

Average circular D-distance and circular D-Wiener index of K-regular graphs

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Abstract: The circular D -distance between nodes of a graph is obtained by the sum of detour D -distance and D -distance. The average circular D -distance between the nodes of a graph is the sum of average of the detour D -distance and D -distances. In this paper, we deal with the average circular D -distance between nodes of graph. We compute the relation between circular Wiener index and circular D -Wiener index of k -regular graphs. We obtained results on circular D -Wiener index of some special graphs

Keywords: Average circular D -distance, detour Wiener index, detour D -Wiener index, circular D -Wiener index

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1. Introduction

The idea of distance is one of the significant idea in investigation of graphs. It is used to test the isomorphism of graphs, connectivity problems and convexity of graphs etc.

In addition to the geodesic distance, $d(r, s)$, we have detour distance (introduced by Chartrand et al in [3], average distance and dominating number (introduced by Dankelmann in [4,5]), mean distance in graphs (introduced by Doyale in [6]), superior distance (introduced by Kathiresan and Marimuthu in [7]), signal distance (introduced by Kathiresan and Marimuthu in [8]), average D -distance (introduced by Reddy babu, Varma in [11]), average detour D -distance (introduced by Venkateswara Rao, Varma in [12]).

In previous article (in [15]), the authors presented the idea of circular D -distance in graphs by adding detour D -distance and D -distance. A relation between Wiener index for k -regular graphs and D -index (was obtained by Ahmed Mohammed Ali and Asmaa Salah Aziz in [1]).

In this article we introduce, the concept of average circular D -distance between nodes of a graph and study some of its properties. Further we compute the average circular D -distance of some classes of graphs. We introduce the concept of Wiener detour D -index of a graph H . We obtain a relation between detour Wiener index and detour D -Wiener index of any regular graph H . Further we compute detour D -Wiener index of some special graphs. Finally we obtain a relation between circular Wiener index and circular D -Wiener index of regular graph H . Furthermore, this distance is also used in molecular dynamics of physics, crystallography, lattice statistics and physics.

Throughout this article we consider connected and simple graphs. For any unexplained terminology and symbols, we refer the book [2].

2. Average circular D-distance

In this section we given some definitions for later use.

Definition 2.1 Let H be a connected graph of order n , the *average distance* (respectively, *average detour distance*) between nodes of H , denoted by $\mu(H)$ (respectively, $\mu_D(H)$), is defined as

$$\mu(H) = \frac{1}{2} \binom{n}{2}^{-1} \sum_{r,s \in V(H)} d(r,s) \quad \mu_D(H) = \frac{1}{2} \binom{n}{2}^{-1} \sum_{r,s \in V(H)} D(r,s)$$

(respectively,) where $d(r,s)$ and $D(r,s)$

denote the distance and detour distance between the nodes of r and s respectively.

Definition 2.2. Let H be a connected graph of order n , the *average D-distance* (respectively, *average detour D-distance*) between nodes of H , denoted by $\mu^D(H)$ (respectively, $\mu_D^D(H)$), is defined as

$$\mu^D(H) = \frac{1}{2} \binom{n}{2}^{-1} \sum_{r,s \in V(H)} d^D(r,s) \quad \mu_D^D(H) = \frac{1}{2} \binom{n}{2}^{-1} \sum_{r,s \in V(H)} D^D(r,s)$$

(respectively,) where $d^D(r,s)$ and $D^D(r,s)$

denote the D -distance and detour D -distance between the nodes of r and s respectively.

Definition 2.3 [16]. In a graph H , r, s are any two nodes and *circular distance* between them is defined as

$$cir(r, s) = \begin{cases} 0 & \text{if } r = s \\ d(r, s) + D(r, s) & \text{if } r \neq s \end{cases}$$

Definition 2.4 [15]. In a graph H , r, s are any two nodes and *circular D -distance* between them is defined as

$$cir^D(r, s) = \begin{cases} 0 & \text{if } r = s \\ \min\{l^D(S)\} + \max\{l^D(S)\} & \text{if } r \neq s \end{cases}$$

Now, average circular D -distance of a graph as shown below.

