach of generating Fibonacci numbers through square matrices and study the limiting behavior of n th power of such matrices.

2. Definitions

1. Let $F_0 = 0, F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ (2.1) for $n \geq 0$. The sequence of numbers generated through recurrence relation defined in (2.1) is called Fibonacci numbers and the sequence is called Fibonacci sequence. From (2.1), we notice that each number is sum of two previous numbers except the first two numbers. With this condition, the Fibonacci numbers is given by the sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, . . .

2. Let $FL_0 = a, FL_1 = b$ (where a, b are integers not both zero) and $FL_{n+2} = FL_{n+1} + FL_n$ (2.2) for $n \geq 0$. The sequence of numbers generated through recurrence relation defined in (2.2) is called Fibonacci-Like numbers and the sequence is called Fibonacci-Like sequence. Using (2.2), the Fibonacci-Like numbers are given by the sequence $a, b, a + b, a + 2b, 2a + 3b, 3a + 5b, 5a + 8b, \dots$

We notice that the coefficients of a, b in Fibonacci-Like sequence are precisely the numbers in Fibonacci sequence.

3. Matrices

Let G be a 2×2 square matrix be defined by $G = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ (3.1). In view of Fibonacci numbers notation we

can write G as $G = \begin{pmatrix} F_0 & F_1 \\ F_1 & F_2 \end{pmatrix}$ (3.2). We similarly define another 2×2 square matrix M by

$M = \begin{pmatrix} a & b \\ b & a + b \end{pmatrix}$ (3.3). By the definition of Fibonacci – Like sequence, we can write M as

$M = \begin{pmatrix} FL_0 & FL_1 \\ FL_1 & FL_2 \end{pmatrix}$ (3.4). We now prove some useful theorems.

4. Generating Matrices

4.1 Theorem 1

For any natural number n , $G^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}$ (4.1) where G is the matrix defined in (3.1) and G^n is

product of G performed n times under matrix multiplication.

Proof: We prove by using Mathematical Induction on number of times that G is multiplied namely n . If $n = 1$, then equation (4.1) exactly coincide with the matrix provided in definition (3.2). If $n = 2$, then

$$G^2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} F_1 & F_2 \\ F_2 & F_3 \end{pmatrix} \text{ which matches with (4.1). Thus the result is true for } n = 1,$$

2. By Induction Hypothesis, we now assume that the result is true up to $n = k$. We now try to prove for $n = k + 1$.

$$G^{k+1} = G^k \times G = \begin{pmatrix} F_{k-1} & F_k \\ F_k & F_{k+1} \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} F_k & F_{k-1} + F_k \\ F_{k+1} & F_k + F_{k+1} \end{pmatrix}$$

where G^k is a matrix obtained because of Induction Hypothesis for $n = k$. By equation (2.1), we find that

$$F_{k-1} + F_k = F_{k+1}, F_k + F_{k+1} = F_{k+2}. \text{ Hence, } G^{k+1} = \begin{pmatrix} F_k & F_{k+1} \\ F_{k+1} & F_{k+2} \end{pmatrix}.$$

Thus the result is true for $n = k + 1$ also. Hence by Induction Principle, the result is true for all natural numbers n . This completes the proof.

4.2 Theorem 2

For any natural number n , $MG^{n-1} = \begin{pmatrix} FL_{n-1} & FL_n \\ FL_n & FL_{n+1} \end{pmatrix}$ (4.2) where G and M are the matrices defined in

(3.1) and (3.3) respectively, G^{n-1} is product of G performed $n - 1$ times under matrix multiplication.

Proof: As in the previous theorem, we prove this on induction on the power term n of the matrix M . If $n = 1$, then equation (4.2) exactly coincide with the matrix provided in definition (3.4). If $n = 2$, then

$$MG = \begin{pmatrix} a & b \\ b & a+b \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} b & a+b \\ a+b & a+2b \end{pmatrix} = \begin{pmatrix} FL_1 & FL_2 \\ FL_2 & FL_3 \end{pmatrix} \text{ which is true by (4.2). Thus the result is}$$

true for $n = 1, 2$. By Induction Hypothesis, we now assume that the result is true up to $n = k$. We now try to prove for $n = k + 1$.

$$MG^k = MG^{k-1} \times G = \begin{pmatrix} FL_{k-1} & FL_k \\ FL_k & FL_{k+1} \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} FL_k & FL_{k-1} + FL_k \\ FL_{k+1} & FL_k + FL_{k+1} \end{pmatrix}$$

where MG^{k-1} is a matrix obtained because of Induction Hypothesis for $n = k$. By equation (2.2), we find that

$$FL_{k-1} + FL_k = FL_{k+1}, FL_k + FL_{k+1} = FL_{k+2}. \text{ Hence, } MG^k = \begin{pmatrix} FL_k & FL_{k+1} \\ FL_{k+1} & FL_{k+2} \end{pmatrix}.$$

Thus the result is true for $n = k + 1$ also. Hence by Induction Principle, the result is true for all natural numbers n . This completes the proof.

