Selected Extensions on Eneström-kakeya Theorem

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Abstract: The theorem of Eneström-Kakeya is important within the hypothesis of dissemination of zeros of polynomials. In the literature, it can be found so many extensions on Eneström-Kakeya theorem by giving various relations between the coefficients of polynomial like increasing, decreasing, irregular order etc. This paper mainly deals with some extensions on the Abdul Aziz and B. A. Zargar theorem by giving some relaxations to the hypothesis that the coefficients are real, positive and alternative coefficients must be in increasing order.

Key words: Eneström-Kakeya theorem, location of zeros of polynomials, bounds for zeros, coefficients of polynomials, irregular coefficients of polynomials.

1. Introduction and statement of results

Eneström-kakeya Theorem [7]: Given the real polynomial \( f(z) = \sum_{k=0}^{n} a_k z^k \).
If \( a_0 \geq a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq a_n > 0 \) then \( f(z) \neq 0 \) for \(|z| < 1\).

The literature includes extensions, generalizations and refinements of Eneström-Kakeya theorem ([1-6]).

Theorem-A: If \( p(z) = \sum_{k=0}^{n} a_k z^k \) is a polynomial with \( a_n \neq 0 \), such that
\[ a_n \geq a_{n-2} \geq \cdots \geq a_1 \text{ or } a_0 > 0 \]
\[ a_{n-1} \geq a_{n-3} \geq \cdots \geq a_0 \text{ or } a_1 > 0 \] (according as \( n \) is odd or even) then all the zeros of \( p(z) \) lie in the disc \(|z + \frac{a_{n-1}}{a_n}| \leq 1 + \frac{a_{n-1}}{a_n}\).

Theorem-A is given by Abdul Aziz and B. A. Zargar[1].

The hypothesis of the theorem-A is relaxed and obtained several extensions which are enumerated as follows.

2. Main results

Theorem-1: If \( p(z) = \sum_{k=0}^{n} a_k z^k \) is a polynomial of degree \( n \) with complex coefficients such that
\[ |a_n| \geq |a_{n-2}| \geq \cdots \geq |a_1| \text{ or } |a_0| \]
\[ |a_{n-1}| \geq |a_{n-3}| \geq \cdots \geq |a_0| \text{ or } |a_1| \] (according as \( n \) is odd or even) and \( |\arg a_k - \beta| \leq \alpha \leq \frac{\pi}{2} \) for some real \( \beta \), for \( k = 0(1)n \) then the bound to the location of zeros of \( p(z) \) is
\[ \frac{|(\cos \alpha + \sin \alpha)(|a_n|+|a_{n-1}|)-(\cos \alpha+\sin \alpha)(|a_1|+|a_0|)|+2\sin \alpha \sum_{k=1}^{n} |a_k|}{|a_n|} \leq \frac{1}{|a_n|} \]

Following extensions can be obtained with an assumption that the real parts of the coefficients are non-negative and satisfy the hypothesis of the theorem-A.

Theorem-2: If \( p(z) = \sum_{k=0}^{n} a_k z^k \) with \( a_n \neq 0 \) such that
\[ a_n \geq a_{n-2} \geq \cdots \geq a_1 \text{ or } a_0 \geq 0 \]
\[ a_{n-1} \geq a_{n-3} \geq \cdots \geq a_0 \text{ or } a_1 \geq 0 \] (according as \( n \) is odd or even) and \( a_n > 0 \) where \( a_j = a_j + i \beta_j \), \( j = 0(1)n \) then a bound for zeros of \( p(z) \) is
\[ |z| \leq 1 + 2a_{n-1} + \frac{2}{a_n} \sum_{k=0}^{n} |\beta_k|. \]

Theorem-3: If \( p(z) = \sum_{k=0}^{n} a_k z^k \) with \( a_n \neq 0 \) such that
\[ a_n \geq a_{n-2} \geq \cdots \geq a_1 \text{ or } a_0 \geq 0 \]
\[ a_{n-1} \geq a_{n-3} \geq \cdots \geq a_0 \text{ or } a_1 \geq 0 \] (according as \( n \) is odd or even) and \( a_n > 0 \) where \( a_j = a_j + i \beta_j \), \( j = 0(1)n \) then a sharp bound for the zeros of \( p(z) \) is
\[ R' \leq |z| \leq R \]