Definition 2.5. $\mu_C^D(H)$, call it as *average circular D -distance (AVCDD)* between nodes of H , where H is a connected graph of order n , defined as sum of average detour D -distance and average D -distance, i.e.,

$$\mu_C^D(H) = \frac{1}{2} \binom{n}{2}^{-1} \sum_{r, s \in V(H)} cir^D(r, s),$$

where $cir^D(r, s)$ denotes the circular D -distance between the nodes r and s .

Definition 2.6. Let H be a connected graph of order n with node set $\{r_1, r_2, r_3, \dots, r_n\}$ and having m edges. The *circular D -distance matrix* of H , represented as, $D_C^D(H)$, is defined as $D_C^D(H) = [cir_{i,j}^D]_{n \times n}$ where $cir_{i,j}^D = cir^D(r_i, r_j)$ is the circular D -distance of the nodes r_i and r_j

Definition 2.7. The *total circular D -distance (TCDD)* of graph H is the number given by $\sum_{r,s} cir^D(r, s)$.

3. Results on average circular D -distance

Now we prove some results on AVCDD between nodes. We begin with theorem which connects the number of nodes and AVCDD. This leads to some more results.

Theorem 3.1. Let H_1 and H_2 be two graphs having equal number of nodes and $d_C^D(H_1) < d_C^D(H_2)$. If $|E_1| < |E_2|$ then $\mu_C^D(H_1) < \mu_C^D(H_2)$.

Proof: Since $d_C^D(H_1) < d_C^D(H_2)$, the biggest entry in the circular D -distance matrix of $H_1 < H_2$ and this leads total circular D -distance value to increase. Since orders are same and $|E_1| < |E_2|$. Hence $\mu_C^D(H_1) < \mu_C^D(H_2)$.

Corollary 3.2 Let H_1 and H_2 contains equal number of nodes and $d_C^D(H_1) = d_C^D(H_2)$. If $|E_1| = |E_2|$ then $\mu_C^D(H_1) = \mu_C^D(H_2)$.

Theorem 3.3. Let H_1 and H_2 be two graphs having equal number of nodes and $d_C^D(H_1) < d_C^D(H_2)$. If $\delta(H_1) < \delta(H_2)$ then $\mu_C^D(H_1) < \mu_C^D(H_2)$.

Proof: Let H_1 and H_2 be two connected graphs having equal number of nodes and $d_C^D(H_1) < d_C^D(H_2)$. Then clearly $\delta(H_1) < \delta(H_2) \Rightarrow |E_1| < |E_2|$. Then from **theorem 3.1** we get $\mu_C^D(H_1) < \mu_C^D(H_2)$.

Corollary 3.4 Let H_1 and H_2 contains equal number of nodes and $d_C^D(H_1) = d_C^D(H_2)$. If $\delta(H_1) = \delta(H_2)$ then $\mu_C^D(H_1) = \mu_C^D(H_2)$.

Theorem 3.5. Let H_1 be a spanning subgraph of H_2 then $\mu_C^D(H_1) < \mu_C^D(H_2)$.

Proof: Let H_1 be a spanning subgraph of H_2 . Then the number of nodes of H_1 and H_2 are same and number of edges of H_1 is less than that of H_2 . Then by **theorem 3.1** we get $\mu_C^D(H_1) < \mu_C^D(H_2)$.

4. Results on some families of graphs

Here we compute the *AVCDD* for some families of graphs.

Theorem 4.1. *The AVCDD of the complete graph K_m is $m^2 + 2m - 2$.*

Proof: We know that, K_m is a $m-1$ regular graph and the circular D -distance between each pair of nodes is $m^2 + 2m - 2$ (see theorem 4.1 in [14]). Thus the *TCDD* is $2 \binom{n}{2} (m^2 + 2m - 2)$. Hence

$$\mu_C^D(K_m) = \frac{1}{2} \left[\frac{TCDD}{\binom{n}{2}} \right] = m^2 + 2m - 2$$

Remark 4.2. *In a complete graph K_m , $\mu_C^D(K_m) = d_C^D(K_m) = d_C^D(K_m)$.*