In view of theorems 1 and 2, we see that the matrices G and M are generating matrices for Fibonacci and Fibonacci-Like numbers respectively.

5. Cassini's Identities

We now prove important identities known as Cassini's Identities for Fibonacci and Fibonacci-Like numbers using the generating matrices G and M .

5.1 Theorem 3

(a) If F_n is the n th Fibonacci number then $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$ (5.1)

(b) If FL_n is the n th Fibonacci-Like number then $FL_{n-1}FL_{n+1} - FL_n^2 = (-1)^{n-1}(a^2 + ab - b^2)$ (5.2)

Proof: We use Theorems 1 and 2 to prove these results respectively.

(a) By definition of (3.1), we first note that $|G| = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1$. Also, by property of determinants, we get

$|G^n| = |G|^n = (-1)^n$. Now by equation (4.1) of theorem 1, we get

$$|G^n| = \begin{vmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{vmatrix} \Rightarrow F_{n-1}F_{n+1} - F_n^2 = (-1)^n. \text{ This proves (a).}$$

(b) By definitions (3.1) and (3.3) we have $|G| = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1$. Also, by property of determinants, we get

$|MG^{n-1}| = |G|^{n-1} \times |M| = (-1)^{n-1}(a^2 + ab - b^2)$. Now by equation (4.2) of theorem 2, we get

$$|MG^{n-1}| = \begin{vmatrix} FL_{n-1} & FL_n \\ FL_n & FL_{n+1} \end{vmatrix} \Rightarrow FL_{n-1}FL_{n+1} - FL_n^2 = (-1)^{n-1}(a^2 + ab - b^2). \text{ This proves (b).}$$

This completes the proof.

6. Limiting Matrices

We now prove a basic result from which we can deduce some interesting consequences.

6.1 Theorem 4

If G and M are the generating matrices of Fibonacci and Fibonacci-Like sequences respectively then (a) $G^n = F_n G + F_{n-1} I$ (6.1) and (b) $MG^{n-1} = FL_n G + FL_{n-1} I$ (6.2) where I is the 2×2 unit matrix.

Proof: (a) Using (4.1) of theorem 1, we have

$$F_n G + F_{n-1} I = \begin{pmatrix} 0 & F_n \\ F_n & F_n \end{pmatrix} + \begin{pmatrix} F_{n-1} & 0 \\ 0 & F_{n-1} \end{pmatrix} = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n-1} + F_n \end{pmatrix} = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix} = G^n. \text{ This proves (a).}$$

(b) Using (4.2) of theorem 2, we have

$$FL_n G + FL_{n-1} I = \begin{pmatrix} 0 & FL_n \\ FL_n & FL_n \end{pmatrix} + \begin{pmatrix} FL_{n-1} & 0 \\ 0 & FL_{n-1} \end{pmatrix} = \begin{pmatrix} FL_{n-1} & FL_n \\ FL_n & FL_{n-1} + FL_n \end{pmatrix} = \begin{pmatrix} FL_{n-1} & FL_n \\ FL_n & FL_{n+1} \end{pmatrix} = MG^{n-1}$$

6.2 Golden Ratio

The real number known as Golden Ratio denoted by φ is defined to be the positive real root of the equation

$x^2 - x - 1 = 0$. Since the roots of $x^2 - x - 1 = 0$ are $\frac{1 \pm \sqrt{5}}{2}$ the Golden Ratio is given by $\varphi = \frac{1 + \sqrt{5}}{2}$. We

notice that the Golden Ratio satisfies the equation $\varphi^2 = \varphi + 1$. We know that (see [1] by the corresponding author) the limiting ratio of successive Fibonacci numbers as well as Fibonacci-Like numbers is the Golden

Ratio φ . That is, $\frac{F_n}{F_{n-1}} \rightarrow \varphi, \frac{FL_n}{FL_{n-1}} \rightarrow \varphi$ as $n \rightarrow \infty$ (6.3)

6.3 Theorem 5

If F_n, FL_n are respectively the n th Fibonacci and Fibonacci-Like numbers then

$$(a) \lim_{n \rightarrow \infty} \frac{G^n}{F_{n-1}} = \begin{pmatrix} 1 & \varphi \\ \varphi & \varphi + 1 \end{pmatrix} \text{ (6.4)} \quad (b) \lim_{n \rightarrow \infty} \frac{MG^{n-1}}{FL_{n-1}} = \begin{pmatrix} 1 & \varphi \\ \varphi & \varphi + 1 \end{pmatrix} \text{ (6.5)}$$

Proof: We use equations (6.1) and (6.2) of theorem 4 and (6.3) to prove this theorem.