Where
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\[ R = 1 + \frac{2\alpha_{n-1}}{\alpha_n} + \frac{1}{\alpha_n} \sum_{k=0}^{n} \beta_k \] and \[ R^* = \frac{|\alpha_0|}{R^n(2\alpha_n R^{n-k-1}(\beta_{k+1}-|\alpha_0+\beta_0|))}. \]

One can observe that the theorem-3 is an improvement of the theorem-2.

An extension can be obtained by including the increasing sequences between imaginary coefficients and further drop the restriction that the coefficients are non-negative in the hypothesis of the theorem-3.

**Theorem-4:** If \( p(z) = \sum_{k=0}^{n} a_k z^k \) with \( a_n \neq 0 \) such that
\[
\begin{align*}
\alpha_n \geq \alpha_{n-2} \geq \cdots \geq \alpha_1 & \quad \text{or} \quad \alpha_0 \\
\alpha_{n-1} \geq \alpha_{n-3} \geq \cdots \geq \alpha_0 & \quad \text{or} \quad \alpha_1 \\
\beta_n \geq \beta_{n-2} \geq \cdots \geq \beta_1 & \quad \text{(according as } n \text{ is odd or even)} \quad \text{and} \quad a_n > 0
\end{align*}
\]
then all the zeros of \( p(z) \) lie in the annular ring \( R_2 \leq |z| \leq R_3 \)

where
\[
R_1 = \left( \frac{a_n-1}{2} \left( \frac{1}{\alpha_n} - \frac{1}{M} \right) + \left( \frac{a_n-1}{4} \left( \frac{1}{\alpha_n} - \frac{1}{M} \right)^2 + \frac{M}{\alpha_n} \right)^{\frac{1}{2}} \right)
\]
\[
R_2 = \frac{R_1^2 |a_1| (M_2 - |a_0|) + \left( R_1^4 |a_1|^2 (M_1 - |a_0|)^2 + 4 |a_0| R_1^2 M_1 \right)^{\frac{1}{2}}}{2M_1^{\frac{1}{2}}}
\]

and
\[
M \equiv \alpha_n + \alpha_{n-1} + \beta_n + \beta_{n-1} + (|\alpha_0| - |\alpha_0|) + (|\alpha_1| - |\alpha_1|) + (|\beta_0| - |\beta_0|) + (|\beta_1| - |\beta_1|) + |\alpha_{n-1}| + |\beta_{n-1}|
\]
\[
M_1 = R_1^{n+1} \left( |\alpha_n| + |\beta_n| \right) R_1 + M - |\alpha_0| - |\beta_0|
\]

An extension can be obtained by including both increasing and decreasing sequences between alternative coefficients in the hypothesis of the theorem-4.

**Theorem-5:** If \( p(z) = \sum_{k=0}^{n} a_k z^k \) with \( a_n \neq 0 \) such that
\[
\begin{align*}
\alpha_n \geq \alpha_{n-2} \geq \cdots \geq \alpha_1 & \quad \text{or} \quad \alpha_0 \\
\alpha_{n-1} \geq \alpha_{n-3} \geq \cdots \geq \alpha_0 & \quad \text{or} \quad \alpha_1 \\
\beta_n \geq \beta_{n-2} \geq \cdots \geq \beta_1 & \quad \text{(according as } n \text{ is odd or even)} \quad \text{and} \quad a_n > 0
\end{align*}
\]
then all the zeros of \( p(z) \) lie in the disc \( |z| \leq \frac{M_2}{\alpha_n} \) where
\[
M_2 = \begin{cases}
\alpha_n + (|\alpha_0| + |\beta_0|) + (|\alpha_1| + |\beta_1| - |\alpha_1|) + (|\alpha_{n-1}| + |\beta_{n-1}| - |\alpha_{n-1}| - |\alpha_{n-1}|) + (|\beta_n| - |\beta_1|) \\
\text{or}
\end{cases}
\]
\[
\begin{align*}
\alpha_n + (|\alpha_0| + |\beta_0| - |\alpha_0|) + (|\alpha_1| + |\beta_1| + \alpha_1 + |\beta_1|) + (|\alpha_{n-1}| + |\beta_{n-1}| - |\alpha_{n-1}| - |\alpha_{n-1}|) + (|\beta_n| - |\beta_0|)
\end{align*}
\]
according as \( n \) is odd or even.

