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Research Article

Selected Extensions on Eneström-kakeya Theorem

R. K. Pavan Kumar. Pannala[1]

[1]Department of Mathematics, ChandigrahUniversity, Gharuan-Mohali, Punjab -140413, India.
kamesh9.iit@gmail.com

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Abstract: The theorem of Eneström-Kakeya is important within the hypothesis of dissemination of zeros ofpolynomials. In the literature, it can be found so many extensions on Eneström-Kakeya theorem by giving various relations between the coefficients of polynomial like increasing, decreasing, irregular order etc.This paper mainly deals with some extensions on the Abdul Aziz and B AZargar theorembygiving some relaxations to the hypothesis that the coefficients are real, positive and alternative coefficients must be in increasing order.

Key words: Eneström-Kakeya theorem, location of zeros of polynomials, bounds for zeros, coefficients of polynomials, irregular coefficients of polynomials.

1. Introduction and statement of results

Eneström-kakeya Theorem [7]: Given the real polynomial $f(z) = \sum_{k=0}^{n} a_k z^k$.
If $a_0 \geq a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq a_n > 0$ then $f(z) \neq 0$ for$|z| < 1$.

The literature includes extensions, generalizations and refinements of Eneström-Kakeya theorem ([1-6]).

Theorem-A: If $p(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomialwith $a_n \neq 0$, such that $\left.\begin{matrix} a_n \geq a_{n-2} \geq \cdots \geq a_1 \;\; or \;\; a_0 > 0 \\ a_{n-1} \geq a_{n-3} \geq \cdots \geq a_0 \;\; or \;\; a_1 > 0 \end{matrix}\right\} (according\ as\ n\ is\ odd\ or\ even)$ then all the zeros of $p(z)$ lie in the disc $\left|z + \frac{a_{n-1}}{a_n}\right| \leq 1 + \frac{a_{n-1}}{a_n}$.

Theorem-A is given by Abdul Aziz and B.A.Zargar[1].

The hypothesis of the theorem-A is relaxed and obtained several extensions which are enumerated as follows.

2. Main results

Theorem-1: If$p(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree n with complex coefficients such that $\left.\begin{matrix} |a_n| \geq |a_{n-2}| \geq \cdots \geq |a_1| \; or \;\; |a_0| \\ |a_{n-1}| \geq |a_{n-3}| \geq \cdots \geq |a_0| \;\; or \;\; |a_1| \end{matrix}\right\} (according\ as\ n\ \ is\ odd\ or\ even)$ and $|arg.\, a_k - \beta| \leq \alpha \leq \frac{\pi}{2}$ for some real β, for $k = 0(1)n$ then the bound to the location of zeros of$p(z)$ is$\left|z + \frac{a_{n-1}}{a_n}\right| \leq \frac{[\{(cos\alpha + sin\alpha)(|a_n| + |a_{n-1}|) - \{(cos\alpha + sin\alpha - 1)(|a_1| + |a_0|)\} + 2sin\alpha \sum_{k=0}^{n-2} |a_k|\}]}{|a_n|}$

Following extensions can be obtained with an assumption that the real parts of the coefficients are non-negative and satisfy the hypothesis of the theorem-A.

Theorem-2: If $p(z) = \sum_{k=0}^{n} a_k z^k$with $a_n \neq 0$ such that $\left.\begin{matrix} \alpha_n \geq \alpha_{n-2} \geq \cdots \geq \alpha_1 \;\; or \;\; \alpha_0 \geq 0 \\ \alpha_{n-1} \geq \alpha_{n-3} \geq \cdots \geq \alpha_0 \;\; or \;\; \alpha_1 \geq 0 \end{matrix}\right\} (according\ as\ n\ is\ odd\ or\ even)$ and $\alpha_n > 0$ where $a_j = \alpha_j + i\,\beta_j,\ j = 0(1)n$ then a bound for zeros of $p(z)$ is$|z| \leq 1 + \frac{2\alpha_{n-1}}{\alpha_n} + \frac{2}{\alpha_n}\sum_{k=0}^{n} |\beta_k|$.