Theorem 4.3. *The AVCDD of the cycle graph C_m is $3m + 4$.*

Proof: Each node of the cyclic graph C_m has 2 adjacent nodes. The circular D -distance between each pair of nodes is $3m + 4$ (see theorem 4.3 in [14]). Thus the *TCDD* is $2 \binom{n}{2} (3m + 4)$. Hence

$$\mu_C^D(C_m) = \frac{1}{2} \left[\frac{TCDD}{\binom{n}{2}} \right] = 3m + 4.$$

Remark 4.4: *In a cycle graph C_m , $\mu_C^D(C_m) = d_C^D(C_m) = d_C^D(C_m)$.*

Theorem 4.5: *The AVCDD of a path graph P_n is $\frac{4a_n}{n}$, where $a_n = a_{n-1} + n + 1$ with $a_1 = 0$.*

Proof: Consider the circular D -distance matrix of the path graph P_n , which is $n \times n$ symmetric matrix.

$$\begin{bmatrix} 0 & 8 & 14 & 10 & \cdots & 6n-10 & 6n-4 \\ & 0 & 10 & 16 & \cdots & 6n-14 & 6n-8 \\ & & 0 & 10 & \cdots & 6n-24 & 6n-18 \\ & & & \vdots & \cdots & \vdots & \vdots \\ & & & & 0 & 10 & 14 \\ & & & & & 0 & 8 \\ & & & & & & 0 \end{bmatrix}$$

Summation of all elements in this right triangular matrix, we have the *TCDD* is $4a_n(n-1)$, where a_n is a constant given by $\{3, 7, 12, 18, 25, \dots\}$ or recursively $a_n = a_{n-1} + n + 1$ with $a_1 = 0$. Hence the total circular D -distance is $4a_n(n-1)$.

Thus AVCDD of P_n is
$$\mu_C^D(C_m) = \frac{1}{2} \left[\frac{TCDD}{\binom{n}{2}} \right] = \left[\frac{4a_n(n-1)}{\frac{1}{2}n(n-1)} \right] = \frac{4a_n}{n}$$

Theorem 4.6. The AVCDD of star graph $St_{n,1}$ is $\frac{2n(n+5)}{n-1}$.

Proof: In a star graph $St_{n,1}$, let v_0 be the node which is adjacent to all other nodes. The circular D - distance between any two nodes is either $2n+4$ or $2n+8$ (see theorem 3.14 in [15]). Finally the total circular D -

distance is $2n(2n+4) + 2\binom{n}{2}(2n+8)$. Thus AVCDD of $St_{n,1}$ is

$$\begin{aligned} \mu_C^D(St_{n,1}) &= \frac{1}{2} \left[\frac{TCDD}{\binom{n}{2}} \right] \\ &= \frac{2(2n+4) + (n-1)(2n+8)}{n-1} \\ &= \frac{2n(n+5)}{n-1} \end{aligned}$$

Theorem 4.7.

1. The AVCDD of complete bipartite graph $K_{x,y}$ is
$$\frac{(x+1)y^3 + (3x^2 + 6x + 1)y^2 + (3x^3 + 6x^2 - 7x - 2)y + (x^4 + x^3 - 2x^2)}{y^2 + (2x-1)y + x^2 - x}$$

2. The AVCDD of complete bipartite graph $K_{x,x}$ is
$$\frac{4x^3 + 6x^2 - 4x}{2x-1}$$

Proof: (1) The node set of $K_{x,y}$ can be split as $U_1 \cup U_2$, where $U_1 = \{v_1, v_2, v_3, \dots, v_x\}$, $U_2 = \{w_1, w_2, w_3, \dots, w_y\}$. Then $cir^D(v_i, v_j) = x(x+5) + y(x+1) + 2$, $cir^D(w_i, w_j) = x^2 + xy + 2(x+y)$, $cir^D(v_i, w_j) = x(x+3) + y(x+1)$.

Thus the total circular D - distance is
$$\frac{(x+1)y^3 + (3x^2 + 6x + 1)y^2 + (3x^3 + 6x^2 - 7x - 2)y + (x^4 + x^3 - 2x^2)}{y^2 + (2x-1)y + x^2 - x}$$
. Hence the average circular D - distance

is
$$\mu_C^D(K_{x,y}) = \frac{1}{2} \left[\frac{TCDD}{x+y} \right]$$

$$= \frac{(x+1)y^3 + (3x^2 + 6x + 1)y^2 + (3x^3 + 6x^2 - 7x - 2)y + (x^4 + x^3 - 2x^2)}{y^2 + (2x-1)y + x^2 - x}$$