3. Lemmas

For proving the main results, the following lemmas have used. Lemma 1 owes itself to Govil and Rahman [3].

**Lemma 1:** If \( |\arg. a_k - \beta| \leq \alpha \leq \frac{\pi}{2} \), \( |\arg. a_k - \beta| \leq \alpha \) and \( |a_k| \geq |a_{k-1}| \) then \( |a_k - \alpha_{k-1}| \leq ((|a_k| - |a_{k-1}|) \cos \alpha + (|a_k| + |a_{k-1}|) \sin \alpha) \)

One can observe that the extension of Schwarz’s lemma is the following lemma 2.

**Lemma 2:** If \( h(z) \) is analytic on and inside the unit circle, \( |h(z)| \leq H \) on \( |z| = 1, f(0) = a \) where \( |a| < H \) then \( |h(z)| \leq H \frac{|h(z)+a|}{|z|-h} \) for \( |z| < 1 \).

**Lemma 3:** If \( h(z) \) is analytic in \( |z| < r, \ |h(z)| \leq H \) on \( |z| = r, h(0) = a \) where \( |a| < H \) then \( |h(z)| \leq H \frac{|h(z)+a|}{|z|-Hr} \) for \( |z| \leq r \).

Lemma 3 can be proved from lemma 2 easily.

Govil, et al. [4] are attributed to the following lemma 4.

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Let $g(z) = (1 - z^2)p(z)$

$$= -a_n z^{n+2} - a_{n-1} z^{n+1} + \sum_{k=0}^{n-2} (a_{k+2} - a_k) z^{k+2} + a_1 z + a_0$$

\[|g(z)| \geq |z|^{n+1} |a_n z + a_{n-1}| - \sum_{k=0}^{n-2} |(a_{k+2} - a_k) z^{k+2} + a_1 z + a_0|\]

For $|z| > 1$,

\[|g(z)| \geq |z|^{n+1} |a_n z + a_{n-1}| - |z|^n \left(\sum_{k=0}^{n-2} |a_{k+2} - a_k| + |a_1| + |a_0|\right)\]

Using Lemma 1 we obtain

\[|g(z)| \geq |z|^{n+1} |a_n z + a_{n-1}| - |z|^n \left(\sum_{k=0}^{n-2} |a_{k+2} - a_k| + |a_1| + |a_0|\right) - |z|^{n+1} (|a_n| + |a_{n-1}|) + 2 \sin\alpha \sum_{k=0}^{n-2} |a_k|\]

\[-(\cos\alpha + \sin\alpha - 1)(|a_1| + |a_0|)\]

\[|g(z)| > 0 \text{ if } |z + \left(\frac{a_{n-1}}{a_n}\right)| > \left[\frac{(|a_n| + |a_{n-1}|) + 2 \sin\alpha \sum_{k=0}^{n-2} |a_k| - (\cos\alpha + \sin\alpha - 1)(|a_1| + |a_0|)}{|a_n|}\right] \equiv M \text{ (say)}\]

\[M > \frac{|a_n| + |a_{n-1}| - |a_1| - |a_0| + |a_1| + |a_0|}{|a_n|} = 1 + \frac{|a_{n-1}|}{a_n} \geq 1\]

Let $1 \leq M < R$

Where $R = |z + \left(\frac{a_{n-1}}{a_n}\right)| \leq |z| + \left|\frac{a_{n-1}}{a_n}\right|$

\[|z| \geq R - \left|\frac{a_{n-1}}{a_n}\right| \geq 1 + R - M > 1\]