Theorem-3: If $p(z) = \sum_{k=0}^{n} a_k z^k$with $a_n \neq 0$ such that $\left.\begin{matrix} \alpha_n \geq \alpha_{n-2} \geq \cdots \geq \alpha_1 \;\; or \;\; \alpha_0 \geq 0 \\ \alpha_{n-1} \geq \alpha_{n-3} \geq \cdots \geq \alpha_0 \;\; or \;\; \alpha_1 \geq 0 \end{matrix}\right\} (according\ as\ n\ is\ odd\ or\ even)$ and $\alpha_n > 0$where $a_j = \alpha_j + i\,\beta_j,\ j = 0(1)n$ then a sharp bound for the zeros of $p(z)$is$R^* \leq |z| \leq R$

Where

$R = 1 + \frac{2\alpha_{n-1}}{\alpha_n} + \frac{2}{\alpha_n}\sum_{k=0}^{n}|\beta_k|$ and $R^* = \frac{|a_0|}{R^n\{2\alpha_n R + (R-1)|\beta_n| - (\alpha_0 + |\beta_0|)\}}$.

One can observe that the theorem-3 is animprovement of the theorem-2.

An extension can be obtained by including the increasing sequences between imaginary coefficients and further drop the restriction that the coefficients are non-negativein the hypothesis of the theorem-3.

Theorem-4: If $p(z) = \sum_{k=0}^{n} a_k z^k$with $a_n \neq 0$such that

$$\left.\begin{array}{c} \alpha_n \geq \alpha_{n-2} \geq \cdots \geq \alpha_1 \;\; or \;\; \alpha_0 \\ \alpha_{n-1} \geq \alpha_{n-3} \geq \cdots \geq \alpha_0 \;\; or \;\; \alpha_1 \\ \beta_n \geq \beta_{n-2} \geq \cdots \geq \beta_1 \;\; or \;\; \beta_0 \\ \beta_{n-1} \geq \beta_{n-3} \geq \cdots \geq \beta_0 \;\; or \;\; \beta_1 \end{array}\right\} (according\ as\ n\ is\ odd\ or\ even) \text{ and } \alpha_n > 0 \text{ where}$$

$a_j = \alpha_j + i\,\beta_j,\ j = 0(1)n$ then all the zeros of $p(z)$ lie in the annular ring $R_2 \leq |z| \leq R_1$

where

$$R_1 = \frac{|a_{n-1}|}{2}\left(\frac{1}{\alpha_n} - \frac{1}{M}\right) + \left\{\frac{|a_{n-1}|^2}{4}\left(\frac{1}{\alpha_n} - \frac{1}{M}\right)^2 + \frac{M}{\alpha_n}\right\}^{\frac{1}{2}}$$

$$R_2 = \frac{-R_1^2|a_1|(M_1 - |a_0|) + \{R_1^4|a_1|^2(M_1 - |a_0|)^2 + 4\,|a_0|R_1^2 M_1^3\}^{\frac{1}{2}}}{2M_1^2}$$

and

$$M \equiv \alpha_n + \alpha_{n-1} + \beta_n + \beta_{n-1} + (|\alpha_0| - \alpha_0) + (|\alpha_1| - \alpha_1) + (|\beta_0| - \beta_0) + (|\beta_1| - \beta_1) + |\alpha_{n-1}| + |\beta_{n-1}|$$

$$M_1 = R_1^{n+1}[(|\alpha_n| + |\beta_n|)R_1 + M - |\alpha_0| - |\beta_0|]$$

An extension can be obtained by including both increasing and decreasing sequences between alternative coefficients in the hypothesis of the theorem-4.