(2) The node set of $K_{x,x}$ can be split as $U_1 \cup U_2$, where $U_1 = \{v_1, v_2, v_3, \dots, v_x\}$, $U_2 = \{w_1, w_2, w_3, \dots, w_x\}$. Then $cir^D(v_i, v_j) = 2x^2 + 4x$, $cir^D(w_i, w_j) = 2x^2 + 4x$, $cir^D(v_i, w_j) = 2x^2 + 4x$. Thus the total circular D -

distance is $2 \left[\binom{x}{2} (2x^2 + 4x) + x^2 (2x^2 + 4x) + \binom{x}{2} (2x^2 + 4x) \right] = 2 \left[(2x^2 - x)(2x^2 + 4x) \right]$. Hence the average

$$\mu_C^D(K_{x,x}) = \frac{1}{2} \left[\frac{TCDD}{2x} \right] = \frac{4x^3 + 6x^2 - 4x}{2x - 1}$$

circular D -distance is

5. The index of detour D -distance for connected k -regular graphs

Here we begin with some definitions on Wiener index and detour D -Wiener index.

Definition 5.1. Let H be a graph of order n , the *Wiener index* (respectively, *detour Wiener index*) between

nodes of H , denoted by $W(H)$ (respectively, $D(H)$), is defined as $W(H) = \frac{1}{2} \sum_{\forall r,s \in V(H)} d(r,s)$ (respectively, $D(H) = \frac{1}{2} \sum_{\forall r,s \in V(H)} D(r,s)$), where $d(r,s)$ and $D(r,s)$ denotes the distance and detour distance between the nodes of r and s respectively.

Definition 5.2. The *total D -distance* or *D -index* of a graph H is defined as

$$W^D(H) = \frac{1}{2} \sum_{\forall r,s \in V(H)} d^D(r,s).$$

Definition 5.3. The *total detour D -distance* or *detour D -index* of a graph H is defined as

$$W_D^D(H) = \frac{1}{2} \sum_{\forall r,s \in V(H)} D^D(r,s).$$

Now we prove a relation between detour Wiener index and detour D -Wiener index of k -regular graphs.

$$W_D^D(H) = (k+1)D(H) + k \binom{n}{2}.$$

Theorem 5.4. Let H be a k -regular graph of order n . Then we have

Proof: We have, by definition

$$\begin{aligned} W_D^D(H) &= \frac{1}{2} \sum_{\forall r,s \in V(H)} D^D(r,s). \\ &= \frac{1}{2} \sum_{\forall r,s \in V(H)} \left[\max_p \left\{ l(P) + \sum_{w \in V(P)} \deg(w) \right\} \right] \\ &= \frac{1}{2} \sum_{\forall r,s \in V(H)} \left\{ D(r,s) + \sum_{w \in V(P:r-s \text{ detour path})} \deg(w) \right\} \\ &= \frac{1}{2} \sum_{\forall r,s \in V(H)} D(r,s) + \frac{1}{2} \sum_{\forall r,s \in V(H)} \left\{ \sum_{w \in V(P:r-s \text{ detour path})} \deg(w) \right\} \\ &= D(H) + \frac{1}{2} \sum_{\forall r,s \in V(H)} \left\{ \sum_{w \in V(P:r-s \text{ detour path})} \deg(w) \right\} \end{aligned}$$

Since H is k -regular then $\deg(w) = k$, for all $w \in V(H)$ and between r and s every detour path contains $D(r,s) + 1$ nodes with $\deg(w) = k$ for all w . Thus

$$\begin{aligned} W_D^D(H) &= D(H) + \frac{1}{2} \sum_{\forall r,s \in V(H)} (D(r,s)+1)k \\ &= D(H) + \frac{1}{2} \sum_{\forall r,s \in V(H)} D(r,s)k + \frac{k}{2} \sum_{\forall r,s \in V(H)} 1 \\ &= D(H) + kD(H) + k \binom{n}{2} \\ &= D(H)(k+1) + k \binom{n}{2} \end{aligned}$$

Hence
$$W_D^D(H) = (k+1)D(H) + k \binom{n}{2}.$$

$$W_D^D(H) = \begin{cases} \frac{11n^2 - 14n}{2} & \text{if } n \text{ is even} \\ \frac{9n^3 - 4n^2 - 5n}{8} & \text{if } n \text{ is odd} \end{cases}$$

Theorem 5.5. (1) For the cycle graph C_n ,

(2) For the complete graph K_n ,
$$W_D^D(K_n) = (n^2 - 1) \binom{n}{2}.$$

Proof. This proof follows from the theorem 1 and corollary 1 of [1].