Hence $g(z)$ does not vanish for

\[|z + \left(\frac{a_{n-1}}{a_n}\right)| > \left[\frac{(|a_n| + |a_{n-1}|) + 2 \sin\alpha \sum_{k=0}^{n-2} |a_k| - (\cos\alpha + \sin\alpha - 1)(|a_1| + |a_0|)}{|a_n|}\right]\]

Therefore, those roots of $g(z)$ for which the modulus is greater than one be located in

\[|z + \left(\frac{a_{n-1}}{a_n}\right)| \leq \left[\frac{(|a_n| + |a_{n-1}|) + 2 \sin\alpha \sum_{k=0}^{n-2} |a_k| - (\cos\alpha + \sin\alpha - 1)(|a_1| + |a_0|)}{|a_n|}\right]\]

**Theorem-2 proof:**

Let $g(z) = (1 - z^2)p(z)$

$$= -a_n z^{n+2} + Q(z) \text{ where } Q(z) = -a_{n-1} z^{n+1} + \sum_{k=0}^{n-2} (a_{k+2} - a_k) z^{k+2} + a_1 z + a_0$$

For $|z| = 1$,

\[|Q(z)| \leq |a_0| + |a_1| + |a_{n-1}| + \sum_{k=2}^{n} |a_k - a_{k-2}|\]
\[ |Q(z)| \leq \alpha_0 + |\beta_0| + \alpha_1 + |\beta_1| + \alpha_{n-1} + |\beta_{n-1}| + \sum_{k=2}^{n} (\alpha_k - \alpha_{k-2}) + \sum_{k=2}^{n} (|\beta_k| + |\beta_{k-2}|) \]
\[ = \alpha_n + 2\alpha_{n-1} - |\beta_n| + 2 \sum_{k=0}^{n} |\beta_k| \]
\[ \leq \alpha_n + 2\alpha_{n-1} + 2 \sum_{k=0}^{n} |\beta_k| \]

Hence also
\[ |z^{n+1}Q\left(\frac{1}{z}\right)| \leq \alpha_n + 2\alpha_{n-1} + 2 \sum_{k=0}^{n} |\beta_k| \]

For \(|z| = 1\), by the maximum modulus principle holds inside the unit circle as well.

If \(R > 1\) then \(e^{-\theta}e^{-\theta}\) be located in the unit circle for all real \(\theta\), which implies
\[ |Q(Re^{i\theta})| \leq (\alpha_n + 2\alpha_{n-1} + 2 \sum_{k=0}^{n} |\beta_k|)R^{n+1} \]

for every \(R \geq 1\) and \(\theta\) real.

Thus for \(|z| = R > 1\)
\[ |g(Re^{i\theta})| \geq |\alpha_n|R^{n+2} - |Q(Re^{i\theta})| \]
\[ \geq |\alpha_n|R^{n+2} - \left(\alpha_n + 2\alpha_{n-1} + 2 \sum_{k=0}^{n} |\beta_k|\right)R^{n+1} \]
\[ \geq \alpha_nR^{n+2} - \left(\alpha_n + 2\alpha_{n-1} + 2 \sum_{k=0}^{n} |\beta_k|\right)R^{n+1} \]
\[ |g(Re^{i\theta})| > 0 \text{ if } R > \frac{(\alpha_n + 2\alpha_{n-1} + 2 \sum_{k=0}^{n} |\beta_k|)}{\alpha_n} \]

Theorem-3 proof:

Let \(q(z) = (1 - z^2)p(z)\)
\[ = a_0 + f(z) \text{ where } f(z) = -a_nz^{n+2} - a_{n-1}z^{n+1} + \sum_{k=2}^{n} (a_k - a_{k-2})z^k + a_1z \]

Let \(M(r) = \max_{|z|=r} |f(z)| \)

Then \(M(R) \geq |a_0|\) where \(R = \frac{(\alpha_n + 2\alpha_{n-1} + 2 \sum_{k=0}^{n} |\beta_k|)}{\alpha_n} \)

Clearly, \(|f(z)| \leq |a_0||z|^{n+2} + |a_{n-1}||z|^{n+1} + \sum_{k=2}^{n} |a_k - a_{k-2}| |z|^k + |a_1||z| \) and \(R \geq 1\).