Theorem-5: If $p(z) = \sum_{k=0}^{n} a_k z^k$with $a_n \neq 0$ such that

$$\left.\begin{array}{c} \alpha_n \geq \alpha_{n-2} \geq \cdots \geq \alpha_1 \;\; or \;\; \alpha_0 \\ \alpha_{n-1} \leq \alpha_{n-3} \leq \cdots \leq \alpha_0 \;\; or \;\; \alpha_1 \\ \beta_n \geq \beta_{n-2} \geq \cdots \geq \beta_1 \;\; or \;\; \beta_0 \\ \beta_{n-1} \leq \beta_{n-3} \leq \cdots \leq \beta_0 \;\; or \;\; \beta_1 \end{array}\right\} (according\ as\ n\ is\ odd\ or\ even) \text{ and } \alpha_n > 0 \text{ where}$$

$a_j = \alpha_j + i\,\beta_j,\ j = 0(1)n$ then all the zeros of $p(z)$ lie in the disc $|z| \leq \frac{M_2}{\alpha_n}$ where

$$M_2 = \begin{cases} \alpha_n + (|\alpha_0| + |\beta_0| + \alpha_0 + \beta_0) + (|\alpha_1| + |\beta_1| - \alpha_1) + (|\alpha_{n-1}| + |\beta_{n-1}| - \alpha_{n-1} - \beta_{n-1}) + (\beta_n - \beta_1) \\ \qquad or \\ \alpha_n + (|\alpha_0| + |\beta_0| - \alpha_0) + (|\alpha_1| + |\beta_1| + \alpha_1 + \beta_1) + (|\alpha_{n-1}| + |\beta_{n-1}| - \alpha_{n-1} - \beta_{n-1}) + (\beta_n - \beta_0) \\ \qquad \text{accordning as n is odd or even} \end{cases}$$

3. Lemmas

For proving the main results, the following lemmas have used. Lemma 1 owes itself to Govil and Rahman [3].

Lemma 1: If $|arg.\,a_k - \beta| \leq \alpha \leq \frac{\pi}{2},\ |arg.\,a_{k-1} - \beta| \leq \alpha$ and $|a_k| \geq |a_{k-1}|$ then

$$|a_k - a_{k-1}| \leq \{(|a_k| - |a_{k-1}|)cos\alpha + (|a_k| + |a_{k-1}|)sin\alpha\}$$

One can observe that the extension of Schwarz‘s lemma is the following lemma 2.

Lemma 2: If $h(z)$ is analytic on and inside the unit circle, $|h(z)| \leq H$ on $|z| = 1, f(0) = a$ where $|a| < H$ then $|h(z)| \leq H\frac{H|z| + |a|}{|a||z| + H}$ for $|z| < 1$.

Lemma 3: If $h(z)$ is analytic in $|z| < r,\ |h(z)| \leq H$ on $|z| = r,\ h(0) = a$ where $|a| < H$ then $|h(z)| \leq H\frac{H|z| + |a|r}{|a||z| + Hr}$ for $|z| \leq r$.

Lemma 3 can be proved from lemma 2 easily.

Govil, et al. [4] are attributed to the following lemma 4.

Lemma 4: If $h\ (z)$ is analytic in $|z| \leq 1, h(0) = c$ where $|c| < 1, h^{'}(0) = d, |h(z)| \leq 1$ on $|z| = 1$ then for $|z| \leq 1, |h(z)| \leq \frac{(1-|c|)|z|^2+|d||z|+|c|(1-|c|)}{|c|(1-|c|)|z|^2+|d||z|+(1-|c|)}$.

Lemma 5: If $h(z)$ is analytic in $|z| \leq r, h(0) = 0, h^{'}(0) = b$ and $|h(z)| \leq H$ for $|z| = r$ then for $|z| \leq r$, $|h(z)| \leq \frac{H|z|}{r^2}\frac{H|z|+r^2\,|b|}{H+|z||b|}$

Lemma 5 can be proved from lemma 4 easily.