(a) From (6.1), we have $\frac{G^n}{F_{n-1}} = \frac{F_n}{F_{n-1}}G + I$. Now taking the limit as $n \rightarrow \infty$ and using (6.3) we have

$$\lim \frac{G^n}{F_{n-1}} = \lim \left(\frac{F_n}{F_{n-1}} \right) G + I = \varphi G + I = \begin{pmatrix} 0 & \varphi \\ \varphi & \varphi \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \varphi \\ \varphi & \varphi + 1 \end{pmatrix}. \text{ This proves (a).}$$

(b) From (6.2), we have $\frac{MG^{n-1}}{FL_{n-1}} = \frac{FL_n}{FL_{n-1}}G + I$. Now taking the limit as $n \rightarrow \infty$ and using (6.3) we have

$$\lim \frac{MG^{n-1}}{FL_{n-1}} = \lim \left(\frac{FL_n}{FL_{n-1}} \right) G + I = \varphi G + I = \begin{pmatrix} 0 & \varphi \\ \varphi & \varphi \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \varphi \\ \varphi & \varphi + 1 \end{pmatrix}. \text{ This proves (b).}$$

This completes the proof of the theorem.

In view of theorem 5, we call the matrices $\frac{G^n}{F_{n-1}}$ and $\frac{MG^{n-1}}{FL_{n-1}}$ as limiting matrices. We observe that in the

limiting case as $n \rightarrow \infty$ both matrices leads to the same matrix whose entries depend only on two numbers namely 1 and the Golden Ratio φ .

7. Conclusion

Using the generating matrices G and M for Fibonacci and Fibonacci-Like numbers, we have proved five interesting theorems in this paper. Theorems 1 and 2 have been proved using mathematical induction. Using these results, we have proved Cassini's Identities for both Fibonacci and Fibonacci-Like numbers in theorem 3. In theorem 4, two identities concerning expressing the powers of generating matrices as linear combination of G and the unit matrix I have been derived. We note that the scalars in such linear combinations are n th and $(n-1)$ th Fibonacci and Fibonacci-Like numbers respectively. Using the two identities of theorem 4, we finally proved in theorem 5, that the limiting matrices produce same matrix whose entries depend only on 1 and the Golden Ratio φ . This result is especially amusing in the sense that no matter with what two numbers a, b we begin, if we follow Fibonacci type recurrence relation, then in the limiting case we are sure enough to end with the matrices as shown in equations (6.4) and (6.5) of theorem 5. Using other particular entries of Fibonacci numbers we can similarly, get few more useful results leading to Golden Ratio entries in the final matrices.

References

1. R. Sivaraman, Exploring Metallic Ratios, Mathematics and Statistics, Horizon Research Publications, Volume 8, Issue 4, (2020), pp. 388 – 391.
2. R. Sivaraman, Generalized Lucas, Fibonacci Sequences and Matrices, Purakala, Volume 31, Issue 18, April 2020, pp. 509 – 515.
3. D. Kalman, "Generalized Fibonacci numbers by matrix methods", The Fibonacci Quarterly, **20** (1) (1982), 73-76.
4. F. Koken, D. Bozkurt, "On the Jacobsthal numbers by matrix methods", International Journal of Contemporary Mathematical. Sciences, **3** (13) (2008), 605-614.
5. P.Catarino, "A note Involving Two-by-Two Matrices of the k -Pell and Pell-Lucas Sequences", International Mathematical Forum, **8** (32) (2013), 1561-1568.
6. P.Catarino, "On some identities for k -Fibonacci sequence", International Journal of Contemporary Mathematical Sciences, **9** (1) (2014), 37-42.
7. Markowsky G. Misconceptions about the golden ratio. The College Mathematics Journal, 1992;23:2–19.
8. Fischler R., How to find the golden number without really trying, Fibonacci Quarterly, 1981;19: 406–410.
9. Newell AC, Pennybacker M. Fibonacci patterns: common or rare? Procedia IUTAM. 2013;9(Supplement C):86–109.
10. Krcadinac V., A new generalization of the golden ratio. Fibonacci Quarterly, 2006;44(4):335–340.