Hence,
\[ M(R) = \max_{|z|=R} |f(z)| \leq |a_0|R^{n+2} + |a_{n-1}|R^{n+1} + |a_1|R + \sum_{k=2}^{n} |a_k - a_{k-2}|R^k \]
\[ \leq |a_0|R^{n+2} + |a_{n-1}|R^{n+1} + |a_1|R + R^n\left(\sum_{k=2}^{n} |a_k - a_{k-2}|\right) \]
\[ \leq |a_0|R^{n+2} + R^n\left(|a_{n-1}| + |a_1| + \sum_{k=2}^{n} |a_k - a_{k-2}|\right) \]
\[ \leq (\alpha_n + |\beta_0|)R^{n+2} + R^n(\alpha_n + 2\alpha_{n-1} - |\beta_n| - |\beta_0| + 2 \sum_{k=0}^{n} |\beta_k|) \equiv M \]

Since \(f(0) = 0\), hence for \(|z| \leq R\) we have by Schwarz's lemma,
\[ |f(z)| \leq \frac{M}{R} |z| \]

For \(|z| \leq R\), 
\[ |g(z)| \geq |a_0| - |z|R^n(2\alpha_nR + (R - 1)|\beta_n| - (\alpha_0 + |\beta_0|) \]

\[ |g(z)| > 0 \text{ if } |z| < \frac{|a_0|}{R^n(2\alpha_nR + (R - 1)|\beta_n| - (\alpha_0 + |\beta_0|))} \]

Since 
\[ \frac{2\alpha_nR + (R - 1)|\beta_n| - (\alpha_0 + |\beta_0|)}{|a_0|} > 0 \]

Then 
\[ R^n(2\alpha_nR + (R - 1)|\beta_n| - (\alpha_0 + |\beta_0|)) < R. \]

Theorem-4 proof:

Let \(g(z) = (1 - z^2)p(z)\)
\[ = -a_nz^{n+2} + Q(z) \text{ where } Q(z) = -a_{n-1}z^{n+1} + \sum_{k=2}^{n} (a_k - a_{k-2})z^k + a_1z + a_0 \]

Let \(T(z) = z^{n+1}Q\left(\frac{1}{z}\right) = -a_{n-1} + \sum_{k=2}^{n} (a_k - a_{k-2})z^{-k+1} + a_1z + a_0z^{n+1} \)
For $|z| = 1$, we have

$$|T(z)| \leq |a_0| + |a_1| + |a_{n-1}| + \sum_{k=2}^{n} |a_k - a_{k-2}| \leq M$$

where

$$M \equiv a_n + a_{n-1} + \beta_n + \beta_{n-1} + (|a_0| - a_0) + (|a_1| - a_1) + (|\alpha_0| - \beta_0) + (|\alpha_1| - \beta_1) + |\alpha_{n-1}| + |\beta_{n-1}|$$

By the maximum modulus principle, it holds inside the unit circle as well.

If $R > 1$ then $\frac{1}{R} e^{-i\theta} z$ be located in the unit circle for all real $\theta$, which implies

$$|Q(Re^{i\theta})| \leq M R^{n+1}$$

for every real $R \geq 1$ and real $\theta$.

Thus for $|z| = R > 1$

$$g(R e^{i\theta}) > 0 \text{ if } \frac{g(R e^{i\theta})}{M} = 1 + \frac{1}{R} e^{-i\theta} = \frac{\alpha_n}{a_{n-1}}\left(\alpha_n - \frac{1}{R} e^{-i\theta}\right) + \left(\sum_{k=2}^{n} |a_k - a_{k-2}| R\right)^2 + \frac{1}{a_{n-1}}$$

Hence the concept of maximum modulus, $|T(0)| = |a_{n-1}| < M$

By lemma-2 on the function $T(z)$ we obtain for $|z| \leq 1$,

$$|T(z)| \leq M \frac{|z| + |a_{n-1}|}{|a_{n-1}| |z| + M}$$

This implies that

$$|z^{n+1} Q(z)\left(\frac{1}{z}\right)| \leq M \frac{|z| + |a_{n-1}|}{|a_{n-1}| |z| + M}$$

If $R > 1$, $\frac{1}{R} e^{-i\theta} z$ be located in the unit circle for all real $\theta$, which implies

$$|Q(Re^{i\theta})| \leq M R^{n+1} M + |a_{n-1}| R$$

for every real $R \geq 1$ and real $\theta$.