4. Main results proofs

Theorem-1 proof:
Let$g(z) = (1 - z^2)p(z)$

$$= -a_n z^{n+2} - a_{n-1}z^{n+1} + \sum_{k=0}^{n-2}(a_{k+2} - a_k)z^{k+2} + a_1 z + a_0$$

$$|g(z)| \geq |z|^{n+1}|a_n z + a_{n-1}| - \left|\sum_{k=0}^{n-2}(a_{k+2} - a_k)z^{k+2} + a_1 z + a_0\right|$$

For $|z| > 1$,

$$|g(z)| \geq |z|^{n+1}|a_n z + a_{n-1}| - |z|^n\left\{\sum_{k=0}^{n-2}|(a_{k+2} - a_k)| + |a_1| + |a_0|\right\}$$

Using Lemma-1 we obtain

$$|g(z)| \geq |z|^{n+1}|a_n z + a_{n-1}| - |z|^n[\{\textstyle\sum_{k=0}^{n-2}(|a_{k+2}| - |a_k|)cos\alpha\} + \{\sum_{k=0}^{n-2}(|a_{k+2}| + |a_k|)sin\alpha\} + |a_1|+|a_0|]$$

$$= |z|^{n+1}|a_n z + a_{n-1}| - |z|^n\left[\{(cos\alpha + sin\alpha)(|a_n| + |a_{n-1}|)\} + 2sin\alpha\sum_{k=0}^{n-2}|a_k| - \{(cos\alpha + sin\alpha - 1)(|a_1| + |a_0|)\}\right]$$

$|g(z)| > 0$ if

$$\left|z + \left(\frac{a_{n-1}}{a_n}\right)\right| > \frac{[\{(cos\alpha+sin\alpha)(|a_n|+|a_{n-1}|)\}+2sin\alpha\sum_{k=0}^{n-2}|a_k|-\{(cos\alpha+sin\alpha-1)(|a_1|+|a_0|)\}]}{|a_n|} \equiv M \text{ (say)}$$

$$M > \frac{|a_n| + |a_{n-1}| - |a_1| - |a_0| + |a_1| + |a_0|}{|a_n|} = 1 + \left|\frac{a_{n-1}}{a_n}\right| \geq 1$$

Let $1 \leq M < R$
Where $R = \left|z + \left(\frac{a_{n-1}}{a_n}\right)\right| \leq |z| + \left|\frac{a_{n-1}}{a_n}\right|$

$$|z| \geq R - \left|\frac{a_{n-1}}{a_n}\right| \geq 1 + R - M > 1$$

Hence $g(z)$ does not vanish for

$$\left|z + \left(\frac{a_{n-1}}{a_n}\right)\right| > \frac{[\{(cos\alpha + sin\alpha)(|a_n| + |a_{n-1}|)\} + 2sin\alpha\sum_{k=0}^{n-2}|a_k| - \{(cos\alpha + sin\alpha - 1)(|a_1| + |a_0|)\}]}{|a_n|}$$

Therefore, those roots of $g(z)$for whichthe modulus is greater than one be located in

$$\left|z + \left(\frac{a_{n-1}}{a_n}\right)\right| \leq \frac{[\{(cos\alpha + sin\alpha)(|a_n| + |a_{n-1}|)\} + 2sin\alpha\sum_{k=0}^{n-2}|a_k| - \{(cos\alpha + sin\alpha - 1)(|a_1| + |a_0|)\}]}{|a_n|}$$

Theorem-2 proof:
Let$g(z) = (1 - z^2)p(z)$
$= -a_n z^{n+2} + Q(z)$ where $Q(z) = -a_{n-1}z^{n+1} + \sum_{k=0}^{n-2}(a_{k+2} - a_k)z^{k+2} + a_1 z + a_0$
For $|z| = 1$,

$$|Q(z)| \leq |a_0| + |a_1| + |a_{n-1}| + \sum_{k=2}^{n}|a_k - a_{k-2}|$$

$$|Q(z)| \leq \alpha_0 + |\beta_0| + \alpha_1 + |\beta_1| + \alpha_{n-1} + |\beta_{n-1}| + \sum_{k=2}^{n}(\alpha_k - \alpha_{k-2}) + \sum_{k=2}^{n}(|\beta_k| + |\beta_{k-2}|)$$
$$= \alpha_n + 2\alpha_{n-1} - |\beta_n| + 2\sum_{k=0}^{n}|\beta_k|$$
$$\leq \alpha_n + 2\alpha_{n-1} + 2\sum_{k=0}^{n}|\beta_k|$$