Theorem 5.6. For the complete bipartite graph $K_{x,x}$,
$$W_D^D(K_{x,x}) = 4x^4 + x^3 - 4x^2 + 2x.$$

Proof: The proof is similar to that of 6.5 below.

6. The detour D -index of some special graphs

Now we find the detour D -index of graphs which are not regular.

Now we begin with path graph.

Theorem 6.1. The detour D -index of a path graph P_n is
$$\frac{1}{2}n^3 + n^2 - \frac{7}{2}n + 2.$$

Proof. In a path graph P_n , both D -distance and detour D -distance are equal. Thus see the proof of theorem 4 of [1].

Theorem 6.2. The detour D -index of a star graph $St_{n,1}$ is
$$\frac{1}{2}n^2(n+5).$$

Proof. In a star graph $St_{n,1}$ both D -distance and detour D -distance are equal. Thus see the proof of theorem 4 of [1].

Theorem 6.3. The detour D -index of a friendship graph F_n is
$$\frac{1}{2}[n^3 + 7n^2 - 17n + 9].$$

Proof. Let F_n be a friendship graph whose nodes can be listed as $V(F_n) = \{r_1, r_2, r_3, \dots, r_n\}$, where $deg(r_1) = n-1$ and $deg(v_i) = 2$, for all $i = 2, 3, \dots, n$. Then we have three cases:

Case 1: If $r_1 r_i \in E(F_n)$ then $D^D(r_1, r_i) = n+5$, for $i = 2, 3, \dots, n$.

Case 2: If $r_{2i} r_{2i+1} \in E(F_n)$ then $D^D(r_{2i}, r_{2i+1}) = n+5$, for $i = 2, 3, \dots, \frac{n-1}{2}$ and number of such pairs of (r_{2i}, r_{2i+1}) is $\frac{n-1}{2}$.

Case 3: If $r_{2i-1}r_{2i} \notin E(F_n)$ then the path P consists from $r_{2i-1}, r_i, r_1, r_{2i}$. Hence $D^D(r_{2i}, r_{2i-1}) = n+8$,
 for $i = 2, 3, \dots, \frac{n-1}{2}$. The number of such pairs are $\frac{(n-1)(n-3)}{2}$. Thus we have
 $W_D^D(F_n) = (n-1)(n+5) + \frac{1}{2}(n+5)(n-1) + \frac{1}{2}(n+8)(n-1)(n-3) = \frac{1}{2}[n^3 + 7n^2 - 17n + 9]$.

Theorem 6.4. For a wheel graph $W_{1,n}$, we have $W_D^D(W_{1,n}) = \frac{1}{2}(5n^2(n+1))$ and for $n \geq 3$.

Proof: Let $\{r_0, r_1, \dots, r_n\}$ be the node set of wheel graph. Suppose that r_0 is adjacent to all other nodes. Thus $deg(r_0) = n$ and $deg(r_i) = 3$, for $i = 1, 2, \dots, n$. Hence $D^D(r_i, r_j) = 5n$ for all i, j (theorem 4.5 of [12]). The number of this type of pairs is $\binom{n+1}{2}$. Thus total detour D -distance is $2\binom{n+1}{2}(5n)$. Hence we have
 $W_D^D(W_{1,n}) = \frac{1}{2}(5n^2(n+1))$.

Now, we look at complete bipartite graph.

Theorem 6.5. For a complete bipartite graph $K_{x,y}$, having $x+y$ nodes with $x < y$. Then we have

$$W_D^D(K_{x,y}) = \frac{1}{2}[xy^3 + (3x^2 + 2x)y^2 + (3x^3 + 2x^2 - 5x)y + (x^4 - 3x^2 + 2x)]$$

Proof. In $K_{x,y}$, the node set can be divided into sets U_1 and U_2 . The number of nodes of U_1 and U_2 is x and y respectively.