Thus for $|z| = R > 1$

$$g(Re^{i\theta}) \geq |a_n|R^{n+2} - \left|Q(Re^{i\theta})\right| \geq |a_n|R^{n+2} - M R^{n+1} M + |a_{n-1}| R$$

$$\geq \frac{R^{n+1}}{MR + |a_{n-1}|} [M |\alpha_n|^2 - |a_{n-1}|(M - \alpha_n) R - M^2]$$

$> 0$ if $R > \frac{|a_{n-1}|}{\frac{1}{2} (\alpha_n - \frac{1}{M})} + \left(\frac{1}{2} \frac{|a_{n-1}|^2}{\alpha_n} - \frac{1}{2} + \frac{M}{\alpha_n}\right)^{\frac{1}{2}} \equiv R_1$

Therefore $g(z)$ have all the zeros of located in $|z| \leq R_1$ where $R_1 > 1$.

It means that all zeros of $f(z)$ are located in $|z| \leq R_1$.

Subsequently, it can be shown that no zeros of $g(z)$ are located in $|z| < R_2$.

$$g(z) = a_0 + f(z) = a_0 + a_1 z + \sum_{k=2}^{n} (a_k - a_{k-2}) z^k - a_{n-1} z^{n+1} - a_n z^{n+2}$$

Let $M(R_1) = \max_{|z|=R_1} |f(z)|$

Since $R_1 \geq 1$, $f(1) = -a_0$, we have $M(R_1) \geq |a_0|$

Clearly $|f(z)| \leq |a_n||z|^{n+2} + \sum_{k=2}^{n} (a_k - a_{k-2}) |z|^k + |a_1||z| + |a_{n-1}| |z|^{n+1}$

And hence

$$M(R_1) \leq |a_n|R_1^{n+2} + \sum_{k=2}^{n} (a_k - a_{k-2}) R_1^k + |a_1|R_1 + |a_{n-1}| R_1^{n+1}$$

$$\leq |a_n|R_1^{n+2} + R_1^{n+1} \left(\frac{|a_1|}{R_1} + |a_{n-1}| + \sum_{k=2}^{n} |a_k - a_{k-2}|\right)$$

$$\leq R_1^{n+1} \left(|\alpha_n| + |\beta_n| R_1 + M - |a_0| - |\beta_0|\right) \equiv M_1$$

Further because $f(0) = 0$, $f'(0) = a_1$ we have by lemma-5

$$|f(z)| \leq \frac{M_1 |z| |M_1 z| + R_1^2 |a_1|}{R_1^2} |M_1 + |a_1||z|$$

$$|g(z)| \geq |a_0| - \frac{M_1 |z| |M_1 z| + R_1^2 |a_1|}{R_1^2} |M_1 + |a_1||z| = \frac{-1}{R_1^2 (M_1 + |z| |a_1|)} [z^2 R_1^2 M_1^2 + R_1^2 |a_1||z|(M_1 - |a_0| R_1^2 M_1)]$$

Therefore $g(z)$ $> 0$ if $|z| < \frac{-R_1^2 |a_0| + R_1^2 |a_1|}{2 |M_1|^2} \equiv R_2$ (say) where $R_2 \leq R_1$. 