Hence also

$$\left|z^{n+1}Q\left(\frac{1}{z}\right)\right| \leq \alpha_n + 2\alpha_{n-1} + 2\sum_{k=0}^{n}|\beta_k|$$

For $|z| = 1$, by the maximum modulus principle holds inside the unit circle as well.
If $R > 1$then$\frac{1}{R}e^{-i\theta}$be located in the unit circle for all real θ,which implies
$|Q(Re^{i\theta})| \leq \{\alpha_n + 2\alpha_{n-1} + 2\sum_{k=0}^{n}|\beta_k|\}R^{n+1}$ for every $R \geq 1$ and θ real.
Thus for $|z| = R > 1$

$$|g(Re^{i\theta})| \geq |a_n|R^{n+2} - |Q(Re^{i\theta})|$$
$$\geq |a_n|R^{n+2} - \left\{\alpha_n + 2\alpha_{n-1} + 2\sum_{k=0}^{n}|\beta_k|\right\}R^{n+1}$$
$$\geq \alpha_n R^{n+2} - \left\{\alpha_n + 2\alpha_{n-1} + 2\sum_{k=0}^{n}|\beta_k|\right\}R^{n+1}$$

$|g(Re^{i\theta})| > 0$ if $R > \frac{\{\alpha_n + 2\alpha_{n-1} + 2\sum_{k=0}^{n}|\beta_k|\}}{\alpha_n}$.

Theorem-3 proof:
Let$g(z) = (1 - z^2)p(z)$
$= a_0 + f(z)$ where $f(z) = -a_n z^{n+2} - a_{n-1}z^{n+1} + \sum_{k=2}^{n}(a_k - a_{k-2})z^k + a_1 z$
Let $M(r) = \max_{|z|=r}|f(z)|$
Then $M(R) \geq |a_0|$ where $R = \frac{\{\alpha_n + 2\alpha_{n-1} + 2\sum_{k=0}^{n}|\beta_k|\}}{\alpha_n}$
Clearly, $|f(z)| \leq |a_n||z|^{n+2} + |a_{n-1}||z|^{n+1} + \sum_{k=2}^{n}|a_k - a_{k-2}|\,|z|^k + |a_1||z|$ and $R \geq 1$.
Hence,

$$M(R) = \max_{|z|=R}|f(z)| \leq |a_n|R^{n+2} + |a_{n-1}|R^{n+1} + |a_1|R + \sum_{k=2}^{n}|a_k - a_{k-2}|R^k$$
$$\leq |a_n|R^{n+2} + |a_{n-1}|R^{n+1} + |a_1|R + R^n\{\sum_{k=2}^{n}|a_k - a_{k-2}|\}$$
$$\leq |a_n|R^{n+2} + R^{n+1}\left\{|a_{n-1}| + |a_1| + \sum_{k=2}^{n}|a_k - a_{k-2}|\right\}$$
$$\leq (\alpha_n + |\beta_n|)R^{n+2} + R^{n+1}\{\alpha_n + 2\alpha_{n-1} - \alpha_0 - |\beta_0| - |\beta_n| + 2\sum_{k=0}^{n}|\beta_k|\}$$
$$= R^{n+1}\{2\alpha_n R + (R-1)|\beta_n| - (\alpha_0 + |\beta_0|)\} \equiv M$$

Since $f(0) = 0$, hence for $|z| \leq R$ we have by Schwarz's lemma,

$$|f(z)| \leq \frac{M\,|z|}{R}$$

For $|z| \leq R$, $|g(z)| \geq |a_0| - |z|R^n\{2\alpha_n R + (R-1)|\beta_n| - (\alpha_0 + |\beta_0|)\}$
$|g(z)| > 0$ if $|z| < \frac{|a_0|}{R^n\{2\alpha_n R + (R-1)|\beta_n| - (\alpha_0 + |\beta_0|)\}}$
Since $\frac{2\alpha_n R + (R-1)|\beta_n| - (\alpha_0 + |\beta_0|)}{|a_0|} > 0$
Then$\frac{|a_0|}{R^n\{2\alpha_n R + (R-1)|\beta_n| - (\alpha_0 + |\beta_0|)\}} < R$.