Now, see the three cases:

Case1: If $r, r' \in U_1$ then $D^D(r, r') = x^2 + xy + x - 2$ (theorem 4.6 of [12]), this is true for all $\frac{x(x-1)}{2}$ pairs of nodes.

Case2: $r, r' \in U_2$ then $D^D(r, r') = x^2 + xy + 3x$ (theorem 4.6 of [12]), this is correct for all $\frac{y(y-1)}{2}$ couple of nodes.

Case 3: If $r \in U_1$ and $r' \in U_2$ then $D^D(r, r') = x^2 + 2x + xy - 1$ (theorem 4.6 of [12]), this is correct for all xy couple of nodes. The total detour D -distance is
 $2\left[\binom{x}{2}(x^2 + xy + x - 2) + xy(x^2 + 2x + xy - 1) + \binom{y}{2}(x^2 + xy + 3x)\right]$. Thus we have $W_D^D(K_{x,y})$
 $= \frac{1}{2}[xy^3 + (3x^2 + 2x)y^2 + (3x^3 + 2x^2 - 5x)y + (x^4 - 3x^2 + 2x)]$.

7. Circular D -Wiener index of k -regular graphs

Here we introduce the definition of circular D -Wiener index and compute the connection between circular D -Wiener index of k -regular graphs and circular Wiener index.

Definition 7.1. The circular D -Wiener index (respectively, circular Wiener index) between nodes of H ,

denoted by $W_C^D(H)$ (respectively, $W^C(H)$), is defined as $W_C^D(H) = \frac{1}{2} \sum_{\forall r, s \in V(H)} cir^D(r, s)$ (respectively,

$W^C(H) = \frac{1}{2} \sum_{\forall r, s \in V(H)} cir(r, s)$, where $cir^D(r, s)$ and $cir(r, s)$ denotes the circular D -distance (see in [15]) and circular distance (see in [16]) between the nodes of r and s respectively.

Now we compute the relation between circular D -Wiener index and circular Wiener index.

Theorem 7.2. Let H be a k -regular graph of order n . Then we have $W_C^D(H) = (k+1)W^C(H) + kn(n-1)$

Proof. The index of D -distance for a k -regular graph of H is $W^D(H) = (k+1)W(H) + k \binom{n}{2}$ (see theorem 3 in [1]). The detour D -Wiener index for a connected k -regular graph is $W_D^D(H) = (k+1)D(H) + k \binom{n}{2}$ (see theorem 5.4). Hence the next proof follows from the definition circular D -Wiener index. Hence $W_C^D(H) = (k+1)W^C(H) + kn(n-1)$.

Theorem 7.3. For the complete graph K_m , $W_C^D(K_m) = (m^2 + 2m - 2) \binom{m}{2}$

Proof; See the proof of theorem 4.1

Theorem 7.4. The circular D -Wiener index of a cycle graph C_m is $(3m+4) \binom{m}{2}$.

Proof; See the proof of theorem 4.3

Theorem 7.5. The circular D -Wiener index of a complete bipartite graph $K_{x,x}$ is $4x^4 + 6x^3 - 4x^2$.

Proof; See the proof of theorem 4.7

Theorem 7.6. The circular D -Wiener index of a path graph P_n is $2a_n(n-1)$.

Proof; See the proof of theorem 4.5

Theorem 7.7. The circular D -Wiener index of a star graph $St_{n,1}$ is $n^3 + 5n^2$.

Proof; See the proof of theorem 4.6

Theorem 7.8. The circular D -Wiener index of a complete bipartite graph $K_{x,y}$ is $\frac{1}{2} \left[(x+1)y^3 + (3x^2 + 6x + 1)y^2 + (3x^3 + 6x^2 - 7x - 2)y + (x^4 + x^3 - 2x^2) \right]$.

Proof; See the proof of theorem 4.7

8. Conclusion

In this paper, we discuss properties of $AVCDD$ and we find the average circular D -distance of some families of graphs. Further, we find a formula giving a connection between the detour Wiener index and detour D -Wiener index. We find the connection between circular Wiener index and circular D -Wiener index of k -regular graphs.

In future work, we find average circular D -distance between arcs of a graph and we can find the relation between circular D -Wiener index and average circular D -distance between arcs of a graph.

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