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Theorem-5 proof:
Let \( g(z) = (1 - z^2)p(z) \) 
\[ g(z) = -a_n z^{n+2} + Q(z) \] 
where \( Q(z) = -a_{n-1}z^{n+1} + \sum_{k=2}^{n}(a_k - a_{k-2})z^k + a_1z + a_0 \) 
For \( |z| = 1 \) we have 
\[ |Q(z)| \leq M_2 \]
where 
\[ M_2 = \{ \begin{cases} a_n + (|a_0| + |\beta_0| + a_0 + \beta_0) + (|a_1| + |\beta_1| - a_1) + (|a_{n-1}| + |\beta_{n-1}| - a_{n-1} - \beta_{n-1}) + (\beta_n - \beta_1) & \text{or} \\ a_n + (|a_0| + |\beta_0| - a_0) + (|a_1| + |\beta_1| + a_1 + \beta_1) + (|a_{n-1}| + |\beta_{n-1}| - a_{n-1} - \beta_{n-1}) + (\beta_n - \beta_0) & \text{according as } n \text{ is odd or even} \end{cases} \] 
Hence also for \( |z| = 1 \), 
\[ |z^{n+1} Q \left( \frac{1}{z} \right)| \leq M_2. \]
By the maximum modulus principle it holds inside the unit circle as well.
If \( R > 1 \) then \( \frac{1}{R} e^{-i\theta} \) be located in the unit circle for all real \( \theta \) and follows that 
\[ |g(R e^{i\theta})| \leq M_2 R^{n+1} \] 
for every \( R \geq 1 \) and real \( \theta \).
Thus for \( |z| = R > 1 \)
\[ |g(R e^{i\theta})| > 0 \text{ if } R > \frac{M_2}{a_n} \text{ where } R > 1. \]

If \( a_0, a_1, a_{n-1} \geq 0 \) and \( \beta_0, \beta, \beta_{n-1} \geq 0 \) in theorem-5 then

Corollary-5.1: If \( p(z) = \sum_{k=0}^{n} a_k z^k \) with \( a_n \neq 0 \) such that 
\[ a_n \geq a_{n-2} \geq \cdots \geq a_1 \text{ or } a_0 \geq 0 \] 
\[ 0 \leq a_{n-1} \leq a_{n-3} \leq \cdots \leq a_0 \text{ or } a_1 \] (according as \( n \) is odd or even) and \( a_n > 0 \)
where \( a_j = a_j + i \beta_j, \text{ } j = 0(1)n \) then all the zeros of \( p(z) \) lie in the disc 
\[ |z| \leq \begin{cases} \frac{a_n}{a_n + \beta_n + 2(a_0 + \beta_0)} & \text{ (according as } n \text{ is odd or even) } \\ \frac{a_n}{a_n + \beta_n + 2(a_1 + \beta_1)} & \end{cases} \]

If all the coefficients of the polynomial are real in theorem-5 then

Corollary-5.2: If \( p(z) = \sum_{k=0}^{n} a_k z^k \) is a polynomial of degree \( n \) such that 
\[ a_n \geq a_{n-2} \geq \cdots \geq a_1 \text{ or } a_0 \geq 0 \] 
\[ a_{n-1} \leq a_{n-3} \leq \cdots \leq a_0 \text{ or } a_1 \] (according as \( n \) is odd or even) and \( a_n > 0 \) then all the roots of \( p(z) \) lie in the disc 
\[ |z| \leq \begin{cases} \frac{a_n}{1 + \frac{(a_0| + a_0) + (a_1| - a_1) + (a_{n-1}| - a_{n-1})}{a_n} - \frac{a_n}{a_n} & \text{ (according as } n \text{ is odd or even) } \\ \frac{a_n}{1 + \frac{(a_0| - a_0) + (a_1| + a_1) + (a_{n-1}| - a_{n-1})}{a_n} - \frac{a_n}{a_n} & \end{cases} \]

If all the coefficients of the polynomial are real and non-negative in theorem-5 then

Corollary-5.3: If \( p(z) = \sum_{k=0}^{n} a_k z^k \) with \( a_n \neq 0 \) such that 
\[ a_n \geq a_{n-2} \geq \cdots \geq \text{ } a_1 \text{ or } a_0 \geq 0 \] 
\[ 0 \leq a_{n-1} \leq a_{n-3} \leq \cdots \leq a_0 \text{ or } a_1 \] (according as \( n \) is odd or even) and \( a_n > 0 \) then all the roots of \( p(z) \) lie in the disc 
\[ |z| \leq \begin{cases} 1 + \frac{2a_0}{a_n} & \text{ (according as } n \text{ is odd or even) } \\ 1 + \frac{2a_1}{a_n} & \end{cases} \]
Selected Extensions on Eneström-kakeya Theorem

References