Theorem-4 proof:
Let$g(z) = (1 - z^2)p(z)$
$= -a_n z^{n+2} + Q(z)$ where $Q(z) = -a_{n-1}z^{n+1} + \sum_{k=2}^{n}(a_k - a_{k-2})z^k + a_1 z + a_0$
Let $T(z) = z^{n+1}Q\left(\frac{1}{z}\right) = -a_{n-1} + \sum_{k=2}^{n}(a_k - a_{k-2})z^{n-k+1} + a_1 z^n + a_0 z^{n+1}$

For $|z| = 1$, we have

$$|T(z)| \le |a_0| + |a_1| + |a_{n-1}| + \sum_{k=2}^{n} |a_k - a_{k-2}| \le M$$

where

$$M \equiv \alpha_n + \alpha_{n-1} + \beta_n + \beta_{n-1} + (|\alpha_0| - \alpha_0) + (|\alpha_1| - \alpha_1) + (|\beta_0| - \beta_0) + (|\beta_1| - \beta_1) + |\alpha_{n-1}| + |\beta_{n-1}|$$

By the maximum modulus principle, it holds inside the unit circle as well.
If $R > 1$ then$\frac{1}{R}e^{-i\theta}$be located in the unit circle for all real θ,which implies
$|Q(Re^{i\theta})| \le M\, R^{n+1}$ for every real $R \ge 1$ and real θ.
Thus for $|z| = R > 1$

$$|g(Re^{i\theta})| \ge |a_n|R^{n+2} - |Q(Re^{i\theta})| \ge \alpha_n R^{n+2} - M\, R^{n+1}$$

$|g(Re^{i\theta})| > 0$ if

$$R > \frac{M}{\alpha_n}$$

$$= 1 + \left\{\frac{\alpha_{n-1} + \beta_n + \beta_{n-1} + (|\alpha_0| - \alpha_0) + (|\alpha_1| - \alpha_1) + (|\beta_0| - \beta_0) + (|\beta_1| - \beta_1) + |\alpha_{n-1}| + |\beta_{n-1}|}{\alpha_n}\right\}$$

Hence the concept of maximum modulus, $|T(0)| = |a_{n-1}| < M$
By lemma-2 on the function $T(z)$ we obtain for $|z| \le 1$,

$$|T(z)| \le M\, \frac{M\,|z| + |a_{n-1}|}{|a_{n-1}||z| + M}$$

This implies that

$$\left|z^{n+1}Q\left(\frac{1}{z}\right)\right| \le M\frac{M\,|z| + |a_{n-1}|}{|a_{n-1}||z| + M}$$

If $R > 1$, $\frac{1}{R}e^{-i\theta}$be located in the unit circle for all real θ, which implies

$$|Q(Re^{i\theta})| \le M\, R^{n+1}\,\frac{M + |a_{n-1}|R}{|a_{n-1}| + M\,R}$$

Thus for $|z| = R > 1$

$$|g(Re^{i\theta})| \ge |a_n|R^{n+2} - |Q(Re^{i\theta})| \ge \alpha_n R^{n+2} - M\, R^{n+1}\,\frac{M + |a_{n-1}|R}{|a_{n-1}| + M\,R}$$

$$= \frac{R^{n+1}}{MR + |a_{n-1}|}[M\,\alpha_n R^2 - |a_{n-1}|(M - \alpha_n)R - M^2]$$

> 0 if $R > \frac{|a_{n-1}|}{2}\left(\frac{1}{\alpha_n} - \frac{1}{M}\right) + \left\{\frac{|a_{n-1}|^2}{4}\left(\frac{1}{\alpha_n} - \frac{1}{M}\right)^2 + \frac{M}{\alpha_n}\right\}^{\frac{1}{2}} \equiv R_1$
Therefore $g(z)$have all the zeros of located in $|z| \le R_1$ where $R_1 > 1$.
It means that all zeros of$p(z)$are located in $|z| \le R_1$.
Subsequently, it can be showed that no zeros of$p(z)$are located in$|z| < R_2$.

$$g(z) = a_0 + f(z) = a_0 + a_1 z + \sum_{k=2}^{n}(a_k - a_{k-2})z^k - a_{n-1}z^{n+1} - a_n z^{n+2}$$

Let $M(R_1) = \max_{|z|=R_1}|f(z)|$
Since $R_1 \ge 1, f(1) = -a_0$ we have $M(R_1) \ge |a_0|$
Clearly $|f(z)| \le |a_n||z|^{n+2} + \sum_{k=2}^{n}|a_k - a_{k-2}||z|^k + |a_1||z| + |a_{n-1}||z|^{n+1}$
And hence $M(R_1) \le |a_n|R_1^{n+2} + \sum_{k=2}^{n}|a_k - a_{k-2}|R_1^k + |a_1|R_1 + |a_{n-1}|R_1^{n+1}$

$$\le |a_n|R_1^{n+2} + R_1^{n+1}\left\{|a_1| + |a_{n-1}| + \sum_{k=2}^{n}|a_k - a_{k-2}|\right\}$$

$\le R_1^{n+1}[(|\alpha_n| + |\beta_n|)R_1 + M - |\alpha_0| - |\beta_0|] \equiv M_1$ (say)
Further because $f(0) = 0, f'(0) = a_1$ we have by lemma-5
$|f(z)| \le \frac{M_1\,|z|}{R_1^2}\frac{M_1|z| + |R_1^2|a_1|}{M_1 + |a_1||z|}$ for $|z| \le R_1$

$$|g(z)| \ge |a_0| - \frac{M_1\,|z|}{R_1^2}\frac{M_1|z| + |R_1^2|a_1|}{M_1 + |a_1||z|} = \frac{-1}{R_1^2(M_1 + |z||a_1|)}[|z|^2 M_1^2 + R_1^2|a_1||z|(M_1 - |a_0|R_1^2 M_1]$$

$|g(z)| > 0$ if $|z| < \frac{-R_1^2|a_1|(M_1 - |a_0|) + \{R_1^4|a_1|^2(M_1 - |a_0|)^2 + 4\,|a_0|R_1^2 M_1^3\}^{\frac{1}{2}}}{2M_1^2} \equiv R_2$ (say) where $R_2 \le R_1$.

Theorem-5 proof:
Let $g(z) = (1 - z^2)p(z)$
$= -a_n z^{n+2} + Q(z)$ where $Q(z) = -a_{n-1} z^{n+1} + \sum_{k=2}^{n} (a_k - a_{k-2}) z^k + a_1 z + a_0$
For $|z| = 1$ we have

$$|Q(z)| \le M_2$$

where

$$M_2 = \begin{cases} \alpha_n + (|\alpha_0| + |\beta_0| + \alpha_0 + \beta_0) + (|\alpha_1| + |\beta_1| - \alpha_1) + (|\alpha_{n-1}| + |\beta_{n-1}| - \alpha_{n-1} - \beta_{n-1}) + (\beta_n - \beta_1) \\ or \\ \alpha_n + (|\alpha_0| + |\beta_0| - \alpha_0) + (|\alpha_1| + |\beta_1| + \alpha_1 + \beta_1) + (|\alpha_{n-1}| + |\beta_{n-1}| - \alpha_{n-1} - \beta_{n-1}) + (\beta_n - \beta_0) \\ \text{accordning as n is odd or even} \end{cases}$$

Hence also for $|z| = 1$, $\left| z^{n+1} Q\left(\frac{1}{z}\right) \right| \le M_2$.
By the maximum modulus principle it holds inside the unit circle as well.
If $R > 1$ then $\frac{1}{R} e^{-i\theta}$be located inthe unit circle for all real θ and follows that
$\left| Q\left(Re^{i\theta}\right) \right| \le M_2 R^{n+1}$ for every $R \ge 1$ and real θ.
Thus for $|z| = R > 1$

$$g\left|\left(Re^{i\theta}\right)\right| \ge |a_n| R^{n+2} - \left|Q\left(Re^{i\theta}\right)\right| \ge \alpha_n R^{n+2} - M_2 R^{n+1}$$

$\left| g\left(Re^{i\theta}\right) \right| > 0$ if $R > \frac{M_2}{\alpha_n}$ where $R > 1$.

If$\alpha_0, \alpha_1, \alpha_{n-1} \ge 0$ and $\beta_0, \beta_1, \beta_{n-1} \ge 0$ in theorem-5 then

Corollary-5.1: If $p(z) = \sum_{k=0}^{n} a_k z^k$with $a_n \ne 0$such that

$$\left.\begin{matrix} \alpha_n \ge \alpha_{n-2} \ge \cdots \ge \alpha_1 \;\; or \;\; \alpha_0 \ge 0 \\ 0 \le \alpha_{n-1} \le \alpha_{n-3} \le \cdots \le \alpha_0 \;\; or \;\; \alpha_1 \\ \beta_n \ge \beta_{n-2} \ge \cdots \ge \beta_1 \;\; or \;\; \beta_0 \ge 0 \\ 0 \le \beta_{n-1} \le \beta_{n-3} \le \cdots \le \beta_0 \;\; or \;\; \beta_1 \end{matrix}\right\} \; (according\ as\ n\ \ is\ odd\ or\ even) \text{ and } \alpha_n > 0$$

where $a_j = \alpha_j + i\,\beta_j,\;\; j = 0(1)n$ then all the zeros of $p(z)$ lie in the disc

$$|z| \le \begin{cases} \dfrac{\alpha_n + \beta_n + 2(\alpha_0 + \beta_0)}{\alpha_n} \\ or \\ \dfrac{\alpha_n + \beta_n + 2(\alpha_1 + \beta_1)}{\alpha_n} \end{cases} \quad (according\ as\ \ n\ \ \ is\ odd\ or\ even)$$

If all the coefficients of the polynomial are real in theorem-5 then

Corollary-5.2: If $p(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree n such that

$$\left.\begin{matrix} a_n \ge a_{n-2} \ge \cdots \ge a_1 \;\; or \;\; a_0 \\ a_{n-1} \le a_{n-3} \le \cdots \le a_0 \;\; or \;\; a_1 \end{matrix}\right\} \; (according\ as\ n\ is\ odd\ or\ even) \text{ and } a_n > 0$$

then all the roots of $p(z)$ lie in the disc

$$|z| \le \begin{cases} 1 + \dfrac{(|a_0| + a_0) + (|a_1| - a_1) + (|a_{n-1}| - a_{n-1})}{a_n} \\ 1 + \dfrac{(|a_0| - a_0) + (|a_1| + a_1) + (|a_{n-1}| - a_{n-1})}{a_n} \end{cases} \quad (according\ as\ n\ is\ odd\ or\ even)$$

If all the coefficients of the polynomial are real and non-negative in theorem-5 then

Corollary-5.3: If $p(z) = \sum_{k=0}^{n} a_k z^k$with $a_n \ne 0$such that

$$\left.\begin{matrix} a_n \ge a_{n-2} \ge \cdots \ge a_1 \;\; or \;\; a_0 \ge 0 \\ 0 \le a_{n-1} \le a_{n-3} \le \cdots \le a_0 \;\; or \;\; a_1 \end{matrix}\right\} \; (according\ as\ n\ is\ odd\ or\ even) \text{ and } a_n > 0$$

then all the roots of $p(z)$ lie in the disc $|z| \le \begin{cases} 1 + \frac{2a_0}{a_n} \\ 1 + \frac{2a_1}{a_n} \end{cases} \quad (according\ as\ n\ is\ odd\ or\ even)